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The characterizations and representations for the generalized inverses with prescribed idempotents in Banach algebras

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Abstract. In this paper, we investigate the various different generalized inverses in a Banach algebra with respect to prescribed two idempotents p and q. Some new characterizations and explicit representations for these generalized inverses, such as $a^{(2)}_{p,q}$, $a^{(1,2)}_{p,q}$ and $a^{(2,l)}_{p,q}$ will be presented. The obtained results extend and generalize some well–known results for matrices or operators.

1. Introduction and preliminaries

Let X be a Banach space and $B(X)$ be the Banach algebra of all bounded linear operators on X. For $A \in B(X)$, let T and S be closed subspaces of X. Recall that the out inverse $A^{(2)}_{T,S}$ with prescribed the range T and the kernel S is the unique operator $G \in B(X)$ satisfying $GAG = G$, $\mathrm{Ran}(G) = T$, $\mathrm{Ker}(G) = S$. It is well known that $A^{(2)}_{T,S}$ exists if and only if

$$\mathrm{Ker}(A) \cap T = \{0\}, \qquad AT \dotplus S = Y. \tag{1.1}$$

This type of general inverse was generalized to the case of rings by Djordjević and Wei in [4]. Let $\mathcal{R}$ be a unital ring and let $\mathcal{R}^\bullet$ denote the set of all idempotent elements in $\mathcal{R}$. Given $p,\, q \in \mathcal{R}^\bullet$. Recall that an element $a \in \mathcal{R}$ has the (p,q)–outer generalized inverse $b = a^{(2)}_{p,q} \in \mathcal{R}$ if

$$bab = b, \quad ba = p, \quad 1 - ab = q.$$

If $b = a^{(2)}_{p,q}$ also satisfies the equation $aba = a$, then we say $a \in \mathcal{R}$ has the (p,q)–generalized inverse b, in this case, written $b = a^{(1,2)}_{p,q}$. If an outer generalized inverse with prescribed idempotents exists, it is necessarily unique [4]. According to this definition, many generalized inverses such as the Moore–Penrose inverses in a C^*–algebra and (generalized) Drazin inverses in a Banach algebra can be expressed by some (p,q)–outer generalized inverses [4, 5].

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Now we give some notations in this paper. Throughout this paper, $\mathscr{A}$ is always a complex Banach algebra with the unit 1. Let $a \in \mathscr{A}$. If there exists $b \in \mathscr{A}$ such that $aba = a$, then a is called inner regular and b is called an inner generalized inverse (or {1} inverse) of a, denoted by $b = a^-$. If there is an element $b \in \mathscr{A}$ such that $bab = b$, then b is called an outer generalized inverse (or {2} inverse) of a. We say that b is a (reflexive) generalized inverse(or {1, 2} inverse) of a, if b is both an inner and an outer generalized inverse of a (certainly such an element b is not unique). In this case, we let a^+ denote one of the generalized inverses of a. Let $Gi(\mathscr{A})$ denote the set of a in $\mathscr{A}$ such that a^+ exists. It is well–known that if b is an inner generalized inverse of a, then bab is a generalized inverse of a (cf. [1, 11, 16]).

Let $\mathscr{A}^\bullet$ denote the set of all idempotent elements in $\mathscr{A}$. If $a \in Gi(\mathscr{A})$, then a^+a and $1 - aa^+$ are all idempotent elements. For $a \in \mathscr{A}$, set

$$K_r(a) = \{x \in \mathscr{A} \mid ax = 0\}, \quad R_r(a) = \{ax \mid x \in \mathscr{A}\};$$
$$K_l(a) = \{x \in \mathscr{A} \mid xa = 0\}, \quad R_l(a) = \{xa \mid x \in \mathscr{A}\}.$$

Clearly, if $p \in \mathscr{A}^\bullet$, then $\mathscr{A}$ has the direct sum decompositions:

$$\mathscr{A} = K_r(p) \dotplus R_r(p) \quad or \quad \mathscr{A} = K_l(p) \dotplus R_l(p).$$

In this paper, we give a new definition of the generalized inverse with prescribed idempotents and discuss the existences of various different generalized inverses with prescribed idempotents in a Banach algebra. We also give some new characterizations and explicit representations for these generalized inverses. Also, we corrected Theorem 1.4 from [5].

2. Some existence conditions for the (p, q)–outer generalized inverse

Theorem 1.4 of [5] gives three equivalent characterizations of the (p, q)–outer generalized inverse $a^{(2)}_{p,q}$ which says:

Let $a \in \mathscr{A}$ and p, $q \in \mathscr{A}^\bullet$. Then the following statements are equivalent:

(i) $a^{(2)}_{p,q}$ exists;

(ii) There exists some element $b \in \mathscr{A}$ satisfying

$$bab = b, \quad R_r(b) = R_r(p), \quad K_r(b) = R_r(q);$$

(iii) There exists some element $b \in \mathscr{A}$ satisfying

$$bab = b, \quad b = pb, \quad p = bap, \quad b(1 - q) = b, \quad 1 - q = (1 - q)ab.$$

We see that **bab** = **b** is redundant in Statement (iii). In fact, by using some other equations in (iii), we can check that $bab = bapb = pb = b$. Also from the definition of $a^{(2)}_{p,q}$, it is easy to check that if $a^{(2)}_{p,q}$ exists, then Statements (ii) and (iii) hold. But we can show by following example that Statement (iii) does not imply Statement (i).

Example 2.1. *Consider the matrix algebra* $\mathscr{A} = M_2(\mathbb{C})$*, Let*

$$a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; p = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \; 1_2 - q = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is obvious that p *and* $1_2 - q$ *are idempotents. Moreover, we have*

$$pb = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = b, \quad bap = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = p,$$

$$b(1_2 - q) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = b, \quad (1_2 - q)ab = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 1_2 - q,$$

i.e., a, b, p, q satisfy Statement (iii) of Theorem 1.4 in [5], where 1_2 is the unit of $\mathscr{A}$. But

$$ba = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq p, \quad ab = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq 1_2 - q.$$

Therefore, by the uniqueness of $a^{(2)}_{p,q}$, we see $a^{(2)}_{p,q}$ does not exist.

By using the following auxiliary lemma, without equation $bab = b$, we prove that statements (ii) and (iii) in [5, Theorem 1.4] are equivalent, and we can also prove that the b is unique if it exists.

