

## On generalized topological groups

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**Abstract.** In this paper, we initiate the study of generalized topological groups. A generalized topological group has the algebraic structure of groups and the topological structure of a generalized topological space defined by Á. Császár [2] and they are joined together by the requirement that multiplication and inversion are  $\mathcal{G}$ -continuous. Every topological group is a  $\mathcal{G}$ -topological group whereas converse is not true in general. Quotients of generalized topological groups are defined and studied.

### 1. Introduction

In his paper [2], Császár introduced the notions of generalized neighborhood systems and generalized topological spaces. He also introduced the notions of associated interior and closure operators and continuous mappings on generalized neighborhood systems and generalized topological spaces. Császár investigated characterizations of generalized continuous mappings in [2] and studied the separation axioms in the generalized topologies in [4]. Product of generalized topologies was investigated in [5]. Many other topologists [6–8] continued to study and investigate different aspects of the generalized topological structures.

In this paper, the notion of generalized topological groups is introduced. Generalized topological groups are the groups with generalized topologies, multiplication and inversion being generalized continuous. Some basic properties are investigated. Generalized connectedness in generalized topological groups and quotients of generalized topological groups are studied. To start with, some notions required in the sequel are recalled.

**Definition 1.1.** ([2]) Let  $X$  be any set and let  $\mathcal{G} \subset \mathcal{P}(X)$  be a subfamily of power set of  $X$ . Then  $\mathcal{G}$  is a *generalized topology* if,  $\emptyset \in \mathcal{G}$  and for any index set  $I$ ,  $\cup_{i \in I} O_i \in \mathcal{G}$  whenever  $O_i \in \mathcal{G}$ ,  $i \in I$ . A  $\mathcal{G}$ -topological space  $X$  is called *strong* if the set  $X$  itself is in  $\mathcal{G}$ . Generalized topology will be denoted by  $\mathcal{G}$ -topology.

The elements of  $\mathcal{G}$  are called  *$\mathcal{G}$ -open sets*. Similarly, a generalized closed set, or  $\mathcal{G}$ -closed set, is defined as complement of a  $\mathcal{G}$ -open set.

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Let  $X$  and  $Y$  be two  $\mathcal{G}$ -topological spaces. A mapping  $f : X \rightarrow Y$  is called  $\mathcal{G}$ -continuous on  $X$  if for any  $\mathcal{G}$ -open set  $O$  in  $Y$ ,  $f^{-1}(O)$  is  $\mathcal{G}$ -open in  $X$ .

The bijective mapping  $f$  is called a  $\mathcal{G}$ -homeomorphism from  $X$  to  $Y$  if both  $f$  and  $f^{-1}$  are  $\mathcal{G}$ -continuous. If there is a  $\mathcal{G}$ -homeomorphism between  $X$  and  $Y$  they are said to be  $\mathcal{G}$ -homeomorphic denoted by  $X \cong_{\mathcal{G}} Y$ .

**Definition 1.2.** ([2]) Let  $\psi : X \rightarrow \exp(\exp(X))$  satisfy  $x \in V$  for  $V \in \psi(x)$ .  $V$  is called *generalized neighborhood* ( $\mathcal{G}$ -neighborhood) of  $x \in X$  and  $\psi$  is called a *generalized neighborhood system* ( $\mathcal{G}$ -neighborhood system) on  $X$ . The collection of all  $\mathcal{G}$ -neighborhood systems on  $X$  will be denoted by  $\Psi(X)$ .

The following result gives the relation between  $\mathcal{G}$ -open sets and  $\mathcal{G}$ -neighborhood systems by combining 1.2 and 1.3 of [2].

**Lemma 1.3.** Let  $\Psi$  be a  $\mathcal{G}$ -neighborhood system on  $X$ , let  $\mathcal{G} \subset \mathcal{P}(X)$ . For  $O \in \mathcal{G}$  and  $x \in O$  there exists a subset  $V \in \psi(x)$ ,  $\psi \in \Psi x \in V \subset O$  iff  $\mathcal{G}$  is a  $\mathcal{G}$ -topology on  $X$ .

**Definition 1.4.** ([2]) Let  $X$  be any set and let  $\Psi$  and  $\mathcal{G}$  be  $\mathcal{G}$ -neighborhood system and  $\mathcal{G}$ -topology on  $X$  respectively. Let  $A \subset X$ . A point  $x \in X$  is called  $\mathcal{G}$ -interior point of  $A$  if there exists a subset  $V \in \psi(x)$ ,  $V \subset A$ .

Collection of all  $\mathcal{G}$ -interior points of  $A \subset X$  is called  $\mathcal{G}$ -interior of  $A$ . It is denoted by  $Int_{\mathcal{G}}(A)$ . By definition it is obvious that  $Int_{\mathcal{G}}(A) \subset A$ . Actually  $\mathcal{G}$ -interior of  $A$ ,  $Int_{\mathcal{G}}(A)$  is equal to union of all  $\mathcal{G}$ -open sets contained in  $A$ . Similarly we can define  $\mathcal{G}$ -closure of  $A$  as intersection of all  $\mathcal{G}$ -closed sets containing  $A$ . It is denoted by  $Cl_{\mathcal{G}}(A)$ .

By definition  $\mathcal{G}$ -interior of a set is a  $\mathcal{G}$ -open set while  $\mathcal{G}$ -closure of a set is a  $\mathcal{G}$ -closed set.

The following definition [2] gives the notion of point-wise  $\mathcal{G}$ -continuity using the  $\mathcal{G}$ -neighborhood systems.

**Definition 1.5.** Let  $(X, \mathcal{G}_X, \Psi(X))$  and  $(Y, \mathcal{G}_Y, \Psi(Y))$  be two  $\mathcal{G}$ -topological spaces with  $\mathcal{G}$ -neighborhood systems. The mapping  $f : X \rightarrow Y$  is *point-wise  $\mathcal{G}$ -continuous* at  $x \in X$  if for every  $\mathcal{G}_Y$ -neighborhood  $V \in \psi(f(x))$ , there exists a  $\mathcal{G}_X$ -neighborhood  $U \in \psi(x)$  such that  $f(U) \subset V$ .

$f$  is  $(\Psi_X, \Psi_Y)$ -**continuous** on  $X$  if  $f$  is point-wise  $\mathcal{G}$ -continuous at every point  $x \in X$ . This continuity is more general than global continuity.

**Definition 1.6.** Let  $X$  be a  $\mathcal{G}$ -topological space, and let  $A \subset X$ .  $x \in X$  is a point in  $\mathcal{G}$ -closure of  $A$  if any  $\mathcal{G}$ -open set  $U$  containing  $x$  intersects  $A$ .

