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Some causal connections between stochastic dynamic systems

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Abstract. In this paper we consider a problem that follows directly from realization problem: how to find Markovian representations , even minimall, for a given family of Hilbert spaces, understood as outputs of a stochastic dynamic system S_1, provided it is in a certain causality relationship with another family of Hilbert spaces , i. e. with some informations about states of a stochastic dynamic system S_2.

This paper is continuation of the papers Gill and Petrović [7] and Petrović [16, 17].

1. Introduction

After this introduction, in Section 2, we present different concepts of causality between flows of information that are represented by families of Hilbert spaces. Then we give a generalization of a causality relationship "**G** is a cause of **E** within **H**" which (in terms of σ-algebras) was first given in [14] and which is based on Granger's definition of causality (see [8]).

The study of Granger-causality has been mainly preoccupied with time series. We, however, concentrate on continuous time processes. Many of systems to which it is natural to apply tests of causality, take place in continuous time. For example, this is generally the case within physics and within economy. Causality concepts expressed in terms of orthogonality in Hilbert spaces of square integrable random variables was studied by Hosoya [12], Gill and Petrović [7].

In Section 3, we relate concepts of causality to the stochastic realization problem. The approach adopted in this paper is that of [13]. However, since our results do not depend on probability distribution, we deal with arbitrary Hilbert spaces instead of those generated by Gaussian processes.

It is clear that all results from this paper can be extended on the σ-algebras generated by finite dimensional Gaussian random variables. But, in the case that σ-algebras are arbitrary, the extensions of the proofs from this paper is nontrivial because one can not take an orthogonal complement with respect to a σ-algebra as one can with respect to subspaces in Hilbert space.

2. Preliminaries and notations

Let $\mathbf{F} = (F_t)$, $t \in \mathbf{R}$ be a family of Hilbert spaces. We shall think about F_t as a basis for approximation an information available at time t, or as a basis for approximation current information. *Total information* $F_{<\infty}$ carried by $\mathbf{F}$ is defined by $F_{<\infty} = \vee_{t \in R} F_t$, while *past and future information* of $\mathbf{F}$ at t is defined as $F_{\leq t} = \vee_{s \leq t} F_s$

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and $F_{\geq t} = \vee_{s\geq t} F_s$, respectively. It is to be understood that $F_{<t} = \vee_{s<t} F_s$ and $F_{>t} = \vee_{s>t} F_s$ do not have to coincide with $F_{\leq t}$ and $F_{\geq t}$ respectively; $F_{<t}$ and $F_{>t}$ are sometimes referred to as the *real past* and *real future* of $\mathbf{F}$ at t. Analogous notation will be used for families $\mathbf{H} = (H_t)$, $\mathbf{G} = (G_t)$ and $\mathbf{E} = (E_t)$.

If F_1 and F_2 are arbitrary subspaces of a Hilbert space $\mathcal{H}$ then $P(F_1|F_2)$ will denote the orthogonal projection of F_1 onto F_2 and $F_1 \ominus F_2$ will denote a Hilbert space generated by all elements $x - P(x|F_2)$, where $x \in F_1$. If $F_2 \subseteq F_1$, then $F_1 \ominus F_2$ coincides with $F_1 \cap F_2^{\perp}$, where $F_2^{\perp}$ is the orthogonal complement of F_2 in $\mathcal{H}$.

Possibly the weakest form of causality can be introduced in the following way.

Definition 2.1. It is said that $\mathbf{H}$ is *submitted* to $\mathbf{G}$ (and written as $\mathbf{H} \subseteq \mathbf{G}$) if $H_{\leq t} \subseteq G_{\leq t}$ for each t.

It will be said that families $\mathbf{H}$ and $\mathbf{G}$ are *equivalent* (and written as $\mathbf{H} = \mathbf{G}$) if $\mathbf{H} \subseteq \mathbf{G}$ and $\mathbf{G} \subseteq \mathbf{H}$.

Definition 2.2. It is said that $\mathbf{H}$ is *strictly submitted* to $\mathbf{G}$ (and written as $\mathbf{H} \leq \mathbf{G}$) if $H_t \subseteq G_t$ for each t.

It is easy to see that strict submission implies submission and that converse does not hold.

The notion of minimality of families of Hilbert spaces is specified in the following definition.

Definition 2.3. It will be said that $\mathbf{F}$ is a *minimal* (respectively, *strictly minimal*) family having a certain property if there is no family $\mathbf{F}^*$ having the same property which is submitted (respectively, strictly submitted) to $\mathbf{F}$.

It will be said that $\mathbf{F}$ is a (respectively, *strictly maximal*) family having a certain property if there is no family $\mathbf{F}^*$ having the same property such that family $\mathbf{F}$ is submitted (respectively, strictly submitted) to $\mathbf{F}^*$.

It should be understood that a minimal (respectively, strictly minimal) and maximal (respectively, strictly maximal) family having a certain property are not necessarily unique.

Definition 2.4. (compare with [26], with conditional independence from [24] and [3, 4]) If F_1, F_2 and F are arbitrary Hilbert spaces, then it is said that F is *splitting* for F_1 and F_2 or that F_1 and F_2 are conditionally orthogonal given F (and written as $F_1 \perp F_2|F$) if $F_1 \ominus F \perp F_2 \ominus F$.

When F is trivial, i. e. $F = \{0\}$, this reduces to the usual orthogonality $F_1 \perp F_2$.

The following result gives an alternative way of defining splitting.

Lemma 2.1. (see [7] and [24]) *$F_1 \perp F_2|F$ if and only if $P(F_i|F_j \vee F) \subseteq F$, for $i, j = 1, 2$, $i \neq j$.*

The following results will be used later (for the proof see the given reference).