Lemma 2.2. *Let $x \in \mathscr{A}$ and $p \in \mathscr{A}^\bullet$. Then*

(1) *$K_r(p)$ and $R_r(p)$ are all closed and $K_r(p) = R_r(1-p)$, $R_r(p)\mathscr{A} \subset R_r(p)$;*
(2) *$px = x$ if and only if $R_r(x) \subset R_r(p)$ or $K_l(p) \subset K_l(x)$;*
(3) *$xp = x$ if and only if $K_r(p) \subset K_r(x)$ or $R_l(x) \subset R_l(p)$.*

Proposition 2.3. *Let $a \in \mathscr{A}$ and $p,\ q \in \mathscr{A}^\bullet$. Then the following statements are equivalent:*

(1) *There is $b \in \mathscr{A}$ satisfying $bab = b$, $R_r(b) = R_r(p)$ and $K_r(b) = R_r(q)$;*
(2) *There is $b \in \mathscr{A}$ satisfying $b = pb$, $p = bap$, $b(1-q) = b$, $1-q = (1-q)ab$.*

If there exists some b satisfying (1) *or* (2)*, then it is unique.*

Proof. (1) ⇒ (2) From $bab = b$, we know that $ba,\ ab \in \mathscr{A}^\bullet$. Then by using $R_r(b) = R_r(p)$ and $K_r(b) = R_r(q)$, we get

$$R_r(ba) = R_r(b) = R_r(p), \quad K_r(ab) = K_r(b) = R_r(q) = K_r(1-q).$$

Since $ba,\ ab \in \mathscr{A}^\bullet$, then by Lemma 2.2, we have

$$b = pb, \qquad p = bap, \qquad b(1-q) = b, \qquad 1-q = (1-q)ab.$$

(2) ⇒ (1) We have already known that $bab = bapb = pb = b$.

From $pb = b$ we get $R_r(b) \subset R_r(p)$. Since $p = bap$ and $bab = b$, we get $R_r(p) \subset R_r(ba) = R_r(b)$. Thus $R_r(b) = R_r(p)$.

Similarly, From $b(1-q) = b$, $1-q = (1-q)ab$, we can check that $K_r(b) = R_r(q)$ by using Lemma 2.2.

Now we show b is unique if it exists. In fact, if there exist b_1 and b, then

$$b_1 = pb_1 = bapb_1 = b(1-q)ab_1(1-q) = b(1-q) = b.$$

That is, b is unique. □

By means of Lemma 2.2 and Proposition 2.3, we can give a correct version of [5, Theorem 1.4] as follows.

Theorem 2.4. *Let $a \in \mathscr{A}$ and $p,\ q \in \mathscr{A}^\bullet$. Then the following statements are equivalent:*

(1) *There exists $b \in \mathscr{A}$ satisfying $bab = b$, $ba = p$ and $1 - ab = q$.*
(2) *There exists $b \in \mathscr{A}$ satisfying*

$$bab = b \quad \text{and} \quad \begin{cases} R_r(ba) = R_r(p),\ K_r(ab) = R_r(q) \\ R_l(ba) = R_l(p),\ K_l(ab) = R_l(q). \end{cases}$$

Proof. The implication (1) ⇒ (2) is obvious. We now prove (2) ⇒ (1).

Suppose that (2) holds. Then $bab = b$ is a known equivalent condition. By Proposition 2.3, we have

$$b = pb, \quad p = bap, \quad b(1-q) = b, \quad 1-q = (1-q)ab. \tag{2.1}$$

Since $ba, ab \in \mathscr{A}^\bullet$, then by Lemma 2.2, we have

$$bap = ba, \quad p = pba, \quad ab(1-q) = (1-q), \quad (1-q)ab = ab. \tag{2.2}$$

Then from Eq. (2.1) and Eq. (2.2), we can get $ba = pba = p$ and $ab = (1-q)ab = 1-q$. This completes the proof. □

Obviously, for an operator $A \in B(X)$, if $A^{(2)}_{T,S}$ exists, we can set $P = A^{(2)}_{T,S}A$ and $Q = I - AA^{(2)}_{T,S}$, then we have $\mathrm{Ran}(A^{(2)}_{T,S}) = \mathrm{Ran}(P)$ and $\mathrm{Ker}(A^{(2)}_{T,S}) = \mathrm{Ran}(Q)$. Thus the (p,q)–outer generalized inverse is a natural algebraic extension of the generalized inverse of linear operators with prescribed range and kernel. Similar to some characterizations of the outer generalized inverse $A^{(2)}_{T,S}$ about matrix and operators, we present the following statements relative to the (p,q)–outer generalized inverse $a^{(2)}_{p,q}$.

Statement 2.5. *Let $a \in \mathscr{A}$ and $p, q \in \mathscr{A}^\bullet$. Consider the following four statements:*

(1) *$a^{(2)}_{p,q}$ exists;*
(2) *There exists $b \in \mathscr{A}$ such that $bab = b$, $R_r(b) = R_r(p)$ and $K_r(b) = R_r(q)$;*
(3) *$K_r(a) \cap R_r(p) = \{0\}$ and $\mathscr{A} = aR_r(p) \dotplus R_r(q)$;*
(4) *$K_r(a) \cap R_r(p) = \{0\}$ and $aR_r(p) = R_r(1-q)$.*

If we assume statements (1) in 2.5 holds, i.e., $a^{(2)}_{p,q}$ exists, then we can check easily that the other three statements (2), (3) and (4) in Statements 2.5 will hold. Here we give a proof of (1) ⇒ (4).

Proposition 2.6. *Let $a \in \mathscr{A}$ and $p, q \in \mathscr{A}^\bullet$. If $a^{(2)}_{p,q}$ exists, then $K_r(a) \cap R_r(p) = \{0\}$ and $aR_r(p) = R_r(1-q)$.*

Proof. Suppose that $a^{(2)}_{p,q}$ exists, then from its definition, we have $ba = p$ and $ab = 1 - q$. Let $x \in K_r(a) \cap R_r(p)$, then there exists some $t \in R_r(p)$ such that $x = pt$ and $ax = 0$, it follows that $x = bat = babat = bapt = bax = 0$, i.e., $K_r(a) \cap R_r(p) = 0$.

Let $y \in aR_r(p)$, then $y = aps$ for some $s \in \mathscr{A}$, that is $y = aps = abas = (1-q)as$, thus $y \in R_r(1-q)$. For any $x \in R_r(1-q)$, there is some $t \in R_r(1-q)$ with $x = (1-q)t$, then $x = abt = ababt = apbt$ and hence $x \in aR_r(p)$. This completes the proof. □

It is obvious that (4) ⇒ (3) holds, but the following example shows that (3) and (4) are not equivalent in general.

