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Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities

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Abstract. In the note, the author presents alternative proofs for limit formulas of ratios between derivatives of the gamma function and the digamma function at their singularities.

1. Introduction

It is common knowledge [1, p. 255, 6.1.2] that the classical gamma function $\Gamma(z)$ may be defined by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{\prod_{k=0}^{n}(z+k)} \tag{1.1}$$

for $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ and that the digamma function $\psi(z)$ is defined by

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{1.2}$$

In 2011, A. Prabhu and H. M. Srivastava considered the limits of ratios between two gamma functions and two digamma functions at their singularities $0, -1, -2, \dots$ and, among other things, obtained, by Euler's reflection formulas for the gamma function $\Gamma(z)$ and the digamma function $\psi(z)$, the following limit formulas.

Theorem 1.1 ([2, Theorems 1 and 2]). *For any non-negative integer k and all positive integers n and q, the limit formulas*

$$\lim_{z\to -k} \frac{\Gamma(nz)}{\Gamma(qz)} = (-1)^{(n-q)k} \frac{q}{n} \cdot \frac{(qk)!}{(nk)!} \tag{1.3}$$

and

$$\lim_{z\to -k} \frac{\psi(nz)}{\psi(qz)} = \frac{q}{n}. \tag{1.4}$$

are valid.

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If we appeal appropriately to the L'Hôspital's limit formula for indeterminate quotients, the following limit formulas would follow as rather immediate consequences of the assertions (1.3) and (1.4).

Theorem 1.2. *For $n, q \in \mathbb{N}$ and $i, k \in \{0\} \cup \mathbb{N}$, we have*

$$\lim_{z\to -k} \frac{\Gamma^{(i)}(nz)}{\Gamma^{(i)}(qz)} = (-1)^{(n-q)k}\left(\frac{q}{n}\right)^{i+1} \frac{(qk)!}{(nk)!} \tag{1.5}$$

and

$$\lim_{z\to -k} \frac{\psi^{(i)}(nz)}{\psi^{(i)}(qz)} = \left(\frac{q}{n}\right)^{i+1}. \tag{1.6}$$

The object of this note is to provide alternative proofs for limit formulas in the above stated theorems.

2. An alternative proof of limit formulas (1.3) and (1.5)

It is well known [1, p. 255, 6.1.3] that the gamma function $\Gamma(z)$ is single valued and analytic over the entire complex plane, save for the points $z = -n$, with $n \in \{0\} \cup \mathbb{N}$, where it possesses simple poles with residue $\frac{(-1)^n}{n!}$. Its reciprocal $\frac{1}{\Gamma(z)}$ is an entire function possessing simple zeros at the points $z = -n$, with $n \in \{0\} \cup \mathbb{N}$. This implies that

$$\Gamma(z) = \frac{(-1)^n}{n!(z+n)} f_n(z) \tag{2.1}$$

is valid on the neighbourhood

$$D\left(-n, \frac{1}{4}\right) = \left\{z : |z+n| < \frac{1}{4}\right\} \tag{2.2}$$

of the points $z = -n$ with $n \in \{0\} \cup \mathbb{N}$, where $f_n(z)$ is analytic on $D\left(-n, \frac{1}{4}\right)$ and satisfies $\lim_{z\to -n} f_n(z) = 1$ for all $n \in \{0\} \cup \mathbb{N}$.

Differentiating $i \ge 0$ times on both sides of (2.1) yields

$$\Gamma^{(i)}(z) = \frac{(-1)^n}{n!} \sum_{\ell=0}^{i} \binom{i}{\ell} \frac{(-1)^\ell \ell!}{(z+n)^{\ell+1}} f_n^{(i-\ell)}(z).$$

Therefore, we have

$$\begin{aligned}\lim_{z\to -k} \frac{\Gamma^{(i)}(nz)}{\Gamma^{(i)}(qz)} &= \lim_{z\to -k}\left\{\left[\frac{(-1)^{nk}}{(nk)!} \sum_{\ell=0}^{i} \binom{i}{\ell} \frac{(-1)^\ell \ell!}{(nz+nk)^{\ell+1}} f_{nk}^{(i-\ell)}(nz)\right] \Big/ \left[\frac{(-1)^{qk}}{(qk)!} \sum_{\ell=0}^{i} \binom{i}{\ell} \frac{(-1)^\ell \ell!}{(qz+qk)^{\ell+1}} f_{qk}^{(i-\ell)}(qz)\right]\right\}\\ &= (-1)^{(n-q)k}\left(\frac{q}{n}\right)^{i+1} \frac{(qk)!}{(nk)!}.\end{aligned}$$

The proof of limit formulas (1.3) and (1.5) is completed.