**Lemma 1.7.** ([6]) Let  $(X, \mathcal{G}_X)$  and  $(Y, \mathcal{G}_Y)$  be two  $\mathcal{G}$ -topological spaces. A mapping  $f : X \rightarrow Y$  is  $\mathcal{G}$ -continuous  $\iff Cl_{\mathcal{G}_X}(f^{-1}(B)) \subset f^{-1}(Cl_{\mathcal{G}_Y}(B))$  for all  $B \subset Y \iff f(Cl_{\mathcal{G}_X}(A)) \subset Cl_{\mathcal{G}_Y}(f(A))$  for all  $A \subset X$ .

**Definition 1.8.** ([7]) Let  $(X, \mathcal{G}_X)$  be a  $\mathcal{G}$ -topological space and  $Y \neq \emptyset$  be a set and  $\pi : X \rightarrow Y$  a surjective mapping. Then  $\mathcal{G}_Y = \{O \subset Y : \pi^{-1}(O) \in \mathcal{G}_X\}$  is a  $\mathcal{G}$ -topology on  $Y$ , known as  $\mathcal{G}$ -quotient topology induced on  $Y$  by  $\pi$  and  $(Y, \mathcal{G}_\pi)$  is called the  $\mathcal{G}$ -quotient space of  $X$ .  $\pi$  is the  $\mathcal{G}$ -quotient map.

Note that the  $\mathcal{G}$ -quotient topology  $\mathcal{G}_\pi$  is the largest  $\mathcal{G}$ -topology on  $Y$  making  $\pi$   $\mathcal{G}$ -continuous.

**Proposition 1.9.** ([7]) Let  $(X, \mathcal{G}_X)$  and  $(Y, \mathcal{G}_Y)$  be two  $\mathcal{G}$ -topological spaces and  $\pi : X \rightarrow Y$  a surjective mapping. If  $\pi$  is  $\mathcal{G}$ -continuous and either  $\mathcal{G}$ -open or  $\mathcal{G}$ -closed, then the  $\mathcal{G}$ -topology  $\mathcal{G}_Y$  on  $Y$  coincides with the  $\mathcal{G}$ -quotient topology  $\mathcal{G}_\pi$ . Conversely, if  $\mathcal{G}_Y = \mathcal{G}_\pi$ , then  $\pi$  is  $\mathcal{G}$ -continuous.

**Proposition 1.10.** ([7]) If  $Y$  is endowed with the  $\mathcal{G}$ -quotient topology induced by a mapping  $\pi$  of  $X$  onto  $Y$ , then  $f : (Y, \mathcal{G}_Y) \rightarrow (Z, \mathcal{G}_Z)$  is  $\mathcal{G}$ -continuous iff  $f\pi$  is  $\mathcal{G}$ -continuous.

**Theorem 1.11.** ([8]) Let  $(X, \mathcal{G})$  be the product of  $\mathcal{G}$ -topological spaces  $(X_k, \mathcal{G}_k)$ ,  $k \in K$ . If every  $(X_k, \mathcal{G}_k)$  is  $\mathcal{G}$ -compact, then so is  $(X, \mathcal{G})$ .

**Definition 1.12.** ([4]) A  $\mathcal{G}$ -topological space  $(X, \mathcal{G})$  is said to be  $T_1$  if for  $x, y \in X$ ,  $x \neq y$ , there exists  $O \in \mathcal{G}$  such that  $x \in O$  and  $y \notin O$ .

The fundamental reference for topological groups and their properties is [1].

## 2. $\mathcal{G}$ -topological groups

In this section, we will introduce  $\mathcal{G}$ -topological groups and give basic properties of this structure. We also discuss similarities with and differences from topological groups.

**Definition 2.1.** A  $\mathcal{G}$ -topological group  $G$  is a group which is also a  $\mathcal{G}$ -topological space such that the multiplication map of  $G \times G$  into  $G$  sending  $x \times y$  into  $x \cdot y$ , and the inverse map of  $G$  into  $G$  sending  $x$  into  $x^{-1}$ , are  $\mathcal{G}$ -continuous maps.

**Example 2.2.** There is a  $\mathcal{G}$ -topological group which is not a topological group.

$(\mathbb{R}, +)$  is a group and it forms a  $\mathcal{G}$ -topological space under the  $\mathcal{G}$ -topology  $\mathcal{G}$  generated by the basis  $\mathcal{B} = \{(-\infty, a), (b, \infty), a, b \in \mathbb{R}\}$ . Then  $(\mathbb{R}, +, \mathcal{G})$  is a  $\mathcal{G}$ -topological group but not a topological group.

**Proposition 2.3.** Any subgroup  $H$  of a  $\mathcal{G}$ -topological group  $G$  is a  $\mathcal{G}$ -topological group again, called  $\mathcal{G}$ -topological subgroup of  $G$ .

We define morphisms of  $\mathcal{G}$ -topological groups.

**Definition 2.4.** Let  $\varphi : G \rightarrow G'$  be a mapping. Then  $\varphi$  is called a *morphism of  $\mathcal{G}$ -topological groups* (briefly,  *$\mathcal{G}$ -morphism*) if  $\varphi$  is both  $\mathcal{G}$ -continuous and group homomorphism.

$\varphi$  is a  *$\mathcal{G}$ -topological group isomorphism* (briefly  *$\mathcal{G}$ -isomorphism*) if it is  $\mathcal{G}$ -homeomorphism and group homomorphism.

If we have a  $\mathcal{G}$ -isomorphism between two  $\mathcal{G}$ -topological groups  $G$  and  $G'$  then we say that they are  $\mathcal{G}$ -isomorphic and we denote them by  $G \cong_{\mathcal{G}} G'$ .

It is obvious that composition of two  $\mathcal{G}$ -morphisms of  $\mathcal{G}$ -topological groups is again a  $\mathcal{G}$ -morphism. Also the identity map is a  $\mathcal{G}$ -isomorphism. So  $\mathcal{G}$ -topological groups and  $\mathcal{G}$ -morphisms form a category.

**Definition 2.5.** A  $\mathcal{G}$ -topological space  $X$  is said to be  *$\mathcal{G}$ -homogeneous* if for any  $x, y \in X$ , there is a  $\mathcal{G}$ -homeomorphism  $\varphi : X \rightarrow X$  such that  $\varphi(x) = y$ .

**Theorem 2.6.** Let  $G$  be a  $\mathcal{G}$ -topological group and let  $g \in G$ . Then left (right) translation map  $L_g (R_g) : G \rightarrow G$ , defined by  $L_g(x) = gx (R_g(x) = xg)$ , is a  $\mathcal{G}$ -topological homeomorphism.