Example 2.7. *We also consider the matrix algebra $\mathscr{A} = M_2(\mathbb{C})$, and take the same elements $a, p, q \in \mathscr{A}$ as in Example 2.1. Then*

$$aR_r(p) = \left\{\begin{bmatrix} 0 & 0 \\ s & t \end{bmatrix} \Big| s, t \in \mathbb{C}\right\}, \quad R_r(1_2 - q) = \left\{\begin{bmatrix} s & t \\ s & t \end{bmatrix} \Big| s, t \in \mathbb{C}\right\},$$
$$K_r(a) = \left\{\begin{bmatrix} 0 & 0 \\ s & t \end{bmatrix} \Big| s, t \in \mathbb{C}\right\}, \qquad R_r(p) = \left\{\begin{bmatrix} s & t \\ 0 & 0 \end{bmatrix} \Big| s, t \in \mathbb{C}\right\}.$$

It follows that $aR_r(p) \neq R_r(1-q)$ when $s \neq t \neq 0$ and $\mathscr{A} = aR_r(p) \dotplus R_r(q)$.

Therefore, from above arguments, (3) and (4) in Statement 2.5 are not equivalent in general.

But similar to the outer inverse $A^{(2)}_{T,S}$, as described by Eq. (1.1), we can prove (2) and (3) in Statements 2.5 are equivalent.

Theorem 2.8. *Let $a \in \mathscr{A}$ and $p, q \in \mathscr{A}^\bullet$. Then the following statements are equivalent:*

(1) *There exists $b \in \mathscr{A}$ such that $bab = b$, $R_r(b) = R_r(p)$ and $K_r(b) = R_r(q)$,*
(2) *$K_r(a) \cap R_r(p) = \{0\}$ and $\mathscr{A} = aR_r(p) \dotplus R_r(q)$.*

Proof. (1) ⇒ (2) Suppose that there exists $b \in \mathscr{A}$ satisfying (1). Since $bab = b$, then ab is idempotent. Thus we have $\mathscr{A} = R_r(ab) \dotplus K_r(ab)$. It is easy to check that

$$R_r(ab) = aR_r(b) = aR_r(p) \text{ and } K_r(ab) = K_r(b) = R_r(q).$$

Hence $\mathscr{A} = aR_r(b) \dotplus R_r(q)$.

Next we show that $K_r(a) \cap R_r(p) = 0$. Let $x \in K_r(a) \cap R_r(p)$. Since $R_r(p) = R_r(b)$, then there exists some $y \in \mathscr{A}$ such that $x = by$ and $ax = 0$. So $x = by = baby = bax = 0$. Therefore, we have $K_r(a) \cap R_r(p) = \{0\}$.

(2) ⇒ (1) Suppose that (2) is true. From $\mathscr{A} = aR_r(p) \dotplus R_r(q)$ and $\mathscr{A}$ is a Banach algebra, we see that $aR_r(p) = R_r(ap)$ is closed in $\mathscr{A}$. Let $L_a : \mathscr{A} \to \mathscr{A}$ be the left multiplier on $\mathscr{A}$, i.e., $L_a(x) = ax$ for all $x \in \mathscr{A}$. Let ϕ be the restriction of L_a on $R_r(p)$, Since $\|L_a\| \le \|a\|$, we have $\phi \in B(R_r(p), aR_r(p))$ with $\mathrm{Ker}(\phi) = \{0\}$ and $\mathrm{Ran}(\phi) = aR_r(p)$. Thus $\phi^{-1} : aR_r(p) \to R_r(p)$ is bounded. Since for any $x \in R_r(p)$ and $z \in \mathscr{A}$, $xz \in R_r(p)$, it follows that $\phi(xz) = \phi(x)z, \forall x \in R_r(p)$ and $z \in \mathscr{A}$, and then $\phi^{-1}(yz) = \phi^{-1}(y)z$ for any $y \in aR_r(p)$ and $z \in \mathscr{A}$.

Let $Q : \mathscr{A} \to aR_r(p)$ be the bounded idempotent mapping. Since

$$\mathscr{A} = aR_r(p) \dotplus R_r(q),\ R_r(q)\mathscr{A} \subset R_r(q) \text{ and } aRr(p)\mathscr{A} \subset aRr(p),$$

it follows that for any $x, z \in \mathscr{A}$, $x = x_1 + x_2$ and $xz = x' + x_2 z$, where $x' = x_1 z \in aR_r(p)$ and $x_2 \in R_r(q)$, and hence $Q(xz) = Q(x)z$. Set $W = \phi^{-1} \circ Q$. Then

$$W(xz) = W(x)z, \quad (WL_aW)(x) = W(x), \quad \forall\, x,\ z \in \mathscr{A}.$$

Put $b = W(1)$. Then from the above arguments, we get that $bab = b$. Since $W(x) = W(1)x$ for any $x \in \mathscr{A}$, we have

$$R_r(b) = R_r(W(1)) = \mathrm{Ran}(W) = R_r(p),$$
$$K_r(b) = K_r(W(1)) = \mathrm{Ker}(W) = R_r(q).$$

Thus (1) holds. □

Now for four statements in Statement 2.5, we have (1) ⇒ (4) ⇒ (3) ⇔ (2), and in general, (3) ⇏ (4). The following example also shows that statements (1) and (4) in Statement 2.5 are not equivalent in general.

Example 2.9. *Let $\mathscr{A} = M_2(\mathbb{C})$ and let $a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $p = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $1_2 - q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $aR_r(p) = R_r(1_2 - q)$ and $K_r(a) \cap R_r(p) = \{0\}$, i.e., Statement (4) holds.*

If $a^{(2)}_{p,q}$ exists, that is, there is $b \in \mathscr{A}$ such that $bab = b$, $ba = p$ and $1_2 - ab = q$. But we could not find $b \in \mathscr{A}$ such that $ba = p$. Thus (4)⇏(1).

Based on above arguments, we give the following new definition with respect to the outer generalized inverse with prescribed idempotents in a general Banach algebra.

Definition 2.10. *Let $a \in \mathscr{A}$ and $p,\ q \in \mathscr{A}^{\bullet}$. An element $c \in \mathscr{A}$ satisfying*

$$cac = c, \qquad R_r(c) = R_r(p), \qquad K_r(c) = R_r(q)$$

will be called the (p, q, l)–outer generalized inverse of a, written as $a^{(2,l)}_{p,q} = c$.

