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Fixed and periodic point theorems for T-contractions on cone metric spaces

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Abstract. Recently, Filipović et al. [M. Filipović, L. Paunović, S. Radenović, M. Rajović, Remarks on "Cone metric spaces and fixed point theorems of T-Kannan and T-Chatterjea contractive mappings", Math. Comput. Modelling. 54 (2011) 1467-1472] proved several fixed and periodic point theorems for solid cones on cone metric spaces. In this paper several fixed and periodic point theorems for T-contraction of two maps on cone metric spaces with solid cone are proved. The results of this paper extend and generalize well-known comparable results in the literature.

1. Introduction and preliminaries

In 1922, Banach proved the following famous fixed point theorem [3]. Suppose that (X, d) is a complete metric space and a self-map T of X satisfies $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$ where $\lambda \in [0, 1)$; that is, T is a contractive mapping. Then T has a unique fixed point. Afterward, other people considered various definitions of contractive mappings and proved several fixed point theorems [4, 8, 11, 12, 17]. In 2007, Huang and Zhang [9] introduced cone metric space and proved some fixed point theorems. Several fixed and common fixed point results on cone metric spaces were introduced in [1, 15, 16, 18, 19].

Recently, Morales and Rajes [14] introduced T-Kannan and T-Chatterjea contractive mappings in cone metric spaces and proved some fixed point theorems. Later, Filipović et al. [6] defined T-Hardy-Rogers contraction in cone metric space and proved some fixed and periodic point theorems. In this work we prove several fixed and periodic point theorems for a T-contraction of two maps on cone metric spaces. Our results extend various comparable results of Abbas and Rhoades [2], Filipović et al. [6] and, Morales and Rajes [14].

We begin with some important definitions.

Definition 1.1. (*See [7, 9]*). *Let E be a real Banach space and P a subset of E. Then P is called a cone if and only if*
(a) *P is closed, non-empty and $P \neq \{\theta\}$;*
(b) *$a, b \in R, a, b \geq 0, x, y \in P$ implies $ax + by \in P$;*
(c) *if $x \in P$ and $-x \in P$, then $x = \theta$.*

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Given a cone $P \subset E$, a partial ordering $\preceq$ with respect to P is defined by
$x \preceq y \Longleftrightarrow y - x \in P$.

We shall write $x \prec y$ to mean $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is the interior of P). If $intP \neq \emptyset$, the cone P is called solid. A cone P is called normal if there exists a number $K > 0$ such that, for all $x, y \in E$,

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P.

Example 1.2. (*See [16]*).
(*i*) *Let* $E = C_{\mathbf{R}}[0, 1]$ *with the supremum norm and* $P = \{f \in E : f \geq 0\}$. *Then P is a normal cone with normal constant* $K = 1$.
(*ii*) *Let* $E = C^2_{\mathbf{R}}[0, 1]$ *with the norm* $\|f\| = \|f\|_\infty + \|f'\|_\infty$ *and consider the cone* $P = \{f \in E : f \geq 0\}$ *for every* $K \geq 1$. *Then P is a non-normal cone.*

Definition 1.3. (*See [9]*). *Let X be a nonempty set. Suppose that the mapping* $d : X \times X \to E$ *satisfies*
($d1$) $\theta \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$;
($d2$) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;
($d3$) $d(x, z) \preceq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.
Then, d is called a cone metric on X and (X, d) *is called a cone metric space.*

Example 1.4. (*See [9]*). *Let* $E = R^2$, $P = \{(x, y) \in E | x, y \geq 0\} \subset R^2$, $X = R$ *and* $d : X \times X \to E$ *is such that* $d(x, y) = (|x - y|, \alpha|x - y|)$, *where* $\alpha \geq 0$ *is a constant. Then* (X, d) *is a cone metric space.*