*Proof.* Here we will prove that  $L_g$  is a  $\mathcal{G}$ -homeomorphism; similarly it can be shown that  $R_g$  is a  $\mathcal{G}$ -homeomorphism. First we will show that  $L_g : G \rightarrow G$ , is  $\mathcal{G}$ -continuous. Since  $L_g : G \rightarrow G$  is equal to the composition

$$G \xrightarrow{i_g} G \times G \xrightarrow{m} G,$$

where  $i_g(x) = (g, x), x \in G$  and  $m$  is the multiplication map in  $G$  then  $L_g$  is  $\mathcal{G}$ -continuous because  $i_g$  and  $m$  are  $\mathcal{G}$ -continuous. Here we should verify that the map  $i_g : G \rightarrow G \times G$  is  $\mathcal{G}$ -continuous. For any  $\mathcal{G}$ -open set  $U \times V$ , where  $U, V$  are  $\mathcal{G}$ -open sets in  $G$ ,

$$i_g^{-1}(U \times V) = \begin{cases} V, & \text{if } g \in U, \\ \emptyset, & \text{if } g \notin U, \end{cases}$$

We know that any  $\mathcal{G}$ -open set in the product  $\mathcal{G}$ -topology of  $G \times G$  can be written as a union of  $\mathcal{G}$ -open sets of the form  $U \times V$ . Then  $i_g$  is  $\mathcal{G}$ -continuous. Since  $(L_g)^{-1} = L_{g^{-1}}$  is  $\mathcal{G}$ -continuous, the left translation map  $L_g : G \rightarrow G$  is a  $\mathcal{G}$ -homeomorphism.  $\square$

Since for any two points  $g, g' \in G$  there exists a  $\mathcal{G}$ -homeomorphism  $L_{g'g^{-1}} : G \rightarrow G$  such that  $L_{g'g^{-1}}(g) = g'$ , any  $\mathcal{G}$ -topological group is a  $\mathcal{G}$ -homogeneous space.

**Theorem 2.7.** Let  $G$  be a  $\mathcal{G}$ -topological group and let  $e \in G$  be the identity element of  $G$  and  $\mathcal{B}_e$  be a local base at  $e$ . For  $g \in G$ , the local base at  $g$  is equal to

$$\mathcal{B}_g = \{gO : O \in \mathcal{B}_e\}.$$

**Theorem 2.8.** Let  $G$  be a  $\mathcal{G}$ -topological group and  $\mathcal{B}_e$  be a base at the identity  $e$  of  $G$ . Then we have the following properties.

- i) For every  $O \in \mathcal{B}_e$ , there is an element  $V \in \mathcal{B}_e$  such that  $V^2 \subset O$ .
- ii) For every  $O \in \mathcal{B}_e$ , there is an element  $V \in \mathcal{B}_e$  such that  $V^{-1} \subset O$ .
- iii) For every  $O \in \mathcal{B}_e$ , and for every  $x \in O$ , there is an element  $V \in \mathcal{B}_e$  such that  $Vx \subset O$ .
- iv) For every  $O \in \mathcal{B}_e$ , and for every  $x \in G$ , there is an element  $V \in \mathcal{B}_e$  such that  $xVx^{-1} \subset O$ .

*Proof.* If  $G$  is  $\mathcal{G}$ -topological group, then i) and ii) follow from the  $\mathcal{G}$ -continuity of the mappings  $(x, y) \rightarrow xy$  and  $x \rightarrow x^{-1}$  at the identity  $e$ . Property iii) follows from the  $\mathcal{G}$ -continuity of the left translation in  $G$ . Since  $L_x, R_{x^{-1}}$  are  $\mathcal{G}$ -homeomorphisms, their composition conjugation map  $x \rightarrow axa^{-1}$  is also  $\mathcal{G}$ -homeomorphism. By this fact we have the property iv).  $\square$

**Theorem 2.9.** Let  $G$  be a  $\mathcal{G}$ -topological group,  $U$  a  $\mathcal{G}$ -open subset of  $G$ , and  $A$  any subset of  $G$ . Then the set  $AU$  (respectively,  $UA$ ) is  $\mathcal{G}$ -open in  $G$ .

**Theorem 2.10.** Let  $G$  be a  $\mathcal{G}$ -topological group. Then for every subset  $A$  of  $G$  and every  $\mathcal{G}$ -open set  $U$  containing the identity element  $e$ ,  $Cl_{\mathcal{G}}(A) \subset AU$  ( $Cl_{\mathcal{G}}(A) \subset UA$ ).

*Proof.* Since the inversion is  $\mathcal{G}$ -continuous, we can find a  $\mathcal{G}$ -open set  $V$  containing  $e$  such that  $V^{-1} \subset U$ . Take  $x \in Cl_{\mathcal{G}}(A)$ . Then  $xV$  is a  $\mathcal{G}$ -open set containing  $x$ , therefore there is  $a \in A \cap xV$ , that is,  $a = xb$ , for some  $b \in V$ . Then  $x = ab^{-1} \in AV^{-1} \subset AU$ , hence,  $Cl_{\mathcal{G}}(A) \subset AU$ .  $\square$

**Theorem 2.11.** Let  $G$  be a  $\mathcal{G}$ -topological group, and  $\mathcal{B}_e$  a base of the space  $G$  at the identity element  $e$ . Then for every subset  $A$  of  $G$ ,

$$Cl_{\mathcal{G}}(A) = \bigcap \{AU : U \in \mathcal{B}_e\}.$$

*Proof.* In view of Theorem 2.10, we only have to verify that if  $x$  is not in  $Cl_{\mathcal{G}}(A)$ , then there exists  $U \in \mathcal{B}_e$  such that  $x \notin AU$ . Since  $x \notin Cl_{\mathcal{G}}(A)$ , there exists a  $\mathcal{G}$ -open neighborhood  $W$  of  $e$  such that  $(xW) \cap A = \emptyset$ . Take  $U$  in  $\mathcal{B}_e$ , satisfying the condition  $U^{-1} \subset W$ . Then  $(xU^{-1}) \cap A = \emptyset$ , which obviously implies that  $AU$  does not contain  $x$ .  $\square$

**Theorem 2.12.** Let  $f : G \rightarrow H$  be a  $\mathcal{G}$ -morphism. If  $f$  is  $\mathcal{G}$ -continuous at the identity  $e_G$  of  $G$ , then  $f$  is  $\mathcal{G}$ -continuous at every  $g \in G$ .