Remark 2.11. *For $a \in \mathscr{A}$ and $p,\ q \in \mathscr{A}^{\bullet}$. Let $L_a \colon \mathscr{A} \to \mathscr{A}$ be the left multiplier on $\mathscr{A}$. If we set $T = R_r(p)$ and $S = R_r(q)$. Then it is obvious that $a^{(2,l)}_{p,q}$ exists in $\mathscr{A}$ if and only if $(L_a)^{(2)}_{T,S}$ exists in the Banach algebra $B(\mathscr{A})$. This shows that why we use the letter "l" in Definition 2.10.*

Example 2.12. *Let $\mathscr{A} = M_2(\mathbb{C})$ and $a,\ p,\ q \in \mathscr{A}$ be as in Example 2.1. Simple computation shows that $a^{(2,l)}_{p,q} = b$.*

By Lemma 2.2 and Definition 2.10, we have

Corollary 2.13. *Suppose that $a, w \in \mathscr{A}$ and $a^{(2,l)}_{p,q}$ exists. Then*

(1) *$a^{(2,l)}_{p,q} aw = w$ if and only if $R_r(w) \subset R_r(p)$,*
(2) *$waa^{(2,l)}_{p,q} = w$ if and only if $R_r(q) \subset K_r(w)$.*

From Theorem 2.3 and Theorem 2.8, we know that if $a^{(2,l)}_{p,q}$ exists, then it is unique, and the properties of $a^{(2,l)}_{p,q}$ are much more similar to $A^{(2)}_{T,S}$ than $a^{(2)}_{p,q}$. Thus the outer generalized inverse $a^{(2,l)}_{p,q}$ is also a natural extension of the generalized inverses $A^{(2)}_{T,S}$. Also from Definition 2.10, we can see that if $a^{(2,l)}_{p,q} = c$ exists, then we also have $R_r(ca) = R_r(c) = R_r(p)$ and $K_r(ac) = K_r(c) = R_r(q)$.

3. Characterizations for the generalized inverses with prescribed idempotents

By using idempotent elements, firstly, we give the following new characterization of generalized invertible elements in a Banach algebra.

Proposition 3.1 ([16, Theorem 2.4.4])**.** *Let $a \in \mathscr{A}$. Then $a \in Gi(\mathscr{A})$ if and only if there exist $p, q \in \mathscr{A}^\bullet$ such that $K_r(a) = K_r(p)$ and $R_r(a) = R_r(q)$.*

Proof. Suppose that $a \in Gi(\mathscr{A})$. Put $p = a^+a$ and $q = aa^+$. Then $p, q \in \mathscr{A}$. Let $x \in K_r(a)$. Then $ax = 0$ and $px = a^+ax = 0$, that is $x \in K_r(p)$. On the other hand, let $y \in K_r(p)$, then $ay = aa^+ax = apy = 0$. So we have $K_r(a) = K_r(p)$.

Similarly, we can check $R_r(a) = R_r(q)$.

Now assume that $K_r(a) = K_r(p)$ and $R_r(a) = R_r(q)$ for $p, q \in \mathscr{A}^\bullet$. Then

$$\mathscr{A} = R_r(p) \dotplus K_r(p) = R_r(p) \dotplus K_r(a).$$

Let L be the restriction of L_a on $R_r(p)$. Then L is a mapping from $R_r(p)$ to $R_r(a)$ with $\mathrm{Ker}(L) = \{0\}$ and $\mathrm{Ran}(L) = R_r(a) = R_r(q)$. Thus $L^{-1}\colon R_r(a) \to R_r(p)$ is well–defined. Note that $xt \in R_r(p)$ for any $x \in R_r(p)$ and $t \in \mathscr{A}$. So we have

$$L(xt) = L_a(xt) = axt = L_a(x)t = L(x)t$$

and hence $L^{-1}(st) = L^{-1}(s)t$ for any $s \in R_r(a)$ and $t \in \mathscr{A}$.

Put $G = L^{-1} \circ L_q$. Since $q \in \mathscr{A}^\bullet$, we have $L_q\colon \mathscr{A} \to R_r(q)$ is an idempotent mapping. Put $b = G(1)$. It is easy to check that $aba = a$ and $bab = b$, i.e., $a \in Gi(\mathscr{A})$. □

We now present the equivalent conditions about the existence of $a^{(2,l)}_{p,q}$ as follows.

Theorem 3.2. *Let $a \in \mathscr{A}$ and $p, q \in \mathscr{A}^\bullet$. Then the following statements are equivalent:*

(1) *$a^{(2,l)}_{p,q}$ exists,*
(2) *There exists $b \in \mathscr{A}$ such that $bab = b$, $R_r(b) = R_r(p)$ and $K_r(b) = R_r(q)$,*
(3) *$K_r(a) \cap R_r(p) = \{0\}$ and $\mathscr{A} = aR_r(p) \dotplus R_r(q)$.*
(4) *There is some $b \in \mathscr{A}$ satisfying $b = pb$, $p = bap$, $b(1-q) = b$, $1-q = (1-q)ab$.*
(5) *$p \in R_l((1-q)ap) = \{x(1-q)ap \mid x \in \mathscr{A}\}$ and $1-q \in R_r((1-q)ap)$,*
(6) *There exist some $s, t \in \mathscr{A}$ such that $p = t(1-q)ap$, $1-q = (1-q)aps$.*

Proof. (1)⇔(2) comes from the definition of $a^{(2,l)}_{p,q}$. The implication (2)⇔(3) is Theorem 2.8 and the implication (3)⇔(4) is Proposition 2.3. The implication (5)⇒(6) is obvious.

(1)⇒(5) Choose $x = b$, then $p = bap = b(1-q)ap \in R_l((1-q)ap)$. Since $1-q = (1-q)ab = (1-q)abp$, then, $1-q \in R_r((1-q)ap)$.

(6) ⇒ (4) If $p = t(1-q)ap$ and $1-q = (1-q)aps$ for some $s, t \in \mathscr{A}$, then $t(1-q) = ps$. Set $b = t(1-q) = ps$. Then $pb = pps = b$, $bap = t(1-q)ap = p$ and $b(1-q) = t(1-q)(1-q) = b$, $(1-q)ab = (1-q)aps = 1-q$. □

If the generalized inverse $a^{(2,l)}_{p,q}$ satisfies $aa^{(2,l)}_{p,q}a = a$, then we call it the $\{1, 2\}$ generalized inverse of $a \in \mathscr{A}$ with prescribed idempotents p and q. It is denoted by $a^{(l)}_{p,q}$. Obviously, $a^{(l)}_{p,q}$ is unique if it exists. The following theorem gives some equivalent conditions about the existence of $a^{(l)}_{p,q}$.

