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Mathematical Programming Involving (α, ρ)-right upper-Dini-derivative Functions

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Abstract. In this paper, we give some new generalized convexities with the tool–right upper-Dini-derivative which is an extension of directional derivative. Next, we establish not only Karush-Kuhn-Tucker necessary but also sufficient optimality conditions for mathematical programming involving new generalized convex functions. In the end, weak, strong and converse duality results are proved to relate weak Pareto (efficient) solutions of the multi-objective programming problems (VP), (MVD) and (MWD).

1. Introduction

Convexity plays an important role in mathematical optimization theory. The concept of convexity is based upon the possibility of connecting any two points of the space by means of a line segment, which has led to convex and generalized convex functions as well as to convex optimization. Various approaches to replacing the line segment joining two points have been proposed recently, we refer to [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11] for recent developments in the field of vector optimization. Following the ideas of [5], [7] extended the classes of quasi-convex, strictly quasi-convex, strongly quasi-convex, pseudo-convex, and strictly pseudo-convex functions to the corresponding forms of arcwise connected functions and presented some interrelations between them. It was shown by them that the functions belonging to these newly defined classes satisfy certain local or global minimum properties. They further established that these new classes of generalized convex functions are more general than the class of classic convex functions. [12] and [13] also discussed some elementary properties pertaining to arcwise connected sets and functions. Using directional derivatives, [14] investigated some properties of arcwise connected functions.

Note that the directional derivative does not always exist, we present some new generalized convexity notations using upper-right Dini derivative, and consider the non-differentiable multi-objective programming problem under these new generalized convexity. The paper is organized as follows. The formulation of the multi-objective programming problem along with some definitions and notations for generalized convexity are given in Section 2. In Section 3, we obtain optimality conditions which include sufficient and necessary optimality conditions for non-differentiable multi-objective under some assumptions. When the sufficient optimality conditions are utilized, two dual problems are formulated and duality results are presented in Section 4.

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2. Notations and Preliminaries

Throughout the paper, let $\mathbb{R}^n$ be an n-dimensional Euclidean space, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x \geqq 0\}$ and $X \subset \mathbb{R}^n$. Moreover, the following conventions for the vectors in $\mathbb{R}^n$ will be followed:

$x > y$ if and only if $x_i > y_i, i = 1, 2, \cdots, n$;
$x \geqq y$ if and only if $x_i \geq y_i, i = 1, 2, \cdots, n$;
$x \geqslant y$ if and only if $x_i \geq y_i$, for $i \in I = 1, 2, \cdots, n$, but $x \neq y$;
$x \ngtr y$ is the negation of $x > y$.

Let $C \subset \mathbb{R}^n$ be an arcwise connected set in [26] and f be a real-valued function defined on C. Let $x_1, x_2 \in C$ and H_{x_1,x_2} be the arc connecting x_1 and x_2 in C. The right derivative or right differential of f with respect to $H_{x_1,x_2}(t)$ at $t = 0$ is defined as follows:

$$\phi'\left(H_{x_1,x_2}(0+)\right) = \lim_{t \to 0^+} \frac{\phi\left(H_{x_1,x_2}(t)\right) - \phi(u)}{t}. \tag{1}$$

We point out that the directional derivative defined by (1) does not always exist, see the following Example 2.1.

Example 2.1. *Consider $x = (x_1, x_2) \in \mathbb{R}^2$, define*

$$\phi(x_1, x_2) = \begin{cases} -x_1^2 \sin^2 \dfrac{1}{x_2}\left(1 + \dfrac{1}{\sqrt{|x_1 x_2|}}\right), & \text{if } x_1 \neq 0 \text{ and } x_2 \neq 0 \\ 0, & \text{if } x_1 = 0 \text{ or } x_2 = 0 \end{cases}.$$

Obviously, f is not continuous at $\mathbf{0} = (0, 0)$. Further, define

$$H_{y,x}(t) = (tx_1, tx_2) + (y_1, y_2), x = (x_1, x_2), y = (y_1, y_2), t \in [0, 1].$$