Theorem 2.1. ([13]) *The space F is minimal one such that $F_1 \perp F_2|F$ if and only if $F = P(F_1|S)$ for some space S such that $F_2 \subseteq S \subseteq (F_2 \vee P(F_2|F_1)) \oplus (F_1 \vee F_2)^{\perp}$.*

Corollary 2.1.1. ([13]) The space $F \subseteq F_1 \vee F_2$ is a minimal one such that $F_1 \perp F_2|F$ if and only if $F = P(F_1|S)$ for some space S such that $F_2 \subseteq S \subseteq F_2 \vee P(F_2|F_1)$.

In this paper the following definition of markovian property will be used.

Definition 2.5. (compare with [26]) Family $\mathbf{G}$ will be called *Markovian* if $P(G_{\geq t}|G_{\leq t}) = G_t$ for each t.

Now we give a definition of a stochastic dynamic system in terms of Hilbert spaces. The characterizing property is the condition that past informations of outputs and states and future informations of outputs and states are conditionally orthogonal given the current state.

Definition 2.6. (compare with [13] and [24]) A *stochastic dynamic system* (s.d.s.) is a set of two families: $\mathbf{H}$ (outputs) and $\mathbf{G}$ (states), that satisfy the condition

$$H_{<t} \vee G_{<t} \perp H_{>t} \vee G_{>t}|G_t. \tag{1}$$

For given family of outputs $\mathbf{H}$, any family $\mathbf{G}$ satisfying (1) is called a *realization* of a s.d.s. with those outputs.

It is clear that realization of a s.d.s. is Markovian.

The next intuitively justifiable notion of causality has been proposed by Petrović [15].

Definition 2.7. [15] It is said that $\mathbf{G}$ is a *cause of* $\mathbf{E}$ within $\mathbf{H}$ (and written as $\mathbf{E} \not< \mathbf{G}; \mathbf{H}$) if $E_{<\infty} \subseteq H_{\leq\infty}$, $\mathbf{G} \subseteq \mathbf{H}$ and $E_{<\infty} \perp H_{\leq t}|G_{\leq t}$ for each t.

Intuitively, $\mathbf{E} \,|\!\!\!< \mathbf{G}; \mathbf{H}$ means that, for arbitrary t, information about $E_{<\infty}$ provided by $H_{\leq t}$ is not "bigger" than that provided by $G_{\leq t}$. The meaning of this interpretation is specified in the next result.

Lemma 2.2. (see [7])$\mathbf{E} \,|\!\!\!< \mathbf{G}; \mathbf{H}$ *if and only if* $E_{<\infty} \subseteq H_{<\infty}$, $\mathbf{G} \subseteq \mathbf{H}$ *and* $P(E_{<\infty}|H_{\leq t}) = P(E_{<\infty}|G_{\leq t})$ *for each* t.

A definition, analogous to Definition 2.7, formulated in terms of σ-algebras, was first given in [14]; however, a strict Hilbert space version of the definition in [14] contains also the condition $\mathbf{E} \subseteq \mathbf{H}$ (instead of $E_{<\infty} \subseteq H_{<\infty}$) which does not have an intuitive justification.

If $\mathbf{G}$ and $\mathbf{H}$ are such that $\mathbf{G} \,|\!\!\!< \mathbf{G}; \mathbf{H}$, we shall say that $\mathbf{G}$ is its *own cause* within $\mathbf{H}$ (compare with Mykland). It should be mentioned that the notion of *subordination* (as introduced in [25]) is equivalent to the notion of being one's own cause, as defined here.

If $\mathbf{G}$ and $\mathbf{H}$ are such that $\mathbf{G} \,|\!\!\!< \mathbf{G}; \mathbf{G} \vee \mathbf{H}$ (where $\mathbf{G} \vee \mathbf{H}$ is a family determined by $(G \vee H)_t = G_t \vee H_t$), we shall say that $\mathbf{H}$ *does not cause* $\mathbf{G}$. It is clear that the interpretation of Granger-causality is now that $\mathbf{H}$ does not cause $\mathbf{G}$ if $\mathbf{G} \,|\!\!\!< \mathbf{G}; \mathbf{G} \vee \mathbf{H}$ (see [14]). Without difficulty, it can be shown that this term and the term "$\mathbf{H}$ *does not anticipate* $\mathbf{G}$" (as introduced in [26]) are identical.

The analog of Definition 2.7 in terms of σ-algebras is considered in recent papers (see [20], [21], [23] and [27]). Specially, motivated with [6] and recent studies of stochastic systems with memory, the new concept of causality for continuous time stochastic processes which deal with finite horizon of the past in continuous time is given in [21].

Also, having in mind classification of causality concepts given in [5], this analog definition lies in the strong-global group.

Definition 2.7 can be extended from fixed times to stopping times. So, in [20], characterization of causality using σ-fields associated to stopping times is given.

We shall give some properties of causality relationship from Definition 2.7 which will be needed later.

From the following result it follows that relationship "being one's own cause" is the transitive relationship.

Lemma 2.3. (compare with [14]) *From* $\mathbf{G} \,|\!\!\!< \mathbf{G}; \mathbf{H}$ *and* $\mathbf{H} \,|\!\!\!< \mathbf{H}; \mathbf{E}$ *it follows that* $\mathbf{G} \,|\!\!\!< \mathbf{G}; \mathbf{E}$.

Lemma 2.4. ([7]) *If* $G_{<\infty} \subseteq H_{<\infty}$ *and if* $\mathbf{H} \,|\!\!\!< \mathbf{H}\,; \mathbf{E}$ *holds, it follows that* $\mathbf{G} \,|\!\!\!< \mathbf{H}\,; \mathbf{E}$ *holds.*

Now we give some examples to illustrate the notions from this part.

Definition 2.8. It will be said that second order stochastic processes are in a certain relationship if and only if the Hilbert spaces they generate are in this relationship.