3. The first alternative proof of limit formulas (1.4) and (1.6)

In [3, Theorem 1.2], an explicit formula for the n-th derivative of the cotangent function $\cot x$ was inductively established, which may be reformulated as

$$\cot^{(n)} x = \frac{1}{\sin^{n+1} x}\left\{\frac{1}{2} b_{n,\frac{1+(-1)^n}{2}} \cos\left[\frac{1+(-1)^n}{2} x\right] + \sum_{i=1}^{\frac{1}{2}\left[n-1-\frac{1+(-1)^n}{2}\right]} b_{n,2i+\frac{1+(-1)^n}{2}} \cos\left[\left(2i + \frac{1+(-1)^n}{2}\right)x\right]\right\}, \tag{3.1}$$

where

$$b_{p,q}=(-1)^{\frac{1-(-1)^p}{2}}2\sum_{\ell=0}^{\frac{p-q-1}{2}}(-1)^\ell\binom{p+1}{\ell}\left(\frac{p-q-1}{2}-\ell+1\right)^p \tag{3.2}$$

for $0\le q<p$ with $p-q$ being a positive and odd number.

In [1, p. 260, 6.4.7], the reflection formula

$$\psi^{(n)}(1-z)+(-1)^{n+1}\psi^{(n)}(z)=(-1)^n\pi\cot^{(n)}(\pi z),\quad n\ge0 \tag{3.3}$$

is collected. Hence, we have

$$\lim_{z\to-k}\frac{\psi^{(i)}(nz)}{\psi^{(i)}(qz)}=\lim_{z\to-k}\frac{(-1)^i\pi\cot^{(i)}(\pi nz)-\psi^{(i)}(1-nz)}{(-1)^i\pi\cot^{(i)}(\pi qz)-\psi^{(i)}(1-qz)},\quad i\ge0. \tag{3.4}$$

When $i=0$, we have

$$\begin{aligned}\lim_{z\to-k}\frac{\psi(nz)}{\psi(qz)}=\lim_{z\to-k}\frac{\pi\cot(\pi nz)-\psi(1-nz)}{\pi\cot(\pi qz)-\psi(1-qz)}&=\lim_{z\to-k}\left[\frac{\pi\cos(\pi nz)-\sin(\pi nz)\psi(1-nz)}{\pi\cos(\pi qz)-\sin(\pi qz)\psi(1-qz)}\cdot\frac{\sin(\pi qz)}{\sin(\pi nz)}\right]\\&=\lim_{z\to-k}\frac{\pi\cos(\pi nz)-\sin(\pi nz)\psi(1-nz)}{\pi\cos(\pi qz)-\sin(\pi qz)\psi(1-qz)}\lim_{z\to-k}\frac{\sin(\pi qz)}{\sin(\pi nz)}=\frac{q}{n}.\end{aligned}$$

The limit formulas (1.4) and (1.6) for $i=0$ follow.

When $i=1$, by (3.4) and (3.1) applied to $n=1$, we have

$$\begin{aligned}\lim_{z\to-k}\frac{\psi'(nz)}{\psi'(qz)}=\lim_{z\to-k}\frac{-\pi\cot'(\pi nz)-\psi'(1-nz)}{-\pi\cot'(\pi qz)-\psi'(1-qz)}&=\lim_{z\to-k}\frac{-\frac{\pi}{\sin^2(n\pi z)}+\psi'(1-nz)}{-\frac{\pi}{\sin^2(q\pi z)}+\psi'(1-qz)}\\&=\lim_{z\to-k}\left[\frac{-\pi+\sin^2(n\pi z)\psi'(1-nz)}{-\pi+\sin^2(q\pi z)\psi'(1-qz)}\cdot\frac{\sin^2(q\pi z)}{\sin^2(n\pi z)}\right]=\lim_{z\to-k}\frac{\sin^2(q\pi z)}{\sin^2(n\pi z)}=\left(\frac{q}{n}\right)^2.\end{aligned}$$