Then, from the definition of directional derivative defined by (1),

$$d\phi'(H_{\mathbf{0},x}(0+)) = \lim_{t \to 0^+} \frac{\phi(H_{\mathbf{0},x}(t))}{t} = \lim_{t \to 0^+} \frac{\phi((tx_1, tx_2))}{t} = \lim_{t \to 0^+} \frac{-t^2 x_1^2 \sin^2 \dfrac{1}{tx_2}\left(1 + \dfrac{1}{\sqrt{t^2|x_1 x_2|}}\right)}{t} - \lim_{t \to 0^+} \frac{x_1^2 \sin^2 \dfrac{1}{tx_2}}{\sqrt{|x_1 x_2|}}.$$

It is easy to check that this limit does not exist. However, the following right upper-Dini-derivative

$$(d\phi)^+(H_{u,x}(0+)) := \lim_{t \to 0^+} \sup \frac{\phi(H_{u,x}(t)) - \phi(u)}{t} \tag{2}$$

exists and is equal to zero.

Thus, we give some new generalized convexity with this upper-Dini-derivative concept. For convenience, we use the following notations.

Definition 2.2. *A set $X \subset \mathbb{R}^n$ is said to be locally arcwise connected (LAC) at $\bar{x}$ if for any $x \in X$ and $x \neq \bar{x}$ there exists a positive number $a(x, \bar{x})$, with $0 < a(x, \bar{x}) \leqslant 1$, and a continuous arc $H_{\bar{x},x}$ such that $H_{\bar{x},x}(t) \in X$ for any $t \in (0, a(x, \bar{x}))$. The set X is locally arcwise connected on X if X is locally arcwise connected at any $x \in X$.*

Definition 2.3. *Let $X \subset \mathbb{R}^n$ be a LAC set and $\phi : X \to \mathbb{R}$ be a real function defined on X. The function ϕ is said to be (α, ρ)-right upper-Dini-derivative locally arcwise connected with respect to H at u, if there exist real functions $\alpha : X \times X \to \mathbb{R}, \rho : X \times X \to \mathbb{R}$ such that*

$$\phi(x) - \phi(u) \geq \alpha(x, u)(d\phi)^+(H_{u,x}(0+)) + \rho(x, u), \forall\, x \in X. \tag{3}$$

If ϕ is (α, ρ)-right upper-Dini-derivative locally arcwise connected (with respect to H) at u for any $u \in X$, then ϕ is called (α, ρ)-right upper-Dini-derivative locally arcwise connected (with respect to H) on X.

Remark 2.4. *If the directional derivative $\phi'(u, \eta_{u,x}(0+))$ (with respect to arcwise $\eta_{u,x}$) of a function ϕ exists, then*

$$(d\phi)^+(\eta_{u,x}(0+)) = \phi'(u, \eta_{u,x}(0+)).$$

Therefore, any d-ρ-(η,θ)-invex function defined in [23] is $(1, \rho\|\theta\|^2)$ -right upper-Dini-derivative arcwise connected with respect to η. However the following two examples show that there exist functions which are not d-ρ-(η,θ)-invex as defined in [23], d-invex as defined in [22] or directional differentially B-arcwise connected as defined in [26], but (α,ρ)-right upper-Dini-derivative arcwise connected as defined in this paper.

Example 2.5. *Let ϕ and $H_{u,x}$ be defined as in Example 2.1, define*

$$\alpha(x,y) :\equiv 1,\ \rho(x,y) := -2\sin^2\frac{1}{x_2}\left(1+\frac{1}{\sqrt{|x_1x_2|}}\right),$$

respectively. For $u = (0,0)$,

$$\phi(x) - \phi(u) = \phi(x) = \begin{cases} -x_1^2\sin^2\frac{1}{x_2}\left(1+\frac{1}{\sqrt{|x_1x_2|}}\right), & x_1 \neq 0, x_2 \neq 0 \\ 0, & x_1 = 0, x_2 = 0 \end{cases},$$

$$(d\phi)^+(H_{u,x}(0+)) + \rho(x,u) = \rho(x,u) = -2\sin^2\frac{1}{x_2}\left(1+\frac{1}{\sqrt{|x_1x_2|}}\right).$$

Thus

$$\phi(x) - \phi(u) \geq \alpha(x,u)(d\phi)^+(H_{u,x}(0+))) + \rho(x,u).$$

It follows that ϕ is (α,ρ)-right upper-Dini-derivative arcwise connected (with respect to H) at $u = (0,0)$. Note that the directional derivative as defined by (1) does not exist, so ϕ is not d-ρ-(η,θ)-invex, d-invex and directional differentially B-arcwise connected.