Example 2.1. Let $X(t) = \sum_{k=1}^{n} \int_{-\infty}^{t} g_n(t,u)\, dZ_n(u),\ t \in [0,1]$ be a proper canonical (or Hida–Cramer) representation of the stochastic process $X(t),\ t \in [0,1]$. Any process $Z_n(t),\ n = \overline{1,N}$, is its own cause within $X(t)$, i. e. $\mathbf{F^{Z_n}} \,|\!\!\!< \mathbf{F^{Z_n}}; \mathbf{F^X}$ holds for any $n = \overline{1,N}$. If we define the process $Y(t)$ as a non–anticipative transformation of $Z_n(t)$, i. e.

$$Y(t) = \int_0^t h(t,u) Z_n(u)\, du\,,\ \ t \in [0,1],$$

it is easy to see that Z_n is a cause of Y within X, i. e. that $\mathbf{F^Y} \,|\!\!\!< \mathbf{F^{Z_n}}; \mathbf{F^X}$ holds. □

Example 2.2. Let $W(t)$ be a Wiener process defined on $[0,1]$, and let

$$X(t) = W(t), \quad Y(t) = W(t^2),\ \ 0 \leq t \leq 1\,.$$

It is clear that the equality $F^Y_{<1} = F^X_{<1}$ holds, and for any $t < 1$ we have

$$F^Y_{\leq t} = \overline{\mathcal{L}}\{\, W(u^2),\ u \leq t\,\} = \overline{\mathcal{L}}\{\, W(u),\ u \leq t^2\,\} = F^X_{\leq t^2} \subseteq F^X_{\leq t}$$

which means that $\mathbf{F^Y}$ is submitted to $\mathbf{F^X}$.

Let us prove that Y is not its own cause within X, i. e. that $\mathbf{F^Y} \models \mathbf{F^Y}; \mathbf{F^X}$ does not hold. If $x \in F^Y_{<1} \ominus F^Y_{\le t}$, it is easy to see that, because of $F^Y_{\le t} = F^X_{\le t^2}$, element x can have the form

$$x = \int_{t^2}^{1} f(u)\, dW(u)$$

where f is a function from $L_2(dt)$, not identically equal to zero on $(t^2, 1)$. But, such x is not orthogonal to $F^X_{\le t}$, since it is not orthogonal, for example, to the element $\int_{t^2}^{t} f(u)\, dW(u)$, which belongs to $F^X_{\le t}$. Thus we have proved that $F^Y_{<1} \ominus F^Y_{\le t} \perp F^X_{\le t}$ does not hold, i. e. $\mathbf{F^Y} \models \mathbf{F^Y}; \mathbf{F^X}$ does not hold. □

Example 2.3. Let $Z(t),\ 0 \le t \le T$ be a (t, ω)–measurable signal process such that $\int_0^T E|Z(t)|^2\, dt < \infty$ and let

$$Y(t) = \int_0^t Z(s)\, ds + W(t),\ \ 0 \le t \le T,$$

be the observation process where $W(t)$ is a Wiener process such that $W(t) - W(s)$ is orthogonal on $F^W_{\le s} \vee F^Z_{\le s}$ for $0 \le s \le t \le T$. Then $Z(t)$ does not cause $W(t)$, i. e. $\mathbf{F^W} \models \mathbf{F^W}; \mathbf{F^{W \vee Z}}$ holds.

If we suppose that $F^Y_{<T} \subseteq F^W_{<T}$, according to Lemma 2.2 we have that $\mathbf{F^W}$ is a cause of $\mathbf{F^Y}$ within $\mathbf{F^{W \vee Z}}$, i. e. $\mathbf{F^Y} \models \mathbf{F^W}; \mathbf{F^{W \vee Z}}$ holds.

If $\mathbf{F^W} \subseteq \mathbf{F^Y}$, then $\mathbf{F^W} \models \mathbf{F^W}; \mathbf{F^Y}$ and $\mathbf{F^W} \models \mathbf{F^Y}; \mathbf{F^{W \vee Z}}$ hold. □

Remark 1. If stochastic process $Y(t),\ t \in R$ is a realization of a stochastic dynamic system with outputs $X(t)$, then there exists a stochastic process $Z(t)$ with orthogonal increments which is a realization of the same system. Stochastic process $Z(t)$ is not uniquely determined, but its spectral type is uniquely determined. This follows from the fact that realization of a stochastic dynamic system is Markovian, i. e. process with multiplicity one, so process $Y(t)$ is equivalent (in the sense that $F^Y_{\le t} = F^Z_{\le t},\ t \in \mathbf{R}$) to some process with orthogonal increments.

Remark 2. The condition of Granger causality is actually a condition of transitivity largely used in sequential analysis (in statistics), see [2] and [9].

Remark 3. Some special cases of given causality concept links Granger–causality with *adapted distribution*. The consequence of $\mathbf{F}^{X,Z} \models \mathbf{F}^{X,Z}\ ;\ \mathbf{F}\ ;\ P$ is

$$\forall A \in \mathcal{F}^{X,Z}_{<\infty}\ \ P(A \mid \mathcal{F}^{X,Z}_t) = P(A \mid \mathcal{F}_t),$$

which links Granger–causality with the concept of adapted distribution which have been studied by Aldous [1], Hoover [10] and Hoover and Keisler [11]. Some results are given in [19].

The given causality concept is shown to be equivalent to a generalization of the notion of *weak uniqueness* for weak solutions of stochastic differential equations (see [18, 22]).

In [23] it is shown that the given causality concept is closely connected to *extremality of measures* and ımartingale problem. Also, in [27] the given concept of causality is related to the *orthogonality* of martingales and local martingales. This connection is considered for the stopped local martingales, too.