*Proof.* Let  $g \in G$  be any point. Suppose that  $O$  is a  $\mathcal{G}$ -open neighborhood of  $h = f(g)$  in  $H$ . Since left translation  $L_h$  is a  $\mathcal{G}$ -homeomorphism of  $H$ , there exists a  $\mathcal{G}$ -open neighborhood  $V$  of the identity element  $e_H$  in  $H$  such that  $hV \subset O$ . By  $\mathcal{G}$ -continuity of  $f$  at  $e_G$  we have a  $\mathcal{G}$ -open neighborhood  $U$  of  $e_G$  in  $G$  such that  $f(U) \subset V$ . Since  $L_g$  is a  $\mathcal{G}$ -homeomorphism of  $G$  onto itself, the set  $gU$  is a  $\mathcal{G}$ -open neighborhood of  $g$  in  $G$ , and we have that  $f(gU) = hf(U) \subset hV \subset O$ . Hence  $f$  is  $\mathcal{G}$ -continuous at the point  $g$ .  $\square$

**Theorem 2.13.** Let  $G$  be a  $\mathcal{G}$ -topological group and let  $H$  be a subgroup of  $G$ . If  $H$  contains a non-empty  $\mathcal{G}$ -open set, then  $H$  is  $\mathcal{G}$ -open in  $G$ .

*Proof.* Let  $U$  be a non-empty  $\mathcal{G}$ -open subset of  $G$  with  $U \subset H$ . For every  $h \in H$ ,  $L_h(U) = hU$  is  $\mathcal{G}$ -open in  $G$ . Therefore, the subgroup  $H = \bigcup_{h \in H} hU$  is  $\mathcal{G}$ -open in  $G$  by  $Uh \subset H, \forall h \in H$ .  $\square$

**Definition 2.14.** Let  $X$  be a set and let  $\Gamma \subset \mathcal{P}(X)$ . We say that  $\Gamma$  is a covering of  $X$  if  $X = \bigcup_{\gamma \in \Gamma} \gamma$ . If  $X$  is a  $\mathcal{G}$ -topological space and every element of  $\Gamma$  is  $\mathcal{G}$ -open (or  $\mathcal{G}$ -closed) then  $\Gamma$  is called a  $\mathcal{G}$ -open covering (respectively a  $\mathcal{G}$ -closed covering).

If a  $\mathcal{G}$ -topological space has a  $\mathcal{G}$ -open covering then it must be strong.

**Theorem 2.15.** Let  $G$  be a  $\mathcal{G}$ -topological group and let  $H$  be a subgroup of  $G$ . If  $H$  is a  $\mathcal{G}$ -open set, then it is also  $\mathcal{G}$ -closed in  $G$ .

*Proof.* Let  $\Gamma = \{gH : g \in G\}$  be the family of all left cosets of  $H$  in  $G$ . This family is a disjoint  $\mathcal{G}$ -open covering of  $G$  by left translations. Therefore, every element of  $\Gamma$ , being the complement of the union of all other elements, is  $\mathcal{G}$ -closed in  $G$ . In particular,  $H = eH$  is  $\mathcal{G}$ -closed in  $G$ .  $\square$

**Definition 2.16.** Let  $G$  be a  $\mathcal{G}$ -topological group. Then a subset  $U$  of  $G$  is called  $\mathcal{G}$ -symmetric if  $U = U^{-1}$

**Theorem 2.17.** Let  $G$  be a  $\mathcal{G}$ -topological group. Then  $\mathcal{G}$ -closure of a  $\mathcal{G}$ -symmetric subset  $A$  of  $G$  is again  $\mathcal{G}$ -symmetric.

We define  $\mathcal{G}$ -regular space.

**Definition 2.18.** A  $\mathcal{G}$ -topological space  $X$  is said to be  $\mathcal{G}$ -regular if for any  $x \in X$  and any  $\mathcal{G}$ -closed set  $F$  such that  $x \notin F$ , there are two  $\mathcal{G}$ -open sets  $U$  and  $V$  such that  $x \in U$ ,  $F \subset V$  and  $U \cap V = \emptyset$ .

**Theorem 2.19.** If a  $\mathcal{G}$ -topological group  $G$  has a base at identity  $e$  consisting of  $\mathcal{G}$ -symmetric neighborhood, then it is a  $\mathcal{G}$ -regular space.

*Proof.* Let  $U$  be a  $\mathcal{G}$ -open set containing the identity  $e$ . We prove  $Cl_{\mathcal{G}}(V) \subset U$ . By Theorem 2.8, there is a  $\mathcal{G}$ -open set  $V$  containing  $e$  such that  $V^{-1} = V$  and  $V^2 \subset U$ , yet  $x \in Cl_{\mathcal{G}}(V)$ . Then  $Vx \cap V \neq \emptyset$ , hence  $a_1x = a_2$  for some  $a_1, a_2$  in  $V$ . Thus  $x = a_1^{-1}a_2 \in V^{-1}V = V^2 \subset U$ . It means  $Cl_{\mathcal{G}}(V) \subset U$ . Since  $G$  is a  $\mathcal{G}$ -homogeneous space, we get  $\mathcal{G}$ -regularity of  $G$ .  $\square$

**Proposition 2.20.** Let  $X$  be a  $T_2$   $\mathcal{G}$ -topological space. Then for every  $x \in X$  the singleton set  $\{x\}$  is a  $\mathcal{G}$ -closed set in  $X$ .

By the last result, the identity  $\{e\}$  of a  $\mathcal{G}$ -topological group  $G$  is  $\mathcal{G}$ -closed and hence we have the following corollary.

**Corollary 2.21.** Let  $\varphi : G \rightarrow G'$  be a  $\mathcal{G}$ -morphism, let  $e'$  be the identity element of  $G'$ . Then

$$\ker(\varphi) = \{g \in G \mid \varphi(g) = e'\}$$

is a  $\mathcal{G}$ -closed invariant  $\mathcal{G}$ -topological subgroup in  $G$ .

We are ready to prove the following results.

**Theorem 2.22.** Let  $G$  be a  $\mathcal{G}$ -topological group. Then  $\mathcal{G}$ -closure of any subgroup of  $G$  is a  $\mathcal{G}$ -topological subgroup again.