Definition 1.5. (*See [6]*). *Let* (X, d) *be a cone metric space,* $\{x_n\}$ *a sequence in X and* $x \in X$. *Then*
(*i*) $\{x_n\}$ *converges to x if, for every* $c \in E$ *with* $\theta \ll c$ *there exists an* $n_0 \in \mathbf{N}$ *such that* $d(x_n, x) \ll c$ *for all* $n > n_0$. *We denote this by* $\lim_{n\to\infty} d(x_n, x) = \theta$
(*ii*) $\{x_n\}$ *is called a Cauchy sequence if, for every* $c \in E$ *with* $\theta \ll c$ *there exists an* $n_0 \in \mathbf{N}$ *such that* $d(x_n, x_m) \ll c$ *for all* $m, n > n_0$. *We denote this by* $\lim_{n,m\to\infty} d(x_n, x_m) = \theta$.

The notation $\theta \ll c$ for $c \in intP$ of a positive cone is used by Krein and Rutman [13]. Also, a cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. In the sequel we shall always suppose that E is a real Banach space, P is a solid cone in E, and $\preceq$ is a partial ordering with respect to P.

Lemma 1.6. (*See [6]*). *Let* (X, d) *be a cone metric space over an ordered real Banach space E. Then the following properties are often used, particularly when dealing with cone metric spaces in which the cone need not be normal.*
(P_1) *If* $x \preceq y$ *and* $y \ll z$, *then* $x \ll z$.
(P_2) *If* $\theta \preceq x \ll c$ *for each* $c \in intP$, *then* $x = \theta$.
(P_3) *If* $x \preceq \lambda x$ *where* $x \in P$ *and* $0 \leq \lambda < 1$, *then* $x = \theta$.
(P_4) *Let* $x_n \to \theta$ *in E and* $\theta \ll c$. *Then there exists a positive integer* n_0 *such that* $x_n \ll c$ *for each* $n > n_0$.

Definition 1.7. (*See [6]*). *Let* (X, d) *be a cone metric space, P a solid cone and* $S : X \to X$. *Then*
(*i*) *S is said to be sequentially convergent if we have, for every sequence* $\{x_n\}$, *if* $\{Sx_n\}$ *is convergent, then* $\{x_n\}$ *also is convergent.*
(*ii*) *S is said to be subsequentially convergent if, for every sequence* $\{x_n\}$ *that* $\{Sx_n\}$ *is convergent,* $\{x_n\}$ *has a convergent subsequence.*
(*iii*) *S is said to be continuous if* $\lim_{n\to\infty} x_n = x$ *implies that* $\lim_{n\to\infty} Sx_n = Sx$, *for all* $\{x_n\}$ *in X.*

Definition 1.8. (*See [6]*). *Let* (X, d) *be a cone metric space and* $T, f : X \to X$ *be two mappings. A mapping f is said to be a T-Hardy-Rogers contraction, if there exist* $\alpha_i \geq 0$, $i = 1, \cdots, 5$ *with* $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ *such that for all* $x, y \in X$,

$$d(Tfx, Tfy) \preceq \alpha_1 d(Tx, Ty) + \alpha_2 d(Tx, Tfx) + \alpha_3 d(Ty, Tfy) + \alpha_4 d(Tx, Tfy) + \alpha_5 d(Ty, Tfx). \quad (1)$$

In Definition 1.8 *if one assumes that* $\alpha_1 = \alpha_4 = \alpha_5 = 0$ *and* $\alpha_2 = \alpha_3 \neq 0$ (*resp.* $\alpha_1 = \alpha_2 = \alpha_3 = 0$ *and* $\alpha_4 = \alpha_5 \neq 0$), *then one obtains a T-Kannan* (*resp. T-Chatterjea*) *contraction.*(*See [14].*)

2. Fixed point results

The following is the cone metric space version of a contractive condition of Ćirić for an ordinary metric space.