3. Causality and Stochastic Dynamic Systems

3.1. Explanation of the considered problems

Suppose that a stochastic dynamic system S_1 causes, in a certain sense, changes of another stochastic dynamic system S_2. It is natural to assume that outputs $\mathbf{H}$ of system S_1 can be registered and that some information $\mathbf{E}$ about the states (or perhaps states themselves) of system S_2 is given. Results that we shall prove will tell us under which conditions concerning the relationships between $\mathbf{H}$ and $\mathbf{E}$ it is possible to find states $\mathbf{G}$ (i.e. Markovian representations) of system S_1 having certain causality relationship in the sense of Definition 2.7 with $\mathbf{H}$ and $\mathbf{E}$. More precisely, the following cases will be considered:

1° available information about s.d.s. S_2 are a cause of states of a s.d.s. S_1 within outputs of a s.d.s. S_1;

2° outputs of a s.d.s. S_1 are cause of states of the same system within available information about s.d.s. S_2;

3° states of a s.d.s. S_1 are a cause of the available information about s.d.s. S_2 within outputs of a s.d.s. S_1,

4° states of a s.d.s. S_1 are a cause of outputs of the same system within available information about s.d.s. S_2;

5° the available information about S_2 is a cause of outputs of S_1 within states of S_1;

6° outputs of a s.d.s. S_1 are cause of the available information about S_2 within states of a s.d.s. S_1.

We consider different kinds of causality between families **G**, **H** and **E**, while **G** and **H** are in the same relationship, that is, **G** is a realization of an s.d.s. with outputs **H** in all cases.

In all cases 1° - 6° it is of interest to find minimal and maximal realizations that satisfy given conditions. We can see that, in some cases, family of extremal realizations is trivial, so as that extremal realizations is unique.

In the 1° we ask for realizations **G** such that $\mathbf{G} \not\leqslant \mathbf{E}; \mathbf{H}$ holds, and in the case 2° realization **G** such that $\mathbf{G} \not\leqslant \mathbf{H}; \mathbf{E}$ holds. Minimal realization **G** such that $\mathbf{G} \not\leqslant \mathbf{E}; \mathbf{H}$ or $\mathbf{G} \not\leqslant \mathbf{H}; \mathbf{E}$ hold, is defined by $G_t = \{0\}$ for each t, but this family is not a realization of a s.d.s. $\mathbf{S}_1$ with outputs **H** (except in the case when $H_{<t} \perp H_{>t}$ for each t). For these cases the given problem is only partially solved.

For the case 3°, we define some minimal realizations **G** (of a s.d.s. with given outputs **H**) such that $\mathbf{E} \not\leqslant \mathbf{G}; \mathbf{H}$ holds. It is easy to see that maximal families **G** for which $\mathbf{E} \not\leqslant \mathbf{G}; \mathbf{H}$ holds are all families such that $G_{\leq t} = H_{\leq t}$ for each t. One of these families is defined by $G_t = H_{\leq t}$ and it is strictly maximal realization of a s.d.s. with outputs **H** such that $\mathbf{E} \not\leqslant \mathbf{G}; \mathbf{H}$ holds.

For the case 4°, we define some minimal realizations **G** (of a s.d.s. with outputs **H**) such that $\mathbf{H} \not\leqslant \mathbf{G}; \mathbf{E}$ holds. Maximal families **G** such that $\mathbf{H} \not\leqslant \mathbf{G}; \mathbf{E}$ holds are families defined by $G_{\leq t} = E_{\leq t}$ for each t. If $\mathbf{H} \subseteq \mathbf{E}$, family defined by $G_t = E_{\leq t}$, $t \in R$ is strictly maximal realization of a s.d.s. with outputs **H** such that $\mathbf{H} \not\leqslant \mathbf{G}; \mathbf{E}$ holds.

For the cases 5° and 6° we define minimal and strictly maximal realizations such that $\mathbf{H} |< \mathbf{E}; \mathbf{G}$, respectively $\mathbf{E} \not\leqslant \mathbf{H}; \mathbf{G}$ hold. In case 5° we ask for realizations **G** such that $\mathbf{H} \not\leqslant \mathbf{E}; \mathbf{G}$ holds. Minimal families **G** such that $\mathbf{H} |< \mathbf{E}; \mathbf{G}$ holds are all families for which $G_{\leq t} = E_{\leq t}$ for each t. If $\mathbf{H} \subseteq \mathbf{E}$, family defined by $G_t = E_{\leq t}$, $t \in R$ is strictly maximal realization of a s.d.s. with outputs **H** such that $\mathbf{H} \not\leqslant \mathbf{E}; \mathbf{G}$ holds. In case 6° we ask for realizations **G** of a s.d.s. $\mathbf{S}_1$ such that $\mathbf{E} \not\leqslant \mathbf{H}; \mathbf{G}$ holds. Minimal families **G** such that $\mathbf{E} \not\leqslant \mathbf{H}; \mathbf{G}$ holds are all families for which $G_{\leq t} = H_{\leq t}$ for each t. One of these families, defined by $G_t = H_{\leq t}$, $t \in R$ is strictly minimal realization of a s.d.s. with outputs **H** such that $\mathbf{E} \not\leqslant \mathbf{H}; \mathbf{G}$ holds.

In cases 1°, 2°, 3° and 4°, we ask for realizations **G** such that $\mathbf{G} \subseteq \mathbf{E}$, or $\mathbf{G} \subseteq \mathbf{H}$ that is, the given families **E** and **H** are a natural "framework" in which we find realizations **G** of an s.d.s. S_1. However, in the cases 5° and 6°, where $\mathbf{E} \subseteq \mathbf{G}$, respectively, $\mathbf{H} \subseteq \mathbf{G}$ the family **E** and **H** are submitted to unknown family **G**, so that we will assume that all considered families of Hilbert spaces are submitted to some given "framework" family **F** of Hilbert spaces.