*Proof.* Let  $H$  be a subgroup of  $G$ . First we prove that  $Cl_{\mathcal{G}}(H)$  is closed under multiplication  $m$  in  $G$ . Given  $x, y \in Cl_{\mathcal{G}}(H)$  and for any  $\mathcal{G}$ -open set  $U$  containing  $xy$ , by the Definition 1.6, we need to show that  $U \cap H \neq \emptyset$ . Since  $m : G \times G \rightarrow G$  is  $\mathcal{G}$ -continuous, there exist  $\mathcal{G}$ -open sets  $V$  and  $W$  containing  $x$  and  $y$ , respectively, such that  $m(V \times W) \subset U$ . Since  $x, y \in Cl_{\mathcal{G}}(H)$  then we have

$$V \cap H \neq \emptyset \quad \text{and} \quad W \cap H \neq \emptyset.$$

Hence  $\emptyset \neq m(V \times W) \cap H \subset U \cap H$  which implies that  $xy \in Cl_{\mathcal{G}}(H)$ .

Now  $Cl_{\mathcal{G}}(H)$  is closed under the inverse operation because  $(Cl_{\mathcal{G}}(H))^{-1} \subset Cl_{\mathcal{G}}(H^{-1}) = Cl_{\mathcal{G}}(H)$  by Lemma 1.7.

$\square$

**Theorem 2.23.** Let  $G$  be a  $\mathcal{G}$ -topological group. Then  $\mathcal{G}$ -closure of any invariant subgroup of  $G$  is a  $\mathcal{G}$ -topological invariant subgroup again.

*Proof.* Suppose  $H$  is an invariant subgroup in  $G$ . By Theorem 2.22,  $Cl_{\mathcal{G}}(H)$  is a subgroup of  $G$ . Now we prove that  $Cl_{\mathcal{G}}(H)$  is invariant. Given  $g \in G$ , let  $\kappa_g : G \rightarrow G$  be conjugation by  $g$ , i.e.  $\kappa_g(h) = ghg^{-1} = L_g \circ R_{g^{-1}}(h)$ . Then  $\kappa_g$  is a  $\mathcal{G}$ -homeomorphism from  $G$  to itself. By Lemma 1.7, we have

$$\kappa_g(Cl_{\mathcal{G}}(H)) \subset Cl_{\mathcal{G}}(\kappa_g(H)) = Cl_{\mathcal{G}}(H), \quad \forall g \in G.$$

It implies

$$\kappa_g(Cl_{\mathcal{G}}(H)) = gCl_{\mathcal{G}}(H)g^{-1} \subset Cl_{\mathcal{G}}(H), \quad \forall g \in G.$$

Hence  $Cl_{\mathcal{G}}(H)$  is an invariant subgroup of  $G$ .  $\square$

### 3. $\mathcal{G}$ -connectedness for $\mathcal{G}$ -topological groups

In this section, we will discuss basic topological properties of  $\mathcal{G}$ -connectedness in  $\mathcal{G}$ -topological groups. For connectedness in  $\mathcal{G}$ -topological spaces, see [3].

**Definition 3.1.** ([3]) Let  $X$  be a  $\mathcal{G}$ -topological space and let  $U, V \subset X$ . Then we say that the pair  $U, V$  is  $\mathcal{G}$ -separated if  $Cl_{\mathcal{G}}(U) \cap V = Cl_{\mathcal{G}}(V) \cap U = \emptyset$ .

Let  $X$  be a  $\mathcal{G}$ -topological space. A set  $S \subset X$  is  $\mathcal{G}$ -connected if there are no two non-empty  $\mathcal{G}$ -separated sets  $U$  and  $V$  such that  $U \cup V = S$ . The space  $X$  is  $\mathcal{G}$ -connected if it is a  $\mathcal{G}$ -connected subset of itself.

From Theorem 1.2 of [3], we have.

**Corollary 3.2.** A  $\mathcal{G}$ -topological group  $G$ , having a  $\mathcal{G}$ -open subgroup, is not  $\mathcal{G}$ -connected.

The  $\mathcal{G}$ -connectedness for  $\mathcal{G}$ -topological subspaces can be defined by induced subspace  $\mathcal{G}$ -topology. From [3], we have the relation between  $\mathcal{G}$ -continuity and  $\mathcal{G}$ -connectedness.

**Lemma 3.3.** Let  $f : X \rightarrow Y$  be a  $\mathcal{G}$ -continuous mapping between  $\mathcal{G}$ -topological spaces. If  $X$  is  $\mathcal{G}$ -connected, so is  $f(X)$ .

**Definition 3.4.** ([3]) A mapping  $f : X \rightarrow Y$  between  $\mathcal{G}$ -topological spaces is called  $\mathcal{G}$ -open if for every  $\mathcal{G}$ -open set  $U \subset X$ ,  $f(U)$  is  $\mathcal{G}$ -open in  $Y$ .

**Theorem 3.5.** ([3]) Let  $f : X \rightarrow Y$  be a  $\mathcal{G}$ -open and injective mapping between  $\mathcal{G}$ -topological spaces and let  $S \subset X$ . If  $f(S)$  is  $\mathcal{G}$ -connected, so is  $S$ .

Now we will give definition of  $\mathcal{G}$ -component in  $\mathcal{G}$ -topological spaces.

**Definition 3.6.** ([3]) Let  $X$  be a  $\mathcal{G}$ -topological space, let  $A \subset X$ . For  $x \in A$ , the set

$$A_x = \bigcup_{x \in S \subset A} S,$$

where  $S$  is  $\mathcal{G}$ -connected in  $A$ , is called the  $\mathcal{G}$ -component of  $A$  belonging to  $x$ .

From [3] we have the following theorem.

**Theorem 3.7.** The  $\mathcal{G}$ -component of a  $\mathcal{G}$ -closed set  $A$  in  $\mathcal{G}$ -topological space  $X$  is again  $\mathcal{G}$ -closed.

Hence the  $\mathcal{G}$ -component of  $\mathcal{G}$ -topological space  $X$  is again  $\mathcal{G}$ -closed. Then we have the following result.

**Theorem 3.8.** Let  $G$  be a  $\mathcal{G}$ -topological group and let  $e$  be the identity element of  $G$ . Then the  $\mathcal{G}$ -connected component containing  $e$  is a  $\mathcal{G}$ -closed invariant subgroup of  $G$ .

*Proof.* Let  $F$  be the  $\mathcal{G}$ -component of the identity  $e$ . By Theorem 3.7,  $F$  is  $\mathcal{G}$ -closed. Let  $a \in F$ . Since the multiplication and inversion mappings in  $G$  are  $\mathcal{G}$ -continuous, by Lemma 3.3, the set  $aF^{-1}$  is also  $\mathcal{G}$ -connected, and since  $e \in aF^{-1}$  we have  $aF^{-1} \subset F$ . Hence, for every  $b \in F$  we have  $ab^{-1} \in F$ , i.e.  $F$  is a subgroup of  $G$ .