Definition 2.1. *Let (X, d) be a cone metric space. A mapping $f : X \to X$ is said to be a λ-generalized contraction if and only if for every $x, y \in X$, there exist nonnegative functions $q(x, y)$, $r(x, y)$, $s(x, y)$ and $t(x, y)$ such that*

$$\sup_{x,y\in X}\{q(x, y) + r(x, y) + s(x, y) + 2t(x, y)\} \le \lambda < 1$$

and

$$d(fx, fy) \le q(x, y)d(fx, fy) + r(x, y)d(x, fx) + s(x, y)d(y, fy) + 2t(x, y)[d(x, fy) + d(y, fx)]$$

holds for all $x, y \in X$.

Theorem 2.2. *Suppose that (X, d) is a complete cone metric space, P is a solid cone, and $T : X \to X$ is a continuous and one to one mapping. Moreover, let f and g be two mappings of X satisfying*

$$\begin{aligned} d(Tfx, Tgy) \le{}& q(x, y)d(Tx, Ty) + r(x, y)d(Tx, Tfx) + s(x, y)d(Ty, Tgy)] \\ &+ t(x, y)[d(Tx, Tgy) + d(Ty, Tfx)], \end{aligned} \tag{2}$$

for all $x, y \in X$, where q, r, s, and t are nonnegative functions satisfying

$$\sup_{x,y\in X}\{q(x, y) + r(x, y) + s(x, y) + 2t(x, y)\} \le \lambda < 1; \tag{3}$$

that is, f and g are T-contractions. Then
(1) There exists a $z_x \in X$ such that $\lim_{n\to\infty} Tfx_{2n} = \lim_{n\to\infty} Tgx_{2n+1} = z_x$.
(2) If T is subsequentially convergent, then $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ have a convergent subsequence.
(3) There exists a unique $w_x \in X$ such that $fw_x = gw_x = w_x$; that is, f and g have a unique common fixed point.
(4) If T is sequentially convergent, then the sequences $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to w_x.

Proof. Suppose that x_0 is an arbitrary point of X, and define $\{x_n\}$ by
$x_1 = fx_0$, $x_2 = gx_1$, $\cdots$, $x_{2n+1} = fx_{2n}$, $x_{2n+2} = gx_{2n+1}$ *for* $n = 0, 1, 2, \ldots.$
First we shall prove that $\{Tx_n\}$ is a Cauchy sequence. Applying the triangle inequality we get

$$\begin{aligned} d(Tx_{2n+1}, Tx_{2n+2}) &= d(Tfx_{2n}, Tgx_{2n+1}) \\ &\le q(x_{2n}, x_{2n+1})d(Tx_{2n}, Tx_{2n+1}) + r(x_{2n}, x_{2n+1})d(Tx_{2n}, Tfx_{2n}) \\ &\quad + s(x_{2n}, x_{2n+1})d(Tx_{2n+1}, Tgx_{2n+1}) + t(x_{2n}, x_{2n+1})[d(Tx_{2n}, Tgx_{2n+1}) + d(Tx_{2n+1}, Tfx_{2n})] \\ &= q(x_{2n}, x_{2n+1})d(Tx_{2n}, Tx_{2n+1}) + r(x_{2n}, x_{2n+1})d(Tx_{2n}, Tx_{2n+1}) \\ &\quad + s(x_{2n}, x_{2n+1})d(Tx_{2n+1}, Tx_{2n+2}) + t(x_{2n}, x_{2n+1})[d(Tx_{2n}, Tx_{2n+2}) + d(Tx_{2n+1}, Tx_{2n+1})] \\ &\le (q + r + t)(x_{2n}, x_{2n+1})d(Tx_{2n}, Tx_{2n+1}) + (s + t)(x_{2n}, x_{2n+1})d(Tx_{2n+1}, Tx_{2n+2}). \end{aligned}$$

Consequently

$$d(Tx_{2n+1}, Tx_{2n+2}) \le \frac{q(x_{2n}, x_{2n+1}) + r(x_{2n}, x_{2n+1}) + t(x_{2n}, x_{2n+1})}{1 - s(x_{2n}, x_{2n+1}) - t(x_{2n}, x_{2n+1})} d(Tx_{2n}, Tx_{2n+1}). \tag{4}$$