3.2. *Main results*

This paper is continuation of the papers [7, 16, 17]. In these papers cases 1°, 3° and 4° are considered. In the remaining part of this paper we consider cases 2°, 5° and 6°.

The first two theorems deals with case 2°.

Theorem 3.1. (compare with [7]) *If* **G** *is its own cause within* **H**, *then* **G** *is a realization of a s.d.s. with outputs* **H** *if and only if* **G** *is Markovian and* $H_{<t} \perp H_{>t} | G_t$ *for each* t.

This result follows from Theorem 2.1 and Corollary 2.1.1. □

It **H** is its own cause within **E**, then Theorem 3.1 gives one solution for the case 2°, i.e. gives realization **G** (of a s.d.s. with outputs **H**) such that $\mathbf{G} \not\leqslant \mathbf{H}; \mathbf{E}$ holds.

The following result gives strictly minimal realization (of a s.d.s. with outputs **H**) such that $\mathbf{G} \models \mathbf{H}; \mathbf{E}$ holds.

Theorem 3.2. *If* **H** *is its own cause within* **E**, *then* **H** *is a cause of realization* **G** *(of a s.d.s. with outputs* **H***) defined by (4) within* **E**.

The proof follows from Lemma 2.4. □

If families **G**, **H** and **E** are such that e $\mathbf{E} \models \mathbf{E}; \mathbf{H}$. (respectively, $\mathbf{H} \models \mathbf{H}; \mathbf{E}$) and $\mathbf{G} \subseteq \mathbf{E}$ (i.e. $\mathbf{G} \subseteq \mathbf{H}$), then the following holds

$$P(G_{\geq t} \,|\, H_{\leq t}) = P(G_{\geq t} \,|\, E_{\leq t}), \ \ t \in R.$$

That means, if we want to predict realization **G**, then (under conditions given in the above results) we can use any of families **H** or **E**.

In the remaining cases 5° and 6° we will assume that all considered families of Hilbert spaces are submitted to some given "framework" family **F** of Hilbert spaces.

The solutions of these problems follow from the next more general result which gives conditions under which it is possible to find minimal realizations of a s.d.s. S_1, that is a cause for **H** within a family $\mathbf{E}^1 = \mathbf{H} \vee \mathbf{E}$.

The following theorem considers the problem of determining the possible states **G** (of an s.d.s. with outputs **H**) such that the family $\mathbf{E}^1 = \mathbf{H} \vee \mathbf{E}$ is a cause of outputs **H** within **G**.

Theorem 3.3. *(i) Each Markovian family* **G** *such that* $\mathbf{H} \vee \mathbf{E} \models \mathbf{H} \vee \mathbf{E}; \mathbf{G}$ *and* $P(H_{>t}|G_{<t}) \subseteq G_t$ *for each t is a realization (of a s. d. s. with outputs* **H***) and the family* $\mathbf{H} \vee \mathbf{E}$ *is a cause of* **H** *within* **G**.

(ii) If **J** *is a Markovian family such that* $\mathbf{H} \vee \mathbf{E} \models \mathbf{H} \vee \mathbf{E}; \mathbf{J}$ *and* $P(J_{<t}|H_{\leq t} \vee E_{\leq t}) \perp H_{>t}|P(J_t|H_{\leq t} \vee E_{\leq t})$ *for each t, then family* **G**, *defined by*

$$G_t = P(J_t|H_{\leq t} \vee E_{\leq t}), \ \ t \in R\,, \tag{2}$$

is minimal among the realizations (of a s. d. s. with outputs **H***) such that the family* $\mathbf{H} \vee \mathbf{E}$ *is a cause of* **H** *within* **G**.

(iii) If families **E** *and* **H** *are submitted to some given "framework" family* **F** *and if* $\mathbf{H} \vee \mathbf{E} \models \mathbf{H} \vee \mathbf{E}; \mathbf{F}$ *holds, then the family* **G**, *defined by*

$$G_t = F_{\leq t}, \ \ t \in R\,, \tag{3}$$

is strictly maximal among the realizations (of a s. d. s. with outputs **H***) such that* $\mathbf{H} \vee \mathbf{E}$ *is a cause of* **H** *within* **G**.

Proof. (i) According to Lemma 2.1, the assumption $G_{<t} \perp H_{>t}|G_t$ is equivalent to $P(H_{>t}|G_{\leq t}) \subseteq G_t$. From that and the assumption that **G** is Markovian family, we get

$$G_t = P(G_{\geq t}|G_{\leq t}) = P(G_{\geq t} \vee H_{>t}|G_{\leq t})$$

which is equivalent to $G_{\leq t} \perp G_{\geq t} \vee H_{>t}|G_t$. However, since $H_{<t} \subseteq G_{\leq t}$ (which is an obvious consequence of $\mathbf{H} \vee \mathbf{E} \models \mathbf{H} \vee \mathbf{E}; \mathbf{G}$) the last relation means that **G** is a realization of a s. d. s. with outputs **H**. According to Lemma 2.4, from $\mathbf{H} \vee \mathbf{E} \models \mathbf{H} \vee \mathbf{E}; \mathbf{G}$ it follows that $\mathbf{H} \models \mathbf{H} \vee \mathbf{E}; \mathbf{G}$ holds.