If  $g$  is an arbitrary element of  $G$ , then  $L_{g^{-1}} \circ R_g(F) = g^{-1}Fg$  is a  $\mathcal{G}$ -connected subset containing  $e$ . Since  $F$  is a  $\mathcal{G}$ -component,  $g^{-1}Fg \subset F$  for every  $g \in G$ , i.e.  $F$  is an invariant subgroup of  $G$ .  $\square$

**Theorem 3.9.** Let  $G$  be a  $\mathcal{G}$ -connected  $\mathcal{G}$ -topological group, and let  $U$  be a  $\mathcal{G}$ -open set containing identity element  $e$  such that  $U$  contains a  $\mathcal{G}$ -symmetric  $\mathcal{G}$ -open set  $V$  containing  $e$ . Then  $G = \bigcup_{n=1}^{\infty} U^n$ .

*Proof.* By induction on  $n$  and Theorem 2.9, for every positive integer  $n$ ,  $V^n$  is  $\mathcal{G}$ -open. Hence  $\bigcup_{n=1}^{\infty} V^n$  is  $\mathcal{G}$ -open. Let  $H = \bigcup_{n=1}^{\infty} V^n$ . Since  $V$  is  $\mathcal{G}$ -symmetric, for every positive integer  $n$ ,  $V^n$  is  $\mathcal{G}$ -symmetric. Hence  $H = H^{-1}$ . Also we have  $V^k V^l = V^{(k+l)}$ . Hence  $HH = H$ . So,  $H$  is a subgroup of  $G$ . Since  $H$  is a  $\mathcal{G}$ -open subgroup of  $G$ , by Theorem 2.15,  $H$  is also  $\mathcal{G}$ -closed. Since  $G$  is  $\mathcal{G}$ -connected, and  $H$  is non-empty and both  $\mathcal{G}$ -closed and  $\mathcal{G}$ -open,  $G = H$ . As  $V \subset U$ , it follows that  $G = \bigcup_{n=1}^{\infty} U^n$ .  $\square$

**4. Quotients of generalized topological groups**

Suppose that  $(G, \mathcal{G})$  is a  $\mathcal{G}$ -topological group with identity  $e$ , and  $H$  is a  $\mathcal{G}$ -closed subgroup of  $G$ . Denote by  $G/H$  the set of all left cosets  $aH$  of  $H$  in  $G$ , and endow it with the  $\mathcal{G}$ -quotient topology with respect to the canonical mapping  $\pi : G \rightarrow G/H$  defined by  $\pi(a) = aH$ , for each  $a \in G$ . We start with the following statement:

**Theorem 4.1.** *Let  $H$  be a  $\mathcal{G}$ -closed subgroup of a  $\mathcal{G}$ -topological group  $(G, \mathcal{G})$  with identity  $e$ . The family  $\{\pi(xU) : U \in \mathcal{G}, e \in U\}$  is a local base of the space  $G/H$  at the point  $xH \in G/H$ , the mapping  $\pi$  is  $\mathcal{G}$ -open, and  $G/H$  is a  $\mathcal{G}$ -homogeneous  $T_1$  space.*

*Proof.* We can write the set  $xUH (= \pi(xU))$  as the union of left cosets  $yH$  where  $y \in xU$ . Therefore,  $\pi^{-1}\pi(xUH) = xUH$ . Since the set  $xUH$  is  $\mathcal{G}$ -open in  $G$  and the mapping  $\pi$  is  $\mathcal{G}$ -quotient, it follows that  $\pi(xUH)$  is  $\mathcal{G}$ -open in  $G/H$ . Now put  $O = \pi^{-1}(W)$ , where  $W$  is any  $\mathcal{G}$ -open neighborhood of  $xH$  in  $G/H$ .  $\mathcal{G}$ -continuity of  $\pi$  implies that the set  $O$  is  $\mathcal{G}$ -open. Clearly,  $x \in O$ . There is a  $\mathcal{G}$ -open neighborhood  $U$  of  $e$  in  $G$  such that  $xU \subset O$ . Then  $\pi(xU) \subset W$  and hence,  $\pi^{-1}\pi(xU) \subset O$ . Since  $xUH = \pi^{-1}\pi(xU)$ , it follows that  $\pi(xUH) \subset W$ . Thus, the first two statements of the theorem are proved.

For  $\mathcal{G}$ -homogeneity, for any  $a \in G$ , define a mapping  $h_a$  of  $G/H$  to itself by  $h_a(xH) = axH$ . Since  $axH \in G/H$ , this definition is well defined. Since  $G$  is a group, the mapping  $h_a$  is evidently a bijection of  $G/H$  onto  $G/H$ . In fact,  $h_a$  is a  $\mathcal{G}$ -homeomorphism as is evident from the following argument.

Take any  $xH \in G/H$  and any  $\mathcal{G}$ -open neighborhood  $U$  of  $e$ . Then  $\pi(xUH)$  is a basic  $\mathcal{G}$ -neighborhood of  $xH$  in  $G/H$ . Similarly, the set  $\pi(axUH)$  is a basic  $\mathcal{G}$ -neighborhood of  $axH$  in  $G/H$ . Since, obviously,  $h_a(\pi(xUH)) = \pi(axUH)$ , it easily follows that  $h_a$  is a  $\mathcal{G}$ -homeomorphism. It is also clear that  $h_a(xH) = axH$ . Now for any given  $xH$  and  $yH$  in  $G/H$ , we can take  $a = yx^{-1}$ . Then  $h_a(xH) = yH$ . Hence the  $\mathcal{G}$ -quotient space  $G/H$  is  $\mathcal{G}$ -homogeneous. It is a  $T_1$  space, since all right cosets  $xH$  are  $\mathcal{G}$ -closed in  $G$  and the mapping  $\pi$  is  $\mathcal{G}$ -quotient.  $\square$

The space  $G/H$  defined above is called the *left coset  $\mathcal{G}$ -space* of  $G$  with respect to  $H$ . Similarly, one can define the right coset  $\mathcal{G}$ -space  $H \setminus G$ .