Using (3), we have

$$\frac{q(x, y) + r(x, y) + t(x, y)}{1 - s(x, y) - t(x, y)} \le \lambda$$

for all $x, y \in X$. Thus, from (4), it follows that

$$d(Tx_{2n+1}, Tx_{2n+2}) \preceq \lambda d(Tx_{2n}, Tx_{2n+1}),$$

which shows that a generalized contraction is a contraction for certain pairs of points. Following arguments similar to those given above, we obtain

$$d(Tx_{2n+3}, Tx_{2n+2}) \preceq \lambda d(Tx_{2n+2}, Tx_{2n+1}),$$

where

$$\frac{q(x, y) + s(x, y) + t(x, y)}{1 - r(x, y) - t(x, y)} \leq \lambda$$

for all $x, y \in X$. Therefore, for all n,

$$d(Tx_n, Tx_{n+1}) \preceq \lambda d(Tx_{n-1}, Tx_n) \preceq \lambda^2 d(Tx_{n-2}, Tx_{n-1}) \preceq \cdots \preceq \lambda^n d(Tx_0, Tx_1). \tag{5}$$

Now, for any $m > n$ and $\lambda < 1$,

$$\begin{aligned} d(Tx_n, Tx_m) &\preceq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots + d(Tx_{m-1}, Tx_m) \\ &\preceq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d(Tx_0, Tx_1) \\ &\preceq \frac{\lambda^n}{1-\lambda} d(Tx_0, Tx_1) \to \theta \qquad as \qquad n \to \infty. \end{aligned}$$

From (P_4) we have $(\lambda^n/(1-\lambda))d(Tx_0, Tx_1) \ll c$ for all n sufficiently large and $\theta \ll c$. From (P_1), we have $d(Tx_n, Tx_m) \ll c$. It follows that $\{Tx_n\}$ is a Cauchy sequence by Definition 1.5.(ii). Since a cone metric space X is complete, there exists a $z_x \in X$ such that $Tx_n \to z_x$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} Tfx_{2n} = z_x, \qquad \lim_{n\to\infty} Tgx_{2n+1} = z_x. \tag{6}$$

Now, if T is subsequentially convergent, $\{fx_{2n}\}$ (resp. $\{gx_{2n+1}\}$) has a convergent subsequence. Thus, there exist $w_{x_1} \in X$ and $\{fx_{2n_i}\}$ (resp. $w_{x_2} \in X$ and $\{gx_{2n_i+1}\}$) such that

$$\lim_{n\to\infty} fx_{2n_i} = w_{x_1}, \qquad \lim_{n\to\infty} gx_{2n_i+1} = w_{x_2}. \tag{7}$$

Because of the continuity of T, we have

$$\lim_{n\to\infty} Tfx_{2n_i} = Tw_{x_1}, \qquad \lim_{n\to\infty} Tgx_{2n_i+1} = Tw_{x_2}. \tag{8}$$

From (6) and (8) and using the injectivity of T, there exists a $w_x \in X$ (set $w_x = w_{x_1} = w_{x_2}$) such that $Tw_x = z_x$. On the other hand, from (d_3) and (2) we have

$$\begin{aligned} d(Tw_x, Tgw_x) &\preceq d(Tw_x, Tgx_{2n_i+1}) + d(Tgx_{2n_i+1}, Tfx_{2n_i}) + d(Tfx_{2n_i}, Tgw_x) \\ &\preceq d(Tw_x, Tx_{2n_i+2}) + d(Tx_{2n_i+2}, Tx_{2n_i+1}) + q(x_{2n_i}, w_x) d(Tx_{2n_i}, Tw_x) \\ &\quad + r(x_{2n_i}, w_x) d(Tx_{2n_i}, Tx_{2n_i+1}) + s(x_{2n_i}, w_x) d(Tw_x, Tgw_x) \\ &\quad + t(x_{2n_i}, w_x)[d(Tx_{2n_i}, Tgw_x) + d(Tw_x, Tx_{2n_i+1})] \\ &\preceq d(Tw_x, Tx_{2n_i+2}) + d(Tx_{2n_i+2}, Tx_{2n_i+1}) + (q+t)(x_{2n_i}, w_x) d(Tx_{2n_i}, Tw_x) \\ &\quad + r(x_{2n_i}, w_x) d(Tx_{2n_i}, Tx_{2n_i+1}) + t(x_{2n_i}, w_x) d(Tw_x, Tx_{2n_i+1}) \\ &\quad + (s+t)(x_{2n_i}, w_x) d(Tw_x, Tgw_x). \end{aligned} \tag{9}$$