Example 2.6. *Let $X = (-1,1)$ and $\phi : X \to \mathbb{R}$ be defined by*

$$\phi(x) = \begin{cases} 0, & if\ -1 < x \leq 0 \\ \frac{1}{2^{n+1}}, & if\ \frac{1}{2^{n+1}} \leq x < \frac{1}{2^n} \end{cases}.$$

Define $H_{u,x}(t) = tx + u$. Then $(d\phi)^+(H_{0,x}(0+)) = 1$ and the following inequality holds

$$\phi(x) - \phi(u) \geq \alpha(x,u)(d\phi)^+(H_{u,x}(0+)) + \rho(x,u)$$

for any $x \in (-1,1)$, where $\alpha(x,0) = \phi(x), \rho(x,0) = 0$. It follows that ϕ is (α,ρ)-right upper-Dini-derivative arcwise connected (with respect to $H_{u,x}$) at $u = 0$. For the same reason as in Example 2.5, we know that ϕ is not d-ρ-(η,θ)-invex, d-invex and directional differentially B-arcwise connected.

In the rest of the paper, we consider the following multiobjective problem:

$$\begin{array}{lll} \text{(VP)} & \min & f(x) \\ & \text{subject to} & g(x) \leqq 0, x \in X, \end{array}$$

where $f = (f_1, \cdots, f_k) : X \to \mathbb{R}^k$, $g = (g_1, \cdots, g_m) : X \to \mathbb{R}^m$, X is a nonempty open subset of $\mathbb{R}^n$. Let $D = \{x \in X \mid g(x) \leqq 0\}$ be the set of feasible solutions of Problem (VP). Denote $K = \{1, \cdots, k\}$, $M = \{1, \cdots, m\}$, $J(x^*) = \{j \in M | g_j(x^*) = 0\}$, $\bar{J} = M \backslash J(x^*)$.

Definition 2.7. *A k-dimensional vector-valued function $f : X \to \mathbb{R}^k$ is called (α,ρ)-right upper-Dini-derivative arcwise connected (with respect to H) at u, if the i-th component of f is (α_i,ρ_i)-right upper-Dini-derivative arcwise connected (with respect to H) at u for $i \in K$, where $\alpha = (\alpha_1, \cdots, \alpha_k)^T$, $\rho = (\rho_1, \cdots, \rho_k)^T$, and the symbol ^{T} in this paper means the transpose. If f is (α,ρ)-right upper-Dini-derivative arcwise connected (with respect to H) at any $u \in X$, then f is called (α,ρ)-right upper-Dini-derivative arcwise connected (with respect to H) on X.*

Definition 2.8. *A k-dimensional vector-valued function $f : X \to \mathbb{R}^k$ is called preinvex (with respect to η) on X if there exists a vector function η such that,*

$$f(u + t\eta(x,u)) \leqq tf(x) + (1-t)f(u)$$

holds for all x, $u \in X$ and any $t \in [0,1]$.

Proposition 2.9. *If a k-dimensional vector-valued function $f : X \to \mathbb{R}^k$ is preinvex (with respect to η) on X, then for any vector-valued function θ, f is (e,o)-right upper-Dini-derivative arcwise connected (with respect to H) on X, where $e = (1,\cdots,1)^T$,$o = (0,\cdots,0)^T$.*

Proof. By Definition 2.8 and 2.7 we can derive the results directly. □

Definition 2.10. *A k-dimensional vector-valued function $f : X \to \mathbb{R}^k$ is called convexlike (with respect to η) on X if for all x, $u \in X$, and any $t \in [0,1]$, there exists $z \in X$ such that*

$$f(z) \leqq tf(x) + (1-t)f(u).$$

Definition 2.11. *A k-dimensional vector-valued function $f : X \to \mathbb{R}^k$ is called ρ-generalized (strong) pseudo-right upper-Dini-derivative arcwise connected (with respect to H) at u, if there exists vector-valued function ρ such that*

$$f(x) < (\leqq) f(u) \Rightarrow (d\phi)^+(H_{u,x}(0+)) < \rho(x,u), \textit{ for } x \in X;$$

f is called ρ-generalized (weak) quasi-right upper-Dini-derivative arcwise connected (with respect to H) at u, if there exists vector-valued function ρ such that

$$f(x) \leqq (<) f(u) \Rightarrow (d\phi)^+(H_{u,x}(0+)) \leqq \rho(x,u), \textit{ for } x \in X;$$

where $\rho = (\rho_1,\cdots,\rho_k)^T$.