**Proposition 4.2.** *Suppose that  $G$  is a  $\mathcal{G}$ -topological group,  $H$  is a  $\mathcal{G}$ -closed subgroup of  $G$ ,  $\pi$  is the natural  $\mathcal{G}$ -quotient mapping of  $G$  onto the left (right)  $\mathcal{G}$ -quotient space  $G/H$  ( $H \setminus G$ ),  $a \in G$ ,  $L_a$  ( $R_a$ ) is the left (right) translation of  $G$  by  $a$ , and  $g_a$  ( $h_a$ ) is the left (right) translation of  $G/H$  ( $H \setminus G$ ) by  $a$ , that is,  $g_a(xH) = axH$ , for each  $xH \in G/H$   $h_a(Hx) = Hxa$ , for each  $Hx \in H \setminus G$ . Then  $L_a(R_a)$  and  $g_a$  ( $h_a$ ) are  $\mathcal{G}$ -homeomorphisms of  $G$  and  $G/H$  ( $H \setminus G$ ), respectively, and  $\pi \circ L_a = g_a \circ \pi$  ( $\pi \circ R_a = h_a \circ \pi$ ).*

If  $G$  is a  $\mathcal{G}$ -topological group and  $H$  is a  $\mathcal{G}$ -closed invariant subgroup of  $G$ , then each left coset of  $H$  in  $G$  is also a right coset of  $H$  in  $G$ , and a natural multiplication of cosets in  $G/H$  is defined by the rule  $xHyH = xyH$ , for all  $x, y \in G$ . It is well known that this operation turns  $G/H$  into a group called the quotient group of  $G$  with respect to  $H$ . So, we have the following:

**Theorem 4.3.** *Suppose that  $G$  is a  $\mathcal{G}$ -topological group, and that  $H$  is a  $\mathcal{G}$ -closed invariant subgroup of  $G$ . Then  $G/H$  with the  $\mathcal{G}$ -quotient topology and multiplication is a  $\mathcal{G}$ -topological group, and the canonical mapping  $\pi : G \rightarrow G/H$  is a  $\mathcal{G}$ -open  $\mathcal{G}$ -continuous homomorphism.*

**Proposition 4.4.** *Let  $G$  be a  $\mathcal{G}$ -topological group and  $H$  be a  $\mathcal{G}$ -closed invariant subgroup of  $G$ . Then  $G/H$  with the  $\mathcal{G}$ -quotient topology is  $\mathcal{G}$ -discrete if and only if  $H$  is  $\mathcal{G}$ -open in  $G$ .*

**Lemma 4.5.** *Let  $G$  be a  $\mathcal{G}$ -topological group,  $H$  be a  $\mathcal{G}$ -closed subgroup of  $G$ ,  $\pi$  be the natural  $\mathcal{G}$ -quotient mapping of  $G$  onto the  $\mathcal{G}$ -quotient space  $G/H$ , and let  $U$  and  $V$  be  $\mathcal{G}$ -open neighborhoods of the neutral element  $e$  in  $G$  such that  $V^{-1}V \subset U$ . Then  $Cl_{\mathcal{G}}(\pi(V)) \subset \pi(U)$ .*

*Proof.* Take any  $x \in G$  such that  $\pi(x) \in Cl_{\mathcal{G}}(\pi(V))$ . Since  $Vx$  is a  $\mathcal{G}$ -open neighborhood of  $x$  and the mapping  $\pi$  is  $\mathcal{G}$ -open,  $\pi(Vx)$  is a  $\mathcal{G}$ -open neighborhood of  $\pi(x)$ . Therefore,  $\pi(Vx) \cap \pi(V) \neq \emptyset$ . It follows that, for some  $a \in V$  and  $b \in V$ , we have  $\pi(ax) = \pi(b)$ , that is,  $ax = bh$ , for some  $h \in H$ . Hence,  $x = (a^{-1}b)h \in UH$ , since  $a^{-1}b \in V^{-1}V \subset U$ . Therefore,  $\pi(x) \in \pi(UH) = \pi(U)$ .  $\square$

**Theorem 4.6.** For any  $\mathcal{G}$ -topological group  $G$  and any  $\mathcal{G}$ -closed subgroup  $H$  of  $G$ , the  $\mathcal{G}$ -quotient space  $G/H$  is  $\mathcal{G}$ -regular.

**Definition 4.7.** A  $\mathcal{G}$ -closed  $\mathcal{G}$ -continuous mapping with  $\mathcal{G}$ -compact preimages of points is called  $\mathcal{G}$ -perfect.

**Proposition 4.8.** Let  $G, H$  and  $K$  be groups and  $\phi : G \rightarrow H$  and  $\psi : G \rightarrow K$  be homomorphisms such that  $\psi(G) = K$  and  $\ker(\psi) \subset \ker(\phi)$ . Then there exists a homomorphism  $f : K \rightarrow H$  such that  $\phi = f \circ \psi$ . Further, if  $G, H, K$  are  $\mathcal{G}$ -topological groups,  $\phi$  and  $\psi$  are  $\mathcal{G}$ -continuous, and for each  $\mathcal{G}$ -neighborhood  $U$  of the identity  $e_H$  in  $H$  there exists a  $\mathcal{G}$ -neighborhood  $V$  of the identity  $e_K$  in  $K$  such that  $\psi^{-1}(V) \subset \phi^{-1}(U)$ , then  $f$  is  $\mathcal{G}$ -continuous.

*Proof.* Algebraic part is well known. For the  $\mathcal{G}$ -continuity of  $f$ , assume that  $U$  is a  $\mathcal{G}$ -neighborhood of  $e_H$  in  $H$ . By assumption, there exists a  $\mathcal{G}$ -neighborhood  $V$  of  $e_K$  in  $K$  such that  $W = \psi^{-1}(V) \subset \phi^{-1}(U)$ . So  $\phi(W) = f(V) \subset U$ , that is,  $f$  is  $\mathcal{G}$ -continuous at the identity of  $K$ . By Theorem 2.8,  $f$  is  $\mathcal{G}$ -continuous.  $\square$

**Corollary 4.9.** Let  $\phi : G \rightarrow H$  and  $\psi : G \rightarrow K$  be  $\mathcal{G}$ -continuous homomorphisms of  $\mathcal{G}$ -topological groups  $G, H$ , and  $K$  such that  $\psi(G) = K$  and  $\ker(\psi) \subset \ker(\phi)$ . If the homomorphism  $\psi$  is  $\mathcal{G}$ -open, then there exists a  $\mathcal{G}$ -continuous homomorphism  $f : K \rightarrow H$  such that  $\phi = f \circ \psi$ .

**Proposition 4.10.** Let  $G$  and  $H$  be  $\mathcal{G}$ -topological groups and  $p$  be a  $\mathcal{G}$ -isomorphism of  $G$  onto  $H$ . If  $G_0$  is a  $\mathcal{G}$ -closed invariant subgroup of  $G$  and  $H_0 = p(G_0)$ , then the  $\mathcal{G}$ -quotient groups  $G/G_0$  and  $H/H_0$  are  $\mathcal{G}$ -isomorphic. The corresponding  $\mathcal{G}$ -isomorphism  $\Phi : G/G_0 \rightarrow H/H_0$  is given by  $\Phi(xG_0) = yH_0$ , where  $x \in G$  and  $y = p(x)$ .