Now, by (3), (5) and (9) we have

$$\begin{aligned} d(Tw_x, Tgw_x) &\preceq \frac{1}{1-\lambda} d(Tw_x, Tx_{2n_i+2}) + \frac{1}{1-\lambda} d(Tx_{2n_i+2}, Tx_{2n_i+1}) + \frac{\lambda}{1-\lambda} d(Tx_{2n_i}, Tw_x) \\ &\quad + \frac{\lambda}{1-\lambda} d(Tx_{2n_i}, Tx_{2n_i+1}) + \frac{\lambda}{1-\lambda} d(Tw_x, Tx_{2n_i+1}) \\ &= B_1 d(Tw_x, Tx_{2n_i+2}) + B_2 \lambda^{2n_i+1} + B_3 d(Tx_{2n_i}, Tw_x) + B_4 d(Tw_x, Tx_{2n_i+1}), \end{aligned}$$

where

$$B_1 = \frac{1}{1-\lambda} \quad , \quad B_2 = \frac{1}{1-\lambda} d(Tx_0, Tx_1) \quad , \quad B_3 = \frac{\lambda}{1-\lambda} \quad , \quad B_4 = \frac{\lambda}{1-\lambda}.$$

Let $\theta \ll c$. Since $\lambda^{2n_i+1} \to \theta$ and $Tx_{n_i} \to Tw_x$ as $i \to \infty$, there exists a natural number n_0 such that, for each $i \geq n_0$, (by Definition 1.5.(i)) we have

$$d(Tw_x, Tx_{2n_i+2}) \ll \frac{c}{4B_1} \quad , \quad \lambda^{2n_i} \ll \frac{c}{4B_2} \quad , \quad d(Tx_{2n_i}, Tw_x) \ll \frac{c}{4B_3} \quad , \quad d(Tw_x, Tx_{2n_i+1}) \ll \frac{c}{4B_4}.$$

By (P_1), we obtain

$$d(Tw_x, Tgw_x) \ll \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = c.$$

Thus, $d(Tw_x, Tgw_x) \ll c$ for each $c \in intP$. Using (P_2), we obtain $d(Tw_x, Tgw_x) = \theta$; that is, $Tw_x = Tgw_x$. Since T is one to one, $gw_x = w_x$. Now we shall show that $fw_x = w_x$.

$$\begin{aligned} d(Tfw_x, Tw_x) &= d(Tfw_x, Tgw_x) \\ &\leq q(w_x, w_x)d(Tw_x, Tw_x) + r(w_x, w_x)d(Tw_x, Tfw_x) + s(w_x, w_x)d(Tw_x, Tgw_x) \\ &\quad + t(w_x, w_x)[d(Tw_x, Tgw_x) + d(Tw_x, Tfw_x)] \\ &= (r+t)(w_x, w_x)d(Tw_x, Tfw_x) \leq \lambda d(Tw_x, Tfw_x). \end{aligned}$$