When $i=2j$ and $j\in\mathbb{N}$, by (3.4) and (3.1), we have

$$\begin{aligned}\lim_{z\to-k}\frac{\psi^{(2j)}(nz)}{\psi^{(2j)}(qz)}&=\lim_{z\to-k}\frac{(-1)^{2j}\pi\cot^{(2j)}(\pi nz)-\psi^{(2j)}(1-nz)}{(-1)^{2j}\pi\cot^{(2j)}(\pi qz)-\psi^{(2j)}(1-qz)}\\&=\lim_{z\to-k}\frac{\frac{\pi}{\sin^{2j+1}(nz\pi)}\sum_{i=0}^{j-1}b_{2j,2i+1}\cos[(2i+1)nz\pi]-\psi^{(2j)}(1-nz)}{\frac{\pi}{\sin^{2j+1}(qz\pi)}\sum_{i=0}^{j-1}b_{2j,2i+1}\cos[(2i+1)qz\pi]-\psi^{(2j)}(1-qz)}\\&=\lim_{z\to-k}\left[\frac{\pi\sum_{i=0}^{j-1}b_{2j,2i+1}\cos[(2i+1)nz\pi]-\sin^{2j+1}(nz\pi)\psi^{(2j)}(1-nz)}{\pi\sum_{i=0}^{j-1}b_{2j,2i+1}\cos[(2i+1)qz\pi]-\sin^{2j+1}(qz\pi)\psi^{(2j)}(1-qz)}\cdot\frac{\sin^{2j+1}(qz\pi)}{\sin^{2j+1}(nz\pi)}\right]\\&=\lim_{z\to-k}\frac{\sin^{2j+1}(qz\pi)}{\sin^{2j+1}(nz\pi)}\\&=\left(\frac{q}{n}\right)^{2j+1}.\end{aligned}$$

When $i = 2j + 1$ and $j \in \mathbb{N}$, by (3.4) and (3.1) again, we have

$$\begin{aligned}
\lim_{z\to -k} \frac{\psi^{(2j+1)}(nz)}{\psi^{(2j+1)}(qz)} &= \lim_{z\to -k} \frac{(-1)^{2j+1}\pi \cot^{(2j+1)}(\pi nz) - \psi^{(2j+1)}(1-nz)}{(-1)^{2j+1}\pi \cot^{(2j+1)}(\pi qz) - \psi^{(2j+1)}(1-qz)} \\
&= \lim_{z\to -k} \frac{\frac{\pi}{\sin^{2j+2}(nz\pi)} \sum_{i=0}^{j} b_{2j+1,2i} \cos(2inz\pi) + \psi^{(2j+1)}(1-nz)}{\frac{\pi}{\sin^{2j+2}(qz\pi)} \sum_{i=0}^{j} b_{2j+1,2i} \cos(2iqz\pi) + \psi^{(2j+1)}(1-qz)} \\
&= \lim_{z\to -k} \left[\frac{\pi \sum_{i=0}^{j} b_{2j+1,2i} \cos(2inz\pi) - \sin^{2j+2}(nz\pi)\psi^{(2j+1)}(1-nz)}{\pi \sum_{i=0}^{j} b_{2j+1,2i} \cos(2iqz\pi) - \sin^{2j+2}(qz\pi)\psi^{(2j+1)}(1-qz)} \cdot \frac{\sin^{2j+2}(qz\pi)}{\sin^{2j+2}(nz\pi)} \right] \\
&= \lim_{z\to -k} \frac{\sin^{2j+2}(qz\pi)}{\sin^{2j+2}(nz\pi)} \\
&= \left(\frac{q}{n}\right)^{2j+2}.
\end{aligned}$$

In conclusion, the limit formulas (1.4) and (1.6) are proved.

Remark 3.1. It seems that explicit formula (3.1) and those in [3] for the n-th derivatives of the cotangent function $\cot x$ and the tangent function $\tan x$ should exist somewhere. But, to the best of his knowledge, the author can not find them anywhere.

4. The second alternative proof of limit formulas (1.4) and (1.6)

It is well known [1, p. 260, 6.4.1] that the polygamma function $\psi^{(n)}(z)$ for $n \in \{0\} \cup \mathbb{N}$ is single valued and analytic over the entire complex plane, save at the points $z = -m$, with $m \in \{0\} \cup \mathbb{N}$, where it possesses poles of order $n + 1$. From this or (2.1), it follows that the expression

$$\psi^{(n)}(z) = \frac{(-1)^{n+1} n!}{(z+m)^{n+1}} + \left[\frac{f_m'(z)}{f_m(z)}\right]^{(n)} \tag{4.1}$$

for $n \in \{0\} \cup \mathbb{N}$ is valid on $D\left(-m, \frac{1}{4}\right)$ which is defined by (2.2).

In virtue of (4.1), we have

$$\lim_{z\to -k} \frac{\psi^{(i)}(nz)}{\psi^{(i)}(qz)} = \lim_{z\to -k} \left(\left\{ \frac{(-1)^{i+1} i!}{(nz+nk)^{i+1}} + \left[\frac{f_{nk}'(nz)}{f_{nk}(nz)}\right]^{(i)} \right\} \Bigg/ \left\{ \frac{(-1)^{i+1} i!}{(qz+qk)^{i+1}} + \left[\frac{f_{qk}'(qz)}{f_{qk}(qz)}\right]^{(i)} \right\} \right) = \left(\frac{q}{n}\right)^{i+1}.$$

The proof of limit formulas (1.4) and (1.6) is completed.

Remark 4.1. This is a combined version of the preprints [4, 5].

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