Theorem 3.3. *Let $a \in \mathscr{A}$ and $p, q \in \mathscr{A}^\bullet$. Then the following conditions are equivalent:*

(1) *$a^{(l)}_{p,q}$ exists, i.e.,there exists some $b \in \mathscr{A}$ such that*

$$aba = a, \quad bab = b, \quad R_r(b) = R_r(p), \quad K_r(b) = R_r(q),$$

(2) *$\mathscr{A} = R_r(a) \dotplus R_r(q) = K_r(a) \dotplus R_r(p)$,*
(3) *$\mathscr{A} = aR_r(p) \dotplus R_r(q)$, $R_r(a) \cap R_r(q) = \{0\}$, $K_r(a) \cap R_r(p) = \{0\}$.*

Proof. (1) ⇒ (2) From (1), we have that $(ab)^2 = ab$ and $(ba)^2 = ba$. Also we have the following relation:

$$K_r(b) \subset K_r(ab) \subset K_r(bab) = K_r(b),\ K_r(ba) \subset K_r(aba) = K_r(a) \subset K_r(ba),$$
$$R_r(ba) \subset R_r(b) = R_r(bab) \subset R_r(ba),\ R_r(ab) \subset R_r(a) = R_r(aba) \subset R_r(ab).$$

Thus we have

$$R_r(ba) = R_r(b) = R_r(p),\ R_r(ab) = R_r(a),$$
$$K_r(ab) = R_r(b) = R_r(q),\ K_r(ba) = K_r(a).$$

From above equations, we can get

$$\mathscr{A} = R_r(a) \dotplus R_r(q) = K_r(a) \dotplus R_r(p).$$

(2) ⇒ (3) If $\mathscr{A} = R_r(a) \dotplus R_r(q) = K_r(a) \dotplus R_r(p)$, then it is obvious that

$$R_r(a) \cap R_r(q) = \{0\},\ K_r(a) \cap R_r(p) = \{0\}.$$

So we need only to check that $aK_r(p) = R_r(a)$.

Obviously, we have $aR_r(p) \subset R_r(a)$. Now for any $x \in R_r(a)$, we have $x = at$ for some $t \in \mathscr{A}$. Since $\mathscr{A} = K_r(a) \dotplus R_r(p)$, we can write $t = t_1 + t_2$, where $t_1 \in K_r(a)$ and $t_2 \in R_r(p)$. Thus $x = at = at_2 \in aR_r(p)$. Therefore $R_r(a) \subset aR_r(p)$ and $R_r(a) = aR_r(p)$.

(3) ⇒ (1) Suppose that (3) is true, then by Theorem 3.2, we know that $a_{p,q}^{(2,l)}$ exists, and $R_r(b) = R_r(p),\ K_r(b) = R_r(q)$. We need to show $aba = a$. Since $bab = b$, then $b(aba - a) = 0$. it follows that

$$aba - a \subset R_r(a) \cap K_r(b) = R_r(a) \cap R_r(q) = \{0\}.$$

i.e., $aba = a$. This completes the proof. □

The following proposition gives a characterization of $a_{p,q}^{(1,2)}$ in a Banach algebra $\mathscr{A}$.

Proposition 3.4. *Let $a \in \mathscr{A}$ and $p,\ q \in \mathscr{A}^\bullet$. Then the following statements are equivalent:*

(1) *$a_{p,q}^{(1,2)}$ exists,*
(2) *There exists some $b \in \mathscr{A}$ satisfying*

$$\begin{cases} aba = a \\ bab = b \end{cases} \quad \text{and} \quad \begin{cases} R_r(b) = R_r(p),\ K_r(b) = R_r(q) \\ R_r(a) = R_r(1-q),\ K_r(a) = R_r(1-p) \end{cases}.$$

Proof. (1) ⇒ (2) This is obvious by the Definition of $a_{p,q}^{(1,2)}$.
(2) ⇒ (1) Since $aba = a$ and $bab = b$, so by (2) we have

$$\begin{cases} R_r(ba) = R_r(b) = R_r(p), \quad K_r(ab) = K_r(b) = R_r(q) \\ R_r(ab) = R_r(a) = R_r(1-q), \quad K_r(ba) = K_r(a) = R_r(1-p) \end{cases}.$$

Then by Lemma 2.2 and Theorem 3.2, we have

$$b = pb, \quad (1-q) = (1-q)ab, \quad a = (1-q)a, \quad p = pba.$$

Thus, we get

$$ba = pba = p, \quad ab = (1-q)ab = 1-q.$$

Since $a_{p,q}^{(1,2)}$ is unique, it follows that $a_{p,q}^{(1,2)} = b$. □

4. Some representations for the generalized inverses with prescribed idempotents

Let $a \in \mathscr{A}$. If for some positive integer k, there exists an element $b \in \mathscr{A}$ such that

$$(1^k) \quad a^{k+1}b = a^k, \quad (2) \quad bab = b, \quad (5) \quad ab = ba.$$

Then a is Drazin invertible and b is called the Drazin inverse of a, denoted by a^D (cf. [1, 3]). The least integer k is the index of a, denoted by $\mathrm{ind}(a)$. When $\mathrm{ind}(a) = 1$, a^D is called the group inverse of a, denoted by $a^\#$. It is well–known that if the Drazin (group) inverse of a exists, then it is unique. Let $\mathscr{A}^D$ (resp. $\mathscr{A}^g$) denote the set of all Drazin (resp. group) invertible elements in $\mathscr{A}$.

The representations of $A^{(2)}_{T,S}$ of a matrix or an operator have been extensively studied. We know that if $A^{(2)}_{T,S}$ exists, then it can be explicitly expressed by the group inverse of AG or GA for some matrix or an operator G with $\mathrm{Ran}(G) = T$ and $\mathrm{Ker}(G) = S$ (cf. [13–15, 18]). By using the left multiplier representation, we give an explicit representation for the $a^{(2,l)}_{p,q}$ by the group inverse as follows.

Theorem 4.1. *Let $a \in \mathscr{A}$ and $p, q \in \mathscr{A}^\bullet$ such that $a^{(2,l)}_{p,q}$ exists. Let $w \in \mathscr{A}$ such that $R_r(w) = R_r(p)$ and $K_r(w) = R_r(q)$. Then $wa,\ aw \in \mathscr{A}^g$ and $a^{(2,l)}_{p,q} = (wa)^\# w = w(aw)^\#$.*

Proof. Firstly, we show that $a^{(2,l)}_{p,q} = w(aw)^\#$.