Using (P_3), it follows that $d(Tfw_x, Tw_x) = \theta$, which implies the equality $Tfw_x = Tw_x$. Since T is one to one, then $fw_x = w_x$. Thus $fw_x = gw_x = w_x$; that is, w_x is a common fixed point of f and g. Now we shall show that w_x is the unique common fixed point. Suppose that w'_x is another common fixed point of f and g. Then

$$\begin{aligned} d(Tw_x, Tw'_x) &= d(Tfw_x, Tgw'_x) \\ &\leq q(w_x, w'_x)d(Tw_x, Tw'_x) + r(w_x, w'_x)d(Tw_x, Tfw_x) + s(w_x, w'_x)d(Tw'_x, Tgw'_x) \\ &\quad + t(w_x, w'_x)[d(Tw_x, Tgw'_x) + d(Tw'_x, Tfw_x)] \\ &= (q+2t)(w_x, w'_x)d(Tw_x, Tw'_x) \leq \lambda d(Tw_x, Tw'_x). \end{aligned}$$

Using (P_3), it follows that $d(Tw_x, Tw'_x) = \theta$, which implies the equality $Tw_x = Tw'_x$. Since T is one to one, $w_x = w'_x$. Thus f and g have a unique common fixed point.
Ultimately, if T is sequentially convergent, then we can replace n by n_i. Thus we have

$$\lim_{n\to\infty} fx_{2n} = w_x, \qquad \lim_{n\to\infty} gx_{2n+1} = w_x.$$

Therefore if T is sequentially convergent, then the sequences $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to w_x. □

The following results is obtained from Theorem 2.2.

Corollary 2.3. *Suppose that (X, d) is a complete cone metric space, P is a solid cone, and $T : X \to X$ is a continuous and one to one mapping. Moreover, let f and g be two maps of X satisfying*

$$d(Tfx, Tgy) \leq \alpha d(Tx, Ty) + \beta[d(Tx, Tfx) + d(Ty, Tgy)] + \gamma[d(Tx, Tgy) + d(Ty, Tfx)], \tag{10}$$

for all $x, y \in X$, where

$$\alpha, \beta, \gamma \geq 0 \quad \text{and} \quad \alpha + 2\beta + 2\gamma < 1; \tag{11}$$

that is, f and g are T-contractions. Then
(1) *There exists a $z_x \in X$ such that* $\lim_{n\to\infty} Tfx_{2n} = \lim_{n\to\infty} Tgx_{2n+1} = z_x$.
(2) *If T is subsequentially convergent, then $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ have a convergent subsequence.*
(3) *There exists a unique $w_x \in X$ such that $fw_x = gw_x = w_x$; that is, f and g have a unique common fixed point.*
(4) *If T is sequentially convergent, then the sequences $\{fx_{2n}\}$ and $\{gx_{2n+1}\}$ converge to w_x.*

Proof. Corollary 2.3 follows from Theorem 2.2 by setting $q = \alpha$, $r = s = \beta$ and $t = \gamma$ □

Corollary 2.4. *Let (X, d) be a complete cone metric space, P a solid cone and $T : X \to X$ a continuous and one to one mapping. Moreover, let the mapping f be a map of X satisfying*

$$d(Tfx, Tfy) \leq q(x, y)d(Tx, Ty) + r(x, y)d(Tx, Tfx) + s(x, y)d(Ty, Tfy) + t(x, y)[d(Tx, Tfy) + d(Ty, Tfx)], \tag{12}$$

for all $x, y \in X$, where q, r, s and t are nonnegative functions satisfying

$$\sup_{x,y\in X}\{q(x, y) + r(x, y) + s(x, y) + 2t(x, y)\} \leq \lambda < 1; \tag{13}$$

that is, f is a T-contraction. Then

(1) *For each $x_0 \in X$, $\{Tf^n x_0\}$ is a Cauchy sequence, (Define the iterate sequence $\{x_n\}$ by $x_{n+1} = f^{n+1}x_0$).*
(2) *There exists a $z_{x_0} \in X$ such that $\lim_{n\to\infty} Tf^n x_0 = z_{x_0}$.*
(3) *If T is subsequentially convergent, then $\{f^n x_0\}$ has a convergent subsequence.*
(4) *There exists a unique $w_{x_0} \in X$ such that $fw_{x_0} = w_{x_0}$; that is, f has a unique fixed point.*
(5) *If T is sequentially convergent, then, for each $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to w_{x_0}.*