Remark 2.12. *Either convex functions or preinvex functions are convexlike.*

Lemma 2.13. *Let $S \subset \mathbb{R}^n$ be a nonempty set and $\Psi : S \to \mathbb{R}^m$ be a convexlik vector-valued function on S. Then either $\Psi(x) < 0$ has a solution $x \in S$, or there exists $\lambda \in \mathbb{R}^m_+$ such that the system $\lambda^T\Psi(x) \geq 0$ holds for all $x \in S$, but both are never true at the same time.*

Definition 2.14. *A point $x^* \in D$ is a weak efficient solution of Problem (VP) if the relation*

$$f(x) \nless f(x^*)$$

holds for all $x \in D$.

3. Optimality Condition

In this section, we first give some necessary optimality conditions for Problem (VP). Using the concept of (local) weak optimality, then we give some sufficient optimality conditions for Problem (VP).

Theorem 3.1 (Fritz John Type Necessary Condition). *Assume that x^* is a local weak efficient solution for Problem (VP). If $(df)^+(H_{x^*,x}(0+))$ and $(dg)^+_{J(x^*)}(H_{x^*,x}(0+))$ are convexlike on X with respect to the variable x, g_j is upper semi-continuous at x^* for $j \in \bar{J}$, then there exist $\lambda \in \mathbb{R}^k_+$, $\mu \in \mathbb{R}^m_+$, $(\lambda,\mu) \neq 0$ such that*

$$\lambda^T(df)^+(H_{x^*,x}(0+)) + \mu^T(dg)^+(H_{x^*,x}(0+)) \geq 0, \forall x \in X, \tag{4}$$

$$\mu^T g(x^*) = 0. \tag{5}$$

Proof. Firstly, we prove that the following system of inequalities

$$(df)^+(H_{x^*,x}(0+)) < 0,$$
$$(dg)^+_{J(x^*)}(H_{x^*,x}(0+)) < 0,$$

has no solution for $x \in X$. Denote $A = \{x \in X | (df)^+(H_{x^*,x}(0+)) < 0\}$, $B = \{x \in X | (dg)^+(H_{x^*,x}(0+)) < 0\}$. Then the above system has no solution in X if only if $A \bigcap B = \emptyset$. Therefore, it is sufficient to prove $A \bigcap B = \emptyset$. On the contrary, if there exists $\bar{x} \in A \bigcap B$, then

$$(df_i)^+(H_{x^*,\bar{x}}(0+)) = \lim_{t\to 0^+} \sup \frac{f_i(H_{x^*,\bar{x}}(t)) - f_i(x^*)}{\lambda} < 0, \forall i \in K.$$

So for each $i \in K$, there exists $\delta_i > 0$ such that

$$f_i(H_{x^*,\bar{x}}(t)) < f_i(x^*), \text{ for } t \in (0, \delta_i).$$

Similarly, for each $j \in J(x^*)$ there exists $\sigma_j > 0$ such that

$$g_j(H_{x^*,\bar{x}}(t)) < g_j(x^*) = 0, \text{ for } t \in (0, \sigma_j).$$

For $j \in \bar{J}$, since $g_j(x)$ is semi-continuous at x^*, then $g_j(H_{x^*,\bar{x}}(t))$ is semi-continuous at $t = 0$. Hence, for $\epsilon = \frac{1}{2}g_j(x^*) > 0$, there exists σ_j such that

$$g_j(H_{x^*,\bar{x}}(t)) < g_j(x^*) + \epsilon = \frac{1}{2}g_j(x^*), \text{ for } t \in (0, \sigma_j).$$

Denote $\delta = \min\{\delta_i, i \in K, \sigma_j, j \in M\}$, then for $t \in (0, \delta)$ we have

$$f_i(H_{x^*,\bar{x}}(t)) < f_i(x^*), \text{ for } i \in K,$$
$$g_j(H_{x^*,\bar{x}}(t)) < g_j(x^*) = 0, \text{ for } j \in M,$$

which contradicts that x^* is a weak efficient solution. By Lemma 2.13 and the hypothesis that $(df)^+$ and $(dg)^+_{J(x^*)}$ are convex-like on X, we obtain the required result. □