(ii) From (2) it follows that $G_{\leq t} = H_{\leq t} \vee E_{\leq t}$ and immediately we get $\mathbf{H} |< \mathbf{H} \vee \mathbf{E}; \mathbf{G}$ and $\mathbf{G} |< \mathbf{G}; \mathbf{J}$. According to Definition 2.7, it is clear that the family **G**, defined by (2), is a minimal family such that $\mathbf{H} |< \mathbf{H} \vee \mathbf{E}; \mathbf{G}$. From the assumptions that $\mathbf{H} \vee \mathbf{E} |< \mathbf{H} \vee \mathbf{E}; \mathbf{J}$ and the fact that **J** is Markovian we get $P(G_{\geq t}|G_{\leq t}) = P(J_{\geq t}|H_{\leq t} \vee E_{\leq t}) = P(P(J_{\geq t}|J_{\leq t})|H_{\leq t} \vee E_{\leq t}) = P(J_t|H_{\leq t} \vee E_{\leq t}) = G_t$ which means that **G** is Markovian. Now, according to part (i) of this theorem, it follows that the family **G**, defined by (2) is a realization (of a s. d. s. with outputs **H**) such that $\mathbf{H} \models \mathbf{H} \vee \mathbf{E}; \mathbf{G}$.

(iii) Since $G_{\leq t} = F_{\leq t}$, the assumption $\mathbf{H} \vee \mathbf{E} \models \mathbf{H} \vee \mathbf{E}; \mathbf{F}$, is equivalent to $\mathbf{H} \vee \mathbf{E} \models \mathbf{H} \vee \mathbf{E}; \mathbf{G}$, so that according to Lemma 2.4, it follows $\mathbf{H} \models \mathbf{H} \vee \mathbf{E}; \mathbf{G}$. From $G_t = G_{\leq t}$, and $\mathbf{H} \subseteq \mathbf{G}$ immediately follows that **G** is a realization of a s. d. s. with outputs **H**. From the fact that **F** is a "framework" family (i.e., $\mathbf{G} \subseteq \mathbf{F}$) it is clear that **G** is a strictly maximal realization with given properties. □

It is easy to see that for given outputs **H** of a s.d.s. S_1 and information **E** about a s.d.s. S_2, the family **G**, defined by (2), is not an unique minimal realization (of a s. d. s. S_1) such that $\mathbf{H} \models \mathbf{H} \vee \mathbf{E}; \mathbf{G}$. For each family

$\mathbf{J}^* \subseteq \mathbf{F}$ which satisfies conditions from part (ii) of Theorem 3.3., with $G_t^* = P(J_t^*|H_{\le t} \vee E_{\le t}), t \in \mathbf{R}$ is defined a minimal realization (of a s. d. s. S_1) such that $\mathbf{H} \models \mathbf{H} \vee \mathbf{E}; \mathbf{G}^*$. All these minimal realizations have the past information equivalent to $H_{\le t} \vee E_{\le t}, t \in \mathbf{R}$, but their present information at t is different.

The next example shows that family $\mathbf{G}$ defined by (2) is not strictly minimal realization of a s. d. s. with outputs $\mathbf{H}$ such that $\mathbf{H} \models \mathbf{H} \vee \mathbf{E}; \mathbf{G}$.

Example 3.4. Let A and B be arbitrary Hilbert spaces and let $\mathbf{H} = (\mathbf{H_t})$, $\mathbf{E} = (\mathbf{E_t})$ and $\mathbf{J} = (\mathbf{J_t})$, $t \in \{1, 2, 3\}$ be defined by

$$\begin{array}{lll} H_1 = A, & H_2 = B, & H_3 = A \vee B, \\ E_1 = A, & E_2 = B, & E_3 = A, \\ J_1 = A, & J_2 = A \vee B, & J_3 = B. \end{array}$$

It is easy to see that $\mathbf{J}$ is Markovian, $\mathbf{H} \vee \mathbf{E} \models \mathbf{H} \vee \mathbf{E}; \mathbf{J}$ and $P(J_{<t}|H_{\le t} \vee E_{\le t}) \perp H_{>t}|P(J_t|H_{\le t} \vee E_{\le t})$ for each t. If the family $\mathbf{G}$ is defined by (2), then

$$G_1 = A, \qquad G_2 = A \vee B, \qquad G_3 = B.$$

According to part (ii) of the Theorem 3.3, $\mathbf{G}$ is a minimal realization (of a s. d. s. with outputs $\mathbf{H}$) and $\mathbf{H} \models \mathbf{H} \vee \mathbf{E}; \mathbf{G}$. However, family $\mathbf{G}^* = (\mathbf{G}_\mathbf{t}^*)$, $t \in \{1, 2, 3\}$, defined by

$$G_1^* = A, \qquad G_2^* = A \vee B, \qquad G_3^* = \{0\},$$

is another realization of the same s. d. s. and $\mathbf{H} \models \mathbf{H} \vee \mathbf{E}; \mathbf{G}^*$. Obviously, $\mathbf{G}^* \le \mathbf{G}$. □

The problem of determining a strictly minimal realization $\mathbf{G}$ (of a s.d.s. with outputs $\mathbf{H}$) such that $\mathbf{H} \models \mathbf{H} \vee \mathbf{E}; \mathbf{G}$. is still open. If it would be possible to find a strictly minimal family $\mathbf{J^m}$ between families $\mathbf{J}^* \subseteq \mathbf{F}$ that satisfy conditions from part (ii) of Theorem 3.3., this strictly minimal family $\mathbf{J^m}$ would define a strictly minimal family $\mathbf{G^m}$ (with (2)) among all families $\mathbf{G}$ of part (ii) of Theorem 3.3. It is clear that if there exists such strictly minimal family, it can not be necessarily unique, so that a strictly minimal realization with given properties is not necessarily unique.

Especially, if $\mathbf{H} \subseteq \mathbf{E}$, Theorem 3.3. gives realizations of a s.d.s. with outputs $\mathbf{H}$ such that the family $\mathbf{E}$ is a cause of outputs $\mathbf{H}$ within $\mathbf{G}$. More precisely, the next corollary to Theorem 3.3. gives a partial solution of the problem 5° formulated above.