Obviously, $R_r(aw) = aR_r(w) = aR_r(p)$. For any $y \in K_r(aw)$, $awy = 0$ and

$$wy \in K_r(a) \cap R_r(w) = K_r(a) \cap R_r(p) = \{0\}.$$

It follows that $wy = 0$ and $y \in K_r(w)$. Thus $K_r(aw) \subset K_r(w)$ and consequently $K_r(aw) = K_r(w) = R_r(q)$. Therefore, $\mathscr{A} = R_r(aw) \dotplus K_r(aw)$. Set $L = L_{aw}\big|_{R_r(aw)}$. Then $\mathrm{Ker}(L) = \{0\}$ and $\mathrm{Ran}(L) = R_r(aw)$. Let $P\colon \mathscr{A} \to R_r(aw)$ be a projection (idempotent operator). Then $P \in B(\mathscr{A})$ and $L^{-1} \in B(R_r(aw))$. Put $G = L^{-1} \circ P \in B(\mathscr{A})$. Then we have $G(xy) = G(x)y$, $\forall\, x,\ y \in \mathscr{A}$ and

$$GL_{aw}G = G,\ L_{aw}GL_{aw} = L_{aw},\ L_{aw}G = GL_{aw}.$$

Put $c = G(1)$. Then $(aw)c(aw) = aw$, $c(aw)c = c$ and $c(aw) = (aw)c$, i.e., $(aw)^\# = c$. So $R_r(aw(aw)^\#) = R_r(aw) = aR_r(p)$. Put $b = w(aw)^\#$. Then

$$bab = w(aw)^\# aw(aw)^\# = w(aw)^\# = b$$

and $R_r(b) = R_r(w(aw)^\#) \subset R_r(w) = R_r(p)$. On the other hand, since

$$K_r((aw)^\# aw) = K_r(aw(aw)^\#) = K_r(aw) = R_r(q) = K_r(w),$$

it follows from Lemma 2.2 that $w(aw)^\# aw = w$. Thus

$$R_r(p) = R_r(w) = R_r(w(aw)^\# aw) \subset R_r(b) \subset R_r(p).$$

Similarly, we have

$$\begin{aligned} K_r(b) &= K_r(w(aw)^\#) \subset K_r(aw(aw)^\#) = K_r(aw) = R_r(q),\\ R_r(q) &= K_r(w) = K_r(aw(aw)^\#) \subset K_r(w(aw)^\# aw(aw)^\#)\\ &= K_r(w(aw)^\#) = K_r(b). \end{aligned}$$

Thus, $K_r(b) = R_r(q)$. So by the unique of $a^{(2,l)}_{p,q}$, we have $b = w(aw)^\# = a^{(2,l)}_{p,q}$.

Similarly, if we put $d = (wa)^\# w$, then we can prove $d = a^{(2,l)}_{p,q} = b$. □

Corollary 4.2. *Let $a \in \mathscr{A}$ and $p, q \in \mathscr{A}^\bullet$ such that $a^{(2,l)}_{p,q}$ exists. Let $w \in \mathscr{A}$ with $R_r(w) = R_r(p)$ and $K_r(w) = R_r(q)$. Then there exists some $c \in \mathscr{A}$ such that*

$$wawc = w \quad \text{and} \quad a^{(2,l)}_{p,q} awc = a^{(2,l)}_{p,q}. \tag{4.1}$$

Proof. From Theorem 4.1 above, we know that $(aw)^{\#}$ exists. Put $c = (aw)^{\#}$. Then by Lemma 2.2 and Theorem 4.1, we see that

$$K_r(a^{(2,l)}_{p,q}) = R_r(q) = K_r(w) = K_r(aw) = K_r(awc).$$

Thus $c = (aw)^{\#}$ satisfies Eq.(4.1). □

Similar to Theorem 4.1, we have the following easy representation of $a^{(2)}_{p,q}$.

Theorem 4.3. *Let $a \in \mathscr{A}$ and $p, q \in \mathscr{A}^{\bullet}$ such that $a^{(2)}_{p,q}$ exists. Let $w \in \mathscr{A}$ such that $wa = p$ and $aw = 1 - q$. Then $a^{(2)}_{p,q} = (wa)^{\#}w = w(aw)^{\#}$.*

Proof. Obviously, $wa, aw \in \mathscr{A}^g$ for $wa = p$ and $aw = 1 - q$. We also have $(wa)^{\#} = p$ and $(aw)^{\#} = 1 - q$. Then by using the uniqueness of $a^{(2)}_{p,q}$, we can prove this theorem by simple computation. □

From the Definition of $a^{(2)}_{p,q}$ and $a^{(2,l)}_{p,q}$, it is easy to see that if $a^{(2)}_{p,q}$ exists, then $a^{(2,l)}_{p,q}$ exists. In this case, $a^{(2)}_{p,q} = a^{(2,l)}_{p,q} = b$ by the uniqueness. Thus, Theorem 4.1 and Corollary 4.2 also hold if we replace $a^{(2,l)}_{p,q}$ by $a^{(2)}_{p,q}$. The following result also gives some generalizations of [4, Theorem 2.2] and [5, Theorem 1.2].

Corollary 4.4. *Let $a, c \in \mathscr{A}$ and $p, q \in \mathscr{A}^{\bullet}$ such that $a^{(2,l)}_{p,q}$ and $c^{(1,2)}_{1-q,1-p}$ exist. Then ac and ca is group invertible and $a^{(2,l)}_{p,q} = (ca)^{\#}c = c(ac)^{\#}$.*

Proof. Since $c^{(1,2)}_{1-q,1-p}$ exists, then by Proposition 3.4, we have $R_r(c) = R_r(p)$, $K_r(c) = R_r(q)$. Thus, we can get our results by using Theorem 4.1. □

In the following theorem, we give a simple representation of $a^{(2,l)}_{p,q}$ based on {1} inverse. The analogous results about outer generalized inverse with prescribed range and null space of Banach space operators were presented in [20, Theorem 2.1].

Theorem 4.5. *Let $p, q \in \mathscr{A}^{\bullet}$. Let $a, w \in \mathscr{A}$ with $R_r(w) = R_r(p)$ and $K_r(w) = R_r(q)$. Then the following conditions are equivalent:*

(1) *$a^{(2,l)}_{p,q}$ exists;*
(2) *$(aw)^{\#}$ exists and $K_r(a) \cap R_r(w) = \{0\}$;*
(3) *$(wa)^{\#}$ exists and $R_r(w) = R_r(waw)$.*

In this case, waw is regular and

$$a^{(2,l)}_{p,q} = (wa)^{\#}w = w(aw)^{\#} = w(waw)^{-}w. \tag{4.2}$$

Proof. (1) ⇒ (2) follows from Theorem 3.2 and Theorem 4.1.

(2) ⇒ (3) Since $(aw)^{\#}$ exists, we have $a(waw(aw)^{\#} - w) = 0$. So

$$waw(aw)^{\#} - w \in K_r(a) \cap R_r(w) = \{0\},$$

i.e., $waw(aw)^{\#} = w$. Therefore $R_r(w) = R_r(waw)$.