Theorem 3.2 (Karush-Kuhn-Tucker Necessary Condition). *Let $(df)^+(H_{x^*,x}(0+))$ and $(dg)^+_{J(x^*)}(H_{x^*,x}(0+))$ be convex-like on X with respect to the variable x. Assume that g_j is upper semi-continuous at x^* for $j \in \bar{J}$, and $(dg)^+(H_{x^*,x}(0+))$ satisfies the slater constraint qualification: there exists $\bar{x} \in D$ such that*

$$(dg)^+_{J(x^*)}(H_{x^*,\bar{x}}(0+)) < 0.$$

If x^* is a local weak efficient solution for (VP), then there exist $\lambda = (\lambda_1, \cdots, \lambda_k)^T \in \mathbb{R}^k_+$, $\mu \in \mathbb{R}^m_+$ satisfying (4), (5) *and*

$$\sum_{i=1}^{k} \lambda_i = 1. \tag{6}$$

Proof. From Theorem 3.1, it is sufficient to prove that $\lambda = (\lambda_1, \ldots, \lambda_k) \neq 0$. Suppose to the contrary that $\lambda = 0$. Then by (4), we have $\mu \neq 0$ and

$$\mu^T (dg)^+(H_{x^*,x}(0+)) \geq 0, \forall x \in X. \tag{7}$$

By (5), we deduce that $\mu_j = 0$ ($j \in \bar{J}$) and there exists $j_0 \in J(x^*)$ such that $\mu_{j_0} > 0$. According to the slater constraint qualification, we have

$$\mu^T (dg)^+(H_{x^*,\bar{x}}(0+)) = \mu^T (dg)^+_{J(x^*)}(H_{x^*,\bar{x}}(0+)) < 0,$$

which contradicts to (7). Now replacing λ_i by $\frac{\lambda_i}{\sum_{i=1}^k \lambda_i}$ for $i = 1, \ldots, k$, and replacing μ_j by $\frac{\mu_j}{\sum_{i=1}^k \lambda_i}$ for $j = 1, \ldots, m$, we get the required result. □

Theorem 3.3 (Karush-Kuhn-Tucker Necessary Condition). *Let $(df)^+(H_{x^*,x}(0+))$ and $(dg)^+_{J(x^*)}(H_{x^*,x}(0+))$ be convex-like on X with respect to the variable x. Assume that g_j is upper semi-continuous at x^* for $j \in \bar{J}$, $g_{J(x^*)}$ is 0-generalized pseudo-right upper-Dini-derivative arcwise connected (with respect to H) at x^* and satisfies the salter constraint qualification: there exists $\bar{x} \in D$ such that $g_{J(x^*)}(\bar{x}) < 0$. If x^* is a local weak efficient solution for (VP), then there exist $\lambda = (\lambda_1, \cdots, \lambda_k)^T \in \mathbb{R}^k_+$, $\mu \in \mathbb{R}^m_+$ satisfying (4), (5) and (6).*

Proof. Note that $g_{J(x^*)}$ is 0-generalized pseudo-right upper-Dini-derivative arcwise connected (with respect to H) at x^* and satisfies the salter constraint qualification, we can deduce that

$$(dg)^+_{J(x^*)}(H_{x^*,\bar{x}}(0+)) < 0.$$

From Theorem 3.2, we get the required result. □

Now, we establish some Karush-Kuhn-Tucker sufficient optimality conditions for Problem (VP) under the new generalized convexity as defined in this paper.