Corollary 3.3.1. ([17]) (i) *Let $\mathbf{H}$ and $\mathbf{E}$ be such that $\mathbf{H} \subseteq \mathbf{E}$. Each Markovian family $\mathbf{G}$ such that $\mathbf{E} \models \mathbf{E}; \mathbf{G}$ and $G_{<t} \perp H_{>t}|G_t$ for each t is a realization (of a s. d. s. with outputs $\mathbf{H}$) and $\mathbf{E}$ is a cause of $\mathbf{H}$ within $\mathbf{G}$.*

(ii) Let $\mathbf{H}$, $\mathbf{E}$ and $\mathbf{J}$ be such that $\mathbf{H} \subseteq \mathbf{E}$, as well as $\mathbf{E} \models \mathbf{E}; \mathbf{J}$ and $P(J_{<t}|E_{\le t}) \perp H_{>t}|P(J_t|E_{\le t})$ for each t. If $\mathbf{J}$ is Markovian, then the family $\mathbf{G}$, defined by

$$G_t = P(J_t|E_{\le t}), \ \ t \in \mathbf{R},$$

is minimal among the realizations (of a s. d. s. with outputs $\mathbf{H}$) such that $\mathbf{E}$ is a cause of $\mathbf{H}$ within $\mathbf{G}$.

(iii) If $\mathbf{H} \subseteq \mathbf{E}$ and if given "framework" family $\mathbf{F}$ is such that $\mathbf{E} \models \mathbf{E}; \mathbf{F}$, then the family $\mathbf{G}$, defined by

$$G_t = F_{\le t}, \ \ t \in \mathbf{R},$$

is strictly maximal among the realizations (of a s. d. s. with outputs $\mathbf{H}$) such that $\mathbf{E}$ is a cause of $\mathbf{H}$ within $\mathbf{G}$.

Now we consider case 6° formulated above.

It is clear that realizations from Theorem 3.3. are such that i $\mathbf{E} \models \mathbf{E} \vee \mathbf{H}; \mathbf{G}$ holds. Especially, if $\mathbf{E} \subseteq \mathbf{H}$, then for realizations $\mathbf{G}$ holds $\mathbf{E} \models \mathbf{H}; \mathbf{G}$, i.e. in case $\mathbf{E} \subseteq \mathbf{H}$, Theorem 3.3 gives the solutions of case 6°.

The following result does not require that $\mathbf{E} \subseteq \mathbf{H}$, but, in that case we defined only minimal and strictly maximal realizations such that $\mathbf{E} \models \mathbf{H}; \mathbf{G}$ holds.

Theorem 3.4. *(i) Let families $\mathbf{E}$ and $\mathbf{H}$ are such that $E_{<\infty} \subseteq H_{<\infty}$. If $\mathbf{J}$ is markovian family such that $\mathbf{H} \models \mathbf{H}; \mathbf{J}$ holds and $P(J_{<t} \,|\, H_{\le t}) \perp H_{>t}|P(J_t \,|\, H_{\le t})$ for each t, then family $\mathbf{G}$, defined by*

$$G_t = P(J_t \,|\, H_{\le t}), \ \ t \in R, \tag{4}$$

is minimal realization (of a s.d.s. with outputs $\mathbf{H}$*) such that outputs* $\mathbf{H}$ *are a cause for family* $\mathbf{E}$ *within* $\mathbf{G}$.

(ii) Let families $\mathbf{E}$ *and* $\mathbf{H}$ *be such that* $E_{<\infty} \subseteq H_{<\infty}$ *holds. Fog given "framework" family* $\mathbf{F}$ *such that* $\mathbf{H} \mid\!\leqslant \mathbf{H}; \mathbf{F}$ *family* $\mathbf{G}$*, defined by*

$$G_t = F_{\leq t}, \quad t \in R \tag{5}$$

is strictly maximal realization (of a s. d. s. with outputs $\mathbf{H}$*) such that outputs* $\mathbf{H}$ *are a cause for family* $\mathbf{E}$ *within* $\mathbf{G}$.

Proof. (i) From the assumption $\mathbf{H} |< \mathbf{H}; \mathbf{J}$ it follows $G_t = P(J_t \,|\, H_{<\infty})$ and $G_{\leq t} = H_{\leq t}$, so that $\mathbf{E} |< \mathbf{H}; \mathbf{G}$ follows immediately. It is clear (according to Definition 2.7) that family $\mathbf{G}$, defined by (4), minimal family such that $\mathbf{E} |< \mathbf{H}; \mathbf{G}$ holds. From the assumptions $\mathbf{H} |< \mathbf{H}; \mathbf{J}$ and that family $\mathbf{J}$ is markovian, we gets $P(G_{\geq t} \,|\, G_{\leq t}) = P(J_{\geq t} \,|\, H_{\leq t}) = P(P(J_{\geq t} \,|\, J_{\leq t}) \,|\, H_{\leq t}) = P(J_t \,|\, H_{\leq t}) = G_t$ which means that family $\mathbf{G}$ is markovian. This fact, together with assumption $P(J_{<t} \,|\, H_{\leq t}) \perp H_{>t} | P(J_t \,|\, H_{\leq t})$ (or , equivalently, $G_{<t} \perp H_{>t} | G_t$), gives $G_{<t} \perp H_{>t} \vee G_{>t} | G_t$. Now, from $G_{\leq t} = H_{\leq t}$, $t \in R$, it follows that $\mathbf{G}$ is a realization of a s. d. s. with outputs $\mathbf{H}$.