Let $c = w((aw)^{\#})^2 a$, we show that c is the group inverse of wa. In fact,

$$c(wa)c = w((aw)^{\#})^2(aw)(aw)((aw)^{\#})^2 a = w(aw)^{\#}(aw)^{\#}a = c,$$

$$(wa)c(wa) = waw((aw)^{\#})^2 awa = (waw(aw)^{\#})a = wa,$$

$$(wa)c = waw((aw)^{\#})^2 a = w(((aw)^{\#})^2 aw)a = c(wa).$$

i.e., $(wa)^{\#}$ exists and $c = (wa)^{\#}$.

(3) $\Rightarrow$ (1) Since $R_r(w) = R_r(waw)$, $R_r(wa) \subset R_r(w) = R_r(waw) \subset R_r(wa)$. So $R_r(w) = R_r(wa)$. The existence of $(wa)^{\#}$ means that

$$K_r(wa) \cap R_r(wa) = \{0\} \text{ and } wa(wa(wa)^{\#}w - w) = 0.$$

So

$$wa(wa)^{\#}w - w \in K_r(wa) \cap R_r(w) = K_r(wa) \cap R_r(wa) = \{0\}.$$

Hence $w = wa(wa)^{\#}w$. Set $b = (wa)^{\#}w$, then we have

$$\begin{aligned} bab &= (wa)^{\#}wa(wa)^{\#}w = (aw)^{\#}w = b, \\ R_r(p) &= R_r(w) = R_r(wa(wa)^{\#}w) = R_r((wa)^{\#}waw) \subset R_r((wa)^{\#}w) \\ &= R_r(b) = R_r((wa)^{\#}wa(wa)^{\#}w) \\ &= R_r(wa((wa)^{\#})^2w) \subset R_r(w) \\ &= R_r(p), \\ R_r(q) &= K_r(w) \subset K_r((wa)^{\#}w) = K_r(b) \subset K_r(wa(wa)^{\#}w) = K_r(w) \\ &= R_r(q). \end{aligned}$$

Thus, $R_r(b) = R_r(p)$ and $K_r(b) = R_r(q)$ and consequently, by Theorem 3.2, we see $a_{p,q}^{(2,l)}$ exists and $a_{p,q}^{(2,l)} = (wa)^{\#}w$.

Now we show that waw is regular and Eq. (4.2) is true when one of (1), (2) or (3) in the Theorem 4.5 holds. Since

$$R_r((wa)^{\#}wa) = R_r(wa) = R_r(w) \supset R_r(w(aw)^{\#}),$$

therefore, by Lemma 2.2, we have $((wa)^{\#}wa)w(aw)^{\#} = w(aw)^{\#}$. Then

$$\begin{aligned} w(aw)^{\#} &= ((wa)^{\#}wa)w(aw)^{\#} = ((wa)^{\#}wa)^2w(aw)^{\#} \\ &= ((wa)^{\#})^2waw = (wa)^{\#}w. \end{aligned}$$

Thus we have $a_{p,q}^{(2,l)} = w(aw)^{\#}$.

Set $x = a(wa)^{\#})^2$. Then it is easy to check that

$$(waw)x(waw) = (waw)a(wa)^{\#})^2(waw) = waw.$$

i.e., waw is regular and $(waw)^{-} = a(wa)^{\#})^2$. Furthermore, we have the following,

$$w(waw)^{-}w = wa(wa)^{\#})^2w = (wa)^{\#}w = a_{p,q}^{(2,l)}.$$

This completes the proof. $\square$

The following result is an improvement of Theorem 4.5, which removes the existence of the group inverses of wa or aw.

Theorem 4.6. *Let $a, w \in \mathscr{A}$ and $p,\ q \in \mathscr{A}^{\bullet}$ such that $w_{1-q,1-p}^{(1,2)}$ exist (or $R_r(w) = R_r(p)$ and $K_r(w) = R_r(q)$). Then the following statements are equivalent:*

(1) *$a_{p,q}^{(2,l)}$ exists;*
(2) *$(aw)^{(1,5)}$ exists and $K_r(a) \cap R_r(w) = \{0\}$;*
(3) *$(wa)^{(1,5)}$ exists and $R_r(w) = R_r(wa)$.*

In this case, waw is inner regular and

$$a_{p,q}^{(2,l)} = (wa)^{(1,5)}w = w(aw)^{(1,5)} = w(waw)^{-}w.$$

Proof. We only need to follow the line of proof Theorem 2.1 in [5], but make some essential modifications by using Definition 2.10 and Theorem 3.2. Here we omit the detail. $\square$

Based on an explicit representation for $a_{p,q}^{(2,l)}$ by the group inverse, now we can give some limit and integral representations of the (p,q,l)-outer generalized inverse $a_{p,q}^{(2,l)}$, the analogous result is well-known for operators on Banach space (see [16]).

Theorem 4.7. *Let $a \in \mathscr{A}$ and $p, q \in \mathscr{A}^\bullet$ such that $a_{p,q}^{(2,l)}$ exists. Let $w \in \mathscr{A}$ such that $R_r(w) = R_r(p)$ and $K_r(w) = R_r(q)$. Then $a_{p,q}^{(2,l)} = \lim\limits_{\substack{\lambda\to 0\\ \lambda\notin\sigma(-aw)}} w(\lambda 1 + aw)^{-1}$.*

Proof. By Theorem 4.1, we know that $aw \in \mathscr{A}^g$ and $a_{p,q}^{(2,l)} = w(aw)^\#$. Let $f = aa_{p,q}^{(2,l)}$, Then $f \in \mathscr{A}^\bullet$ and $f = aw(aw)^\#$. Also from the proof of Theorem 4.1, we see that aw, $(aw)^\# \in f\mathscr{A}f$ and aw is invertible in $f\mathscr{A}f$ with $(aw|_{f\mathscr{A}f})^{-1} = (aw)^\#$. Consider the decomposition $\mathscr{A} = f\mathscr{A}f \oplus (1-f)\mathscr{A}(1-f)$. Then we can write $\lambda 1 + aw$ as the following matrix form

$$\lambda 1 + aw = \begin{array}{c} \begin{bmatrix} \lambda f + aw & \\ & \lambda(1-f) \end{bmatrix} \end{array} \begin{array}{c} f\mathscr{A}f \\ (1-f)\mathscr{A}(1-f) \end{array} \tag{4.3}$$

It is well-known that if $\lambda \notin \sigma(-aw) \cup \{0\}$, then $\lambda f + aw$ is invertible in $f\mathscr{A}f$. Thus, in the case, by Eq.(4.3) we have

$$(\lambda 1 + aw)^{-1} = \begin{bmatrix} (\lambda f + aw)^{-1} & \\ & \lambda^{-1}(1-f) \end{bmatrix} \begin{array}{c} f\mathscr{A}f \\ (1-f)\mathscr{A}(1-f) \end{array} \tag{4.4}$$

Since $K_r(w) = R_r(q)$, then from the proof of Theorem 4.1, we see that $K_r(w) = R_r(q) = K_r(aw(aw)^\#) = K_r(f) = R_r(1-f)$. So by Eq. (4.4) we have $w(\lambda 1 + aw)^{-1} = w(\lambda 1 + aw)^{-1} f$, where the inverse is taken in $f\mathscr{A}f$. Then we can compute in the following way

$$a_{p,q}^{(2,l)} = w(aw)^\# f = \lim_{\substack{\lambda\to 0\\ \lambda\notin\sigma(-aw)}} w(\lambda 1 + aw)^{-1} f = \lim_{\substack{\lambda\to 0\\ \lambda\notin\sigma(-aw)}} w(\lambda 1 + aw)^{-1}.$$

This completes the proof. $\square$

We need the following lemma in a Banach algebra.