Theorem 3.4 (Karush-Kuhn-Tucker Sufficient Condition). *Let (x^*, λ, μ) satisfy conditions (4), (5) and (6). Assume that f_i $(i \in K)$ and g_j $(j \in J)$ are (α, ρ_i) and $(\alpha, \bar{\rho}_j)$-right upper-Dini-derivative locally arcwise connected at x^* (with respect to the same function H), respectively. If*

$$\alpha(x, x^*) \geq 0, \tag{8}$$

$$\sum_{i=1}^{k} \lambda_i \rho_i(x, x^*) + \sum_{j \in J(x^*)} \mu_j \bar{\rho}_j(x, x^*) \geq 0 \tag{9}$$

hold for any $x \in D$, then x^* is a local weak efficient solution for (VP)

Proof. Suppose to the contrary that x^* is not a local weak efficient solution for (VP). Then there exists $\bar{x} \in D$ such that

$$f(\bar{x}) < f(x^*),\ g_{J(x^*)}(\bar{x}) \leqq g_{J(x^*)}(x^*).$$

Thus we have

$$\sum_{i=1}^{k} \lambda_i (f_i(\bar{x}) - f_i(x^*)) + \sum_{j \in J(x^*)} \mu_j (g_j(\bar{x}) - g_j(x^*)) < 0. \tag{10}$$

Note that f_i $(i \in K)$ and g_j $(j \in J)$ are generalized convex functions, i.e.,

$$f_i(\bar{x}) - f_i(x^*) \geq \alpha(\bar{x}, x^*)(df_i)^+(H_{x^*,\bar{x}}(0+)) + \rho_i(\bar{x}, x^*), i \in K, \tag{11}$$

$$g_j(\bar{x}) - g_j(x^*) \geq \alpha(\bar{x}, x^*)(dg_j)^+(H_{x^*,\bar{x}}(0+)) + \bar{\rho}_j(\bar{x}, x^*), j \in J(x^*). \tag{12}$$

Employing (11) and (12) to (10), we have

$$\alpha(\bar{x}, x^*)\left(\sum_{i=1}^{k} \lambda_i (df_i)^+(H_{x^*,\bar{x}}(0+)) + \sum_{j \in J(x^*)} \mu_j (dg_j)^+(H_{x^*,\bar{x}}(0+))\right) + \left(\sum_{i=1}^{k} \lambda_i \rho_i(\bar{x}, x^*) + \sum_{j \in J(x^*)} \mu_j \bar{\rho}_j(\bar{x}, x^*)\right) < 0. \tag{13}$$

Again, employing (8) and (9) to (13), we obtain

$$\sum_{i=1}^{k} \lambda_i (df_i)^+(H_{x^*,\bar{x}}(0+)) + \sum_{j \in J(x^*)} \mu_j (dg_j)^+(H_{x^*,\bar{x}}(0+)) < 0.$$

Condition (5) and the facts $g_j(\bar{x}) < 0$ $(j \in M)$ can deduce the above inequality to the inequality

$$\sum_{i=1}^{k} \lambda_i (df_i)^+(H_{x^*,\bar{x}}(0+)) + \sum_{j=1}^{m} \mu_j (dg_j)^+(H_{x^*,\bar{x}}(0+)) < 0,$$

which contradicts to (4). We complete the proof. □

Theorem 3.5 (Karush-Kuhn-Tucker Sufficient Condition). *Let (x^*, λ, μ) satisfy the conditions (4), (5) and (6). Assume that $f_i\,(i\in K)$ and $g_j\,(j\in J)$ are ρ_i-generalized pseudo–right upper-Dini-derivative arcwise connected and $\bar{\rho}_j$-generalized quasi–right upper-Dini-derivative arcwise connected at x^* with respect to H, respectively. If inequality*

$$\sum_{i=1}^{k}\lambda_i\rho_i(x,x^*)+\sum_{j\in J(x^*)}\mu_j\bar{\rho}_j(x,x^*)\le 0 \tag{14}$$

holds for any $x\in D$, then x^* is a local weak efficient solution for (VP).

Proof. Suppose to the contrary that x^* is not a local weak efficient solution for (VP). Then there exists $\bar{x}\in D$ such that

$$f(\bar{x})<f(x^*),\ g_{J(x^*)}(\bar{x})\leqq g_{J(x^*)}(x^*).$$

Note that $f_i\,(i\in K)$ and $g_j\ (j\in J)$ are generalized convex at x^*. We have