(ii) From (5) it follows that $G_{\leq t} = F_{\leq t}$, $t \in R$, so from assumption $\mathbf{H} \mid\!\leqslant \mathbf{H}; \mathbf{F}$ we gets $\mathbf{H} \mid\!\leqslant \mathbf{H}; \mathbf{G}$. From this relation and assumption $E_{<\infty} \subseteq H_{<\infty}$, according to Lemma 2.1, it follows $\mathbf{E} \mid\!\leqslant \mathbf{H}; \mathbf{G}$. From $G_t = G_{\leq t}$, $t \in R$, and $\mathbf{H} \subseteq \mathbf{G}$ we get that $\mathbf{G}$ is a realization of a s. d. s. with outputs $\mathbf{H}$. □

It is clear that family $\mathbf{G}$, defined by (4), is not unique minimal realization (of a s. d. s. with outputs $\mathbf{H}$) such that $\mathbf{E} \mid\!\leqslant \mathbf{H}; \mathbf{G}$ holds.

The family $\mathbf{G}$, defined by (4) is not strictly minimal realization (of a s. d. s. with outputs $\mathbf{H}$) such that $\mathbf{E} \mid\!\leqslant \mathbf{H};$ holds. The problem of finding such that realization is still open.

Final remark. It is of an interest to find conditions for the existence of a realization with certain properties less restrictive than those obtained in this paper.

Also, the problems formulated here and in the papers [7, 16, 17] can be considered in the σ-algebraic approach when stochastic dynamic system is defined for σ-algebra families in terms of the conditional independence relation (see [13]).

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References

[1] D.J. Aldous, Weak Convergence and the General Theory of Processes, Unpublished manuscript.

[2] R.R. Bahadur Sufficiency and Statistical Decision Functions, Annals of Mathematical Statistics, Vol. 25, (1954),423–462.

[3] J. P. Florens, M. Mouchart, A Note on Noncausality, Econometrica, Vol.50, (1982), 583–591.

[4] J. P. Florens, M. Mouchart, A Linear Theory for Noncausality, Econometrica, Vol.53, (1985), 157–175.

[5] J. P. Florens, D. Fougères, Noncausality in Continuous Time, Econometrica, Vol. 64,(1996), 1195–1212.

[6] J.P. Florens, M. Mouchart, J.M. Rolin, Noncausality and Marginalization of Markov Process, Econometric Theory, Vol. 9, (1993), 241–262.

[7] J.B. Gill, Lj. Petrović, Causality and Stochastic Dynamic Systems, SIAM J. Appl. Math. **47** No 6, (1987), 1361-1366.

[8] C.W.J. Granger, Investigating Causal Relationship by Econometric Models and Cross Spectral Methods, Econometrica **37**, (1969), 424-438.

[9] W.J. Hall, R.A. Wijsman, J.K. Gosh, The Relationship between Sufficiency and Invariance with Applications in Sequential Analysis, Annals of Mathematical Statistics, Vol. 36, (1965), 575–614.

[10] D.N. Hoover, Synonymity, Generalized Martingales and Subfiltrations, Ann. Prob., **12**, (1984), 703–713

[11] D.N. Hoover, J.H. Keisler, Adapted Probability Distributions, Trans. Am.Math.Soc., **286**, (1984), 159–201

[12] Y. Hosoya, On the Granger Condition for Noncausality, Econometrica **45** (1977), 1735–1736.

[13] A. Lindquist, G. Picci, G. Ruckebush, On Minimal Splitting Subspaces and Markovian Representations, Math. Systems Theory **12**, (1979), 271-279.

[14] P.A. Mykland, Statistical Causality, University of Bergen, (1986).

[15] Lj. Petrović, Causality and Stochastic Realization Problem, Publ. Inst. Math. (Beograd), **45(59)**, (1989), 203-210.

[16] Lj. Petrović, Causality and Markovian Representations, Statistics and Probability Letters, 29, (1996), 223-227.

[17] Lj. Petrović, Causality and Realization Problem, International Journal of Mathematics and Mathematical Sciences, 3, (2005), 349-356.

[18] Lj. Petrović, S. Dimitrijević, Causality and Stochastic Differential Equations Driven by Fractional Brownian Motion, Facta Universitatis, (Niš) Ser. Mathematics and Informatics, (5), (2005), 113–122.

[19] Lj. Petrović, S. Dimitrijević, Czhechoslovak Mathematica Journal, ISSN 0011-4642 Springer Verlag, Vol. 61, No. 3, (2011), 827–843.
[20] Lj. Petrović, S. Dimitrijević, Invariance of statistical causality under convergence, Statistical and Probability Letters, Volume 81, Issue 9, (2011), 1445–1448.
[21] Lj. Petrović, S. Dimitrijević, Causality with finite horizon of the past in continuous time, Statistical and Probability Letters, Volume 82, Issue 7, (2012), 1219–1223.
[22] Lj. Petrović, D. Stanojević, Some Models of Causality and Weak Solutions of Stochastic Differential Equations with Driving Semimartingales, Facta Universitatis, (Niš), Ser. Mathematics and Informatics, (5), (2005), 103–112.
[23] Lj. Petrović, D. Stanojević, Statistical Causality, Extremal Measures and Weak Solutions of Stochastic Differential Equations with Driving Semimartingales, Journal of Mathematical Modelling and Algorithms, Vol.9, No.1, (2010), 113–128.
[24] C. van Putten, J.H. van Schuppen, On Stochastic Dynamic Systems, International Symposium on Mathematical Theory of Networks and Systems, Vol. 3 (Delft 1979), Western Periodical, North Hollywood, Calif., (1979), 350-356.
[25] Yu.A. Rozanov, Theory of Innovation Processes, Monographs in Probabolity Theory and Mathematical Statistics, Izdat. Nauka, Moscow, 1974.
[26] Yu.A. Rozanov, Innovation Processes, V. H. Winston and Sons, New York, 1977.
[27] D. Valjarević, Lj. Petrović, Statistical Causality and Orthogonality of Local Martingales, Statistical and Probability Letters, Volume 82, Issue 7, (2012), 1326–1330.