Lemma 4.8 ([16, Proposition 1.4.17])**.** *Let $a \in \mathscr{A}$. Suppose that $Re(\lambda) < 0$ for every $\lambda \in \sigma(a)$. Then*

$$a^{-1} = \int_0^{+\infty} e^{at}dt \triangleq \lim_{x\to+\infty} \int_0^x e^{at}dt.$$

Theorem 4.9. *Let $a \in \mathscr{A}$ and p, $q \in \mathscr{A}^\bullet$ such that $a_{p,q}^{(2,l)}$ exists. Let $w \in \mathscr{A}$ such that $R_r(w) = R_r(p)$ and $K_r(w) = R_r(q)$. Suppose that* $\mathrm{Re}(\lambda) \geq 0$ *for every $\lambda \in \sigma(aw)$. Then*

$$a_{p,q}^{(2,l)} = \int_0^{+\infty} we^{-(aw)t}dt.$$

Proof. We follow the method of proof [16, Corollary 4.2.11], but by using Theorem 4.1 above. It follows from Theorem 4.1 that $aw \in \mathscr{A}^g$ and $a_{p,q}^{(2,l)} = w(aw)^\#$. Similar to Theorem 4.7, Put $g = aw(aw)^\#$. Then $g \in \mathscr{A}^\bullet$ and aw is invertible in $g\mathscr{A}g$ with $(aw|_{g\mathscr{A}g})^{-1} = (aw)^\#$. Obviously, from the proof of Theorem 4.1, we can write aw and $(aw)^\#$ as the following matrix forms, respectively.

$$aw = \begin{bmatrix} aw & \\ & 0 \end{bmatrix} \begin{array}{c} g\mathscr{A}g \\ (1-g)\mathscr{A}(1-g) \end{array}, \quad (aw)^\# = \begin{bmatrix} (aw)^{-1} & \\ & 0 \end{bmatrix} \begin{array}{c} g\mathscr{A}g \\ (1-g)\mathscr{A}(1-g) \end{array}.$$

Since $\mathrm{Re}(\lambda) \geq 0$ for every $\lambda \in \sigma(aw)$, then we have $\mathrm{Re}(\lambda) > 0$ for every $\lambda \in \sigma(aw|_{g\mathscr{A}g})$. Thus by Lemma 4.8, we have $(aw|_{g\mathscr{A}g})^{-1} = \int_0^{+\infty} e^{-(aw)t}dt$. Note that $K_r(w) = R_r(1-g)$. Thus, we have

$$a_{p,q}^{(2,l)} = w(aw)^{\#} = w\begin{bmatrix} (aw)^{-1} & \\ & 1-g \end{bmatrix} = \int_0^{+\infty} we^{-(aw)t}dt.$$

This completes the proof. $\square$

Let $a \in \mathscr{A}$. The element a^d is the generalized Drazin inverse, or Koliha–Drazin inverse of $a \in \mathscr{A}$ (see [7]), provided that the following hold:

$$(1^{\infty}) \quad a(1-a^d) \text{ is quasinilpotent}, \quad (2) \quad a^d a a^d = a^d, \quad (5) \quad aa^d = d^d a.$$

It is known that $a \in \mathscr{A}$ is generalized Drazin invertible if and only if 0 is not the point of accumulation of the spectrum of a (see [7, 16]).

In the case when $\mathscr{A}$ is a unital C^*–algebra, then the Moore–Penrose inverse of $a \in \mathscr{A}$ (see [8, 12]) is the unique $a^{\dagger} \in \mathscr{A}$ (in the case when it exists), such that the following hold:

$$(1) \quad aa^{\dagger}a = a, \quad (2) \quad a^{\dagger}aa^{\dagger} = a^{\dagger}, \quad (3) \quad (aa^{\dagger})^* = aa^{\dagger}, \quad (4) \quad (a^{\dagger}a)^* = a^{\dagger}a.$$

For an element $a \in \mathscr{A}$, if $a^{\dagger}$ exists, then a is called Moore–Penrose invertible. The set of all $a \in \mathscr{A}$ that possess the Moore–Penrose inverse will be denoted by $\mathscr{A}^{\dagger}$. It is well–known that for $a \in \mathscr{A}$, $a^{\dagger}$ exists if and only if a is inner regular (see [6, 9]). The following corollary shows that for any $a \in \mathscr{A}$, $a^{\dagger}$, a^d and $a^{\#}$, if they exist, are all the special cases of $a_{p,q}^{(2)}$, in this case, we have $a_{p,q}^{(2)} = a_{p,q}^{(2,l)}$.

Corollary 4.10. *Let $\mathscr{A}$ be a unital C^*–algebra and $a \in \mathscr{A}$.*

(1) *If $a \in \mathscr{A}^d$ and $p = 1-q = 1-a^{\pi} = 1-aa^d = 1-a^d a$ is the spectral idempotent of a, then $a^d = a_{1-p,p}^{(2)} = a_{q,1-q}^{(2,l)} = a_{q,1-q}^{(2)}$.*

(2) *If $a \in \mathscr{A}^{\dagger}$, $p = a^{\dagger}a$, $q = 1 - aa^{\dagger}$, then $a^{\dagger} = a_{p,q}^{(2)} = a_{p,q}^{(2,l)}$.*

Proof. It is routine to check these by the definitions of $a^{\dagger}$, a^d, $a_{p,q}^{(2)}$ and $a_{p,q}^{(2,l)}$. $\square$

In this paper, we give some characterizations and representations for the various different generalized inverses with prescribed idempotents in Banach algebras. Obviously, most of our results can be proved in a ring. But in our forthcoming paper, we will use the main results in this paper to discuss the perturbation analysis of the generalized inverses $a_{p,q}^{(2)}$ and $a_{p,q}^{(2,l)}$ in a Banach algebra.

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