Corollary 2.5. *Let (X, d) be a complete cone metric space, P a solid cone and $T : X \to X$ a continuous and one to one mapping. Moreover, let the mapping f be a map of X satisfying*

$$d(Tfx, Tfy) \leq \alpha d(Tx, Ty) + \beta[d(Tx, Tfx) + d(Ty, Tfy)] + \gamma[d(Tx, Tfy) + d(Ty, Tfx)], \tag{14}$$

for all $x, y \in X$, where

$$\alpha, \beta, \gamma \geq 0 \qquad \textit{and} \qquad \alpha + 2\beta + 2\gamma < 1; \tag{15}$$

that is, f be a T-contraction. Then

(1) *For each $x_0 \in X$, $\{Tf^n x_0\}$ is a Cauchy sequence, (Define the iterate sequence $\{x_n\}$ by $x_{n+1} = f^{n+1}x_0$).*
(2) *There exists a $z_{x_0} \in X$ such that $\lim_{n\to\infty} Tf^n x_0 = z_{x_0}$.*
(3) *If T is subsequentially convergent, then $\{f^n x_0\}$ has a convergent subsequence.*
(4) *There exists a unique $w_{x_0} \in X$ such that $fw_{x_0} = w_{x_0}$; that is, f has a unique fixed point.*
(5) *If T is sequentially convergent, then, for each $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to w_{x_0}.*

Example 2.6. (*See [14]*). *Let $X = [0, 1]$, $E = C^2_{\mathbf{R}}[0, 1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, $P = \{f \in E | f \geq 0\}$ and $d(x, y) = |x - y|2^t$ where $2^t \in P \subset E$. Moreover, suppose that $Tx = x^2$ and $fx = x/2$, which map the set X into X. (X, d) is a cone metric space with non-normal solid cone [9, 16]. Also, T is a one to one, continuous mapping, and f is not a Kannan contraction [14]. All of the conditions of Corollary 2.5 are satisfied with $\alpha = \gamma = 0$ and $\beta = \frac{1}{3}$. Therefore, $x = 0$ is the unique fixed point of f.*

Corollary 2.7. *Let (X, d) be a complete cone metric space, P a solid cone and $T : X \to X$ a continuous and one to one mapping. Moreover, let the mapping f be a T-Hardy-Rogers contraction. Then, the results of the previous Corollary hold.*

Proof. See [6]. □

3. Periodic point results

Obviously, if f is a map which has a fixed point z, then z is also a fixed point of f^n for each $n \in \mathbf{N}$. However the converse need not be true [2]. If a map $f : X \to X$ satisfies $Fix(f) = Fix(f^n)$ for each $n \in \mathbf{N}$, where $Fix(f)$ stands for the set of fixed points of f [10], then f is said to have property P. Recall also that two mappings $f, g : X \to X$ are siad to have property Q if $Fix(f) \bigcap Fix(g) = Fix(f^n) \bigcap Fix(g^n)$ for each $n \in N$. The following results extend some theorems of [2, 6].

Theorem 3.1. *Let (X, d) be a cone metric space, P be a solid cone and $T : X \to X$ be a one to one mapping. Moreover, let the mapping f be a map of X satisfyiing*
(i) $d(fx, f^2x) \preceq \lambda d(x, fx)$ for all $x \in X$, where $\lambda \in [0, 1)$, or (ii) with strict inequality, $\lambda = 1$ for all $x \in X$ with $x \neq fx$. If $Fix(f) \neq \emptyset$, then f has property P.