*Proof.* Let  $\phi : G \rightarrow G/G_0$  and  $\psi : H \rightarrow H/H_0$  be the  $\mathcal{G}$ -quotient homomorphisms. Obviously  $\Phi$  is a homomorphism of  $G/G_0$  onto  $H/H_0$ . It follows that  $\psi \circ p = \Phi \circ \phi$ . Since  $p, \phi$ , and  $\psi$  are  $\mathcal{G}$ -open  $\mathcal{G}$ -continuous homomorphisms, so is  $\Phi$ . Let  $xG_0 \in G/G_0$  and set  $y = p(x)$ . If  $\Phi(xG_0) = H_0$  then  $\psi(y) = H_0$ , so that  $y \in H_0$  and  $x \in G_0$ . This shows that the kernel of  $\Phi$  is trivial. Thus  $\Phi$  is an isomorphism, hence a  $\mathcal{G}$ -isomorphism,  $\square$

**Theorem 4.11.** (First Isomorphism)

Let  $G$  and  $H$  be  $\mathcal{G}$ -topological groups with the identity elements  $e_G$  and  $e_H$  respectively. Let  $p$  be a  $\mathcal{G}$ -open  $\mathcal{G}$ -continuous homomorphism of  $G$  onto  $H$ . Then the kernel  $N = p^{-1}(e_H)$  of  $p$  is a  $\mathcal{G}$ -closed invariant subgroup of  $G$ , and the fibers  $p^{-1}(y)$  with  $y \in H$  coincide with the cosets of  $N$  in  $G$ . The mapping  $\Phi : G/N \rightarrow H$  which assigns to a coset  $xN$  the element  $p(x) \in H$  is a  $\mathcal{G}$ -isomorphism.

*Proof.* Assertion about the kernel and fibers are well known. The definition of multiplication in  $G/N$  gives that  $\Phi$  is a homomorphism. Let  $\pi$  be the  $\mathcal{G}$ -quotient homomorphism of  $G$  onto  $G/N$ . From the definition of  $\Phi$ , it follows that  $p = \Phi \circ \pi$  and since  $\pi$  is  $\mathcal{G}$ -open, the homomorphism  $\Phi$  is  $\mathcal{G}$ -continuous. Also, if  $x \in G$  and  $\Phi(xN) = e_H$ , then  $p(x) = \Phi(\pi(x)) = e_H$ . So  $x \in N$  and  $xN = N$ . Thus  $\Phi$  is an isomorphism. Now if  $U$  is a  $\mathcal{G}$ -open set in  $G/N$ , the image  $\Phi(U) = p(\pi^{-1}(U))$  is  $\mathcal{G}$ -open in  $H$ . So  $\Phi$  is a  $\mathcal{G}$ -open  $\mathcal{G}$ -continuous isomorphism, hence a  $\mathcal{G}$ -homeomorphism.  $\square$

**Theorem 4.12.** (Second Isomorphism)

Let  $G$  and  $H$  be  $\mathcal{G}$ -topological groups with the identity elements  $e_G$  and  $e_H$  respectively. Let  $p : G \rightarrow H$  be a  $\mathcal{G}$ -open  $\mathcal{G}$ -continuous homomorphism of  $G$  onto  $H$ . Let  $H_0$  be a  $\mathcal{G}$ -closed invariant subgroup of  $H$ ,  $G_0 = p^{-1}(H_0)$ , and  $N = p^{-1}(e_H)$ . Then the  $\mathcal{G}$ -topological groups  $G/G_0, H/H_0$  and  $(G/N)/(G_0/N)$  are  $\mathcal{G}$ -isomorphic.

*Proof.* Let  $\phi$  be the  $\mathcal{G}$ -quotient homomorphism of  $H$  onto  $H/H_0$ . By Theorem 2.9,  $\phi$  is  $\mathcal{G}$ -open, so the composition  $\phi \circ p$  is a  $\mathcal{G}$ -open  $\mathcal{G}$ -continuous homomorphism of  $G$  onto  $H/H_0$  with kernel  $G_0 = p^{-1}(H_0)$ . Hence the  $\mathcal{G}$ -quotient group  $G/G_0$  is  $\mathcal{G}$ -isomorphic to  $H/H_0$ , by Theorem 4.11. It is clear that  $G_0$  is a  $\mathcal{G}$ -closed invariant subgroup of  $G$ . Note that the mapping  $\Phi : G/N \rightarrow H$  defined by  $\Phi(xN) = p(x)$  is a  $\mathcal{G}$ -isomorphism by Theorem 4.11, and  $\Phi(G_0/N) = H_0$ . Therefore applying Proposition 4.10, we conclude that the group  $(G/N)/(G_0/N)$  is  $\mathcal{G}$ -isomorphic to  $H/H_0$ .  $\square$

**Theorem 4.13.** (Third Isomorphism)

Let  $G$  be a  $\mathcal{G}$ -topological group and  $H$  be a  $\mathcal{G}$ -closed invariant subgroup of  $G$ , and  $M$  be any  $\mathcal{G}$ -topological subgroup of  $G$ . Then the  $\mathcal{G}$ -quotient group  $MH/H$  is  $\mathcal{G}$ -isomorphic to the subgroup  $\pi(M)$  of the  $\mathcal{G}$ -topological group  $G/H$ , where  $\pi : G \rightarrow G/H$  is the natural  $\mathcal{G}$ -quotient homomorphism.

*Proof.* Clearly,  $MH = \pi^{-1}(\pi(M))$ . Since the mapping  $\pi$  is  $\mathcal{G}$ -open and  $\mathcal{G}$ -continuous, the restriction  $\psi$  of  $\pi$  to  $MH$  is a  $\mathcal{G}$ -open  $\mathcal{G}$ -continuous mapping of  $MH$  onto  $\pi(M)$ . Since  $M$  is a subgroup of  $G$  and  $\pi$  is a homomorphism of  $G$  onto  $G/H$ , it follows that  $\pi(M)$  and  $MH$  are subgroups of the groups  $G/H$  and  $G$ , respectively, and  $\psi$  is a homomorphism of  $MH$  onto  $\pi(M)$ . Let  $e$  be the identity of  $G$ . Clearly  $\psi^{-1}(\psi(e)) = \pi^{-1}(\pi(e)) = H$ , that is, the kernel of the homomorphism  $\psi$  is  $H$ . Now it follows from Theorem 4.11 that the  $\mathcal{G}$ -topological groups  $MH/H$  and  $\pi(M)$  are  $\mathcal{G}$ -isomorphic.  $\square$

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