Proof. See [6]. □

Theorem 3.2. *Let (X, d) be a complete cone metric space, and P a solid cone. Suppose that mappings $f, g : X \to X$ satisfy all of the conditions of Corollary 2.3. Then f and g have property Q.*

Proof. From Corollary 2.3, $Fix(f) \bigcap Fix(g) = \{w\}$, where w is the unique common fixed point of f and g. Suppose that $z \in Fix(f^n) \bigcap Fix(g^n)$, where $n > 1$ is arbitrary. Then we have

$$\begin{aligned}
d(Tw, Tz) &= d(Tf^n w, Tg^n z) = d(Tf(f^{n-1}w), Tg(g^{n-1}z)) \\
&\preceq \alpha d(Tf^{n-1}w, Tg^{n-1}z) + \beta[d(Tf^{n-1}w, Tf^n w) + d(Tg^{n-1}z, Tg^n z)] \\
&\quad + \gamma[d(Tf^{n-1}w, Tg^n z) + d(Tg^{n-1}z, Tf^n w)] \\
&= \alpha d(Tw, Tg^{n-1}z) + \beta[\theta + d(Tg^{n-1}z, Tz)] \\
&\quad + \gamma[d(Tw, Tz) + d(Tg^{n-1}z, Tw)] \\
&\preceq \alpha d(Tw, Tg^{n-1}z) + \beta[d(Tg^{n-1}z, Tw) + d(Tw, Tz)] \\
&\quad + \gamma[d(Tw, Tz) + d(Tg^{n-1}z, Tw)],
\end{aligned}$$

which implies that

$$d(Tw, Tz) = d(Tw, Tg^n z) \preceq \lambda d(Tw, Tg^{n-1}z),$$

where $\lambda = (\alpha + \beta + \gamma)/(1 - \beta - \gamma) < 1$ (by relation (11)). Now, we have

$$d(Tw, Tz) = d(Tw, Tg^n z) \preceq \lambda d(Tw, Tg^{n-1}z) \preceq \lambda^2 d(Tw, Tg^{n-2}z) \cdots \preceq \lambda^n d(Tw, Tz).$$

Since $\lambda^n \in [0, 1)$, according to (P_3), we have $d(Tw, Tz) = \theta$; that is, $Tw = Tz$. Since T is one to one, then $w = z$, which implies that f and g have property Q. □

Theorem 3.3. *Let (X, d) be a complete cone metric space, and P a solid cone. Suppose that the mapping $f : X \to X$ satisfies all of the conditions of Corollary 2.5. Then f has property P.*

Proof. From Corollary 2.5, f has a unique fixed point in X. Suppose that $z \in Fix(f^n)$. Then we have

$$\begin{aligned}
d(Tz, Tfz) &= d(Tf(f^{n-1}z), Tf(f^n z)) \\
&\preceq \alpha d(Tf^{n-1}z, Tf^n z) + \beta[d(Tf^{n-1}z, Tf^n z) + d(Tf^n z, Tf^{n+1}z)] \\
&\quad + \gamma[d(Tf^{n-1}z, Tf^{n+1}z) + d(Tf^n z, Tf^n z)] \\
&\preceq \alpha d(Tf^{n-1}z, Tz) + \beta[d(Tf^{n-1}z, Tz) + d(Tz, Tfz)] + \gamma[d(Tf^{n-1}z, Tz) + d(Tz, Tfz)] \\
&= (\alpha + \beta + \gamma)d(Tf^{n-1}z, Tz) + (\beta + \gamma)d(Tz, Tfz),
\end{aligned}$$

which implies that
$d(Tz, Tfz) \preceq \lambda d(Tf^{n-1}z, Tz)$ where $\lambda = (\alpha + \beta + \gamma)/(1 - \beta - \gamma) < 1$, (by relation (15)). Hence, $d(Tz, Tfz) = d(Tf^n z, Tf^{n+1}z) \preceq \lambda d(Tf^{n-1}z, Tz) \preceq \cdots \preceq \lambda^n d(Tfz, Tz)$. Therefore we have $d(Tfz, Tz) = \theta$; that is, $Tfz = Tz$. Since T is one to one, $fz = z$. □

Corollary 3.4. *Let (X, d) be a complete cone metric space, and P be a solid cone. Suppose that the mapping $f : X \to X$ satisfies all of the conditions of Corollary 2.7. Then f has property P.*

Proof. See [6]. □

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