$$(df_i)^+(H_{x^*,\bar{x}}(0+))<\rho_i(\bar{x},x^*),\ i\in K, \tag{15}$$

$$(dg_j)^+(H_{x^*,\bar{x}}(0+))\le\bar{\rho}_j(\bar{x},x^*),\ j\in J. \tag{16}$$

Combining (15), (16), (5), (6) and (14), we obtain

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i(df_i)^+(H_{x^*,\bar{x}}(0+))+\sum_{j=1}^{m}\mu_j(dg_j)^+(H_{x^*,\bar{x}}(0+)) &= \sum_{i=1}^{k}\lambda_i(df_i)^+(H_{x^*,\bar{x}}(0+))+\sum_{j\in J(x^*)}\mu_j(dg_j)^+(H_{x^*,\bar{x}}(0+))\\ &< \left(\sum_{i=1}^{k}\lambda_i\rho_i(\bar{x},x^*)+\sum_{j\in J(x^*)}\mu_j\bar{\rho}_j(\bar{x},x^*)\right)\le 0,\end{aligned}$$

which is a contradiction to (4). We complete the proof. □

4. Duality

4.1. Mixed Type Duality

In a similar manner to that given in [24] and [23], relative to Problem (VP), we consider the following multi-objective dual Problem (MVD):

$$\begin{aligned}&\text{(MVD)}\max_{(y,\xi,\mu)}\ \phi(y,\xi,\mu)=f(y)+\left(\mu^Tg(y)\right)\mathbf{e}\\ &\text{subject to}\\ &\xi^T(df)^+(H_{y,x}(0+))+\mu^T(dg)^+(H_{y,x}(0+))\ge 0,\forall x\in D,\\ &\mu^Tg(y)\ge 0,\\ &\xi\in R^k_+,\xi^T\mathbf{e}=1,\mu\in\mathbb{R}^m_+,\ y\in X.\end{aligned}$$

where $\mathbf{e}$ is k-tuple of 1's. Let W denote the set of all feasible points of (MVD) and denote pr_XW as the projection of the set W on X.

Theorem 4.1 (Weak Duality)). *Let x and (y,ξ,μ) be feasible points for (VP) and (MVD), respectively. Moreover, we assume that $f_i\,(i\in K)$ and $g_j\,(j\in M)$ are (α,ρ_i) and $(\alpha,\bar{\rho}_j)$-right upper-Dini-derivative arcwise connected at y on $D\bigcup pr_XW$ with respect to H, respectively. If*

$$\alpha(x,y)\ge 0, \tag{17}$$

$$\sum_{i=1}^{k}\xi_i\rho_i(x,y)+\sum_{j=1}^{m}\mu_j\bar{\rho}_j(x,y)\ge 0,\ \textit{for all } x\in D, \tag{18}$$

then

$$f(x)\nless\phi(y,\xi,\mu).$$

Proof. We proceed by contradiction. That is $f(x) < \phi(y, \xi, \mu)$. Since x is feasible for (VP) and $\xi \in R^k_+, \xi^T \mathbf{e} = 1, \mu \in \mathbb{R}^m_+$, then

$$\xi^T f(x) + \mu^T g(x) < \xi^T f(y) + \mu^T g(y). \tag{19}$$

Note that f_i $(i \in K)$ and g_j $(j \in M)$ are generalized convex at y, i.e.,

$$f_i(x) - f_i(y) \geq \alpha(x, y)(df_i)^+(H_{y,x}(0+)) + \rho_i(x, y), i \in K,$$
$$g_j(x) - g_j(y) \geq \alpha(x, y)(dg_j)^+(H_{y,x}(0+)) + \bar{\rho}_j(x, y), j \in M.$$

We have

$$\begin{aligned}&\xi^T f(x) + \mu^T g(x) - \left(\xi^T f(y) + \mu^T g(y)\right)\\ &\quad\geq \alpha(x, y)\left(\sum_{i=1}^k \xi_i (df_i)^+(H_{y,x}(0+)) + \sum_{j=1}^m \mu_j (dg_j)^+(H_{y,x}(0+))\right) + \left(\sum_{i=1}^k \xi_i \rho_i(x, y) + \sum_{j=1}^m \mu_j \bar{\rho}_j(x, y)\right).\end{aligned}$$

This, together with (18), follows

$$\xi^T f(x) + \mu^T g(x) \geq \xi^T f(y) + \mu^T g(y),$$

which is a contradiction to (19). We complete the proof. □

Theorem 4.2 (Strong Duality). *Let x be a weak Pareto solution of the multi-objective programming problem (VP) at which for $\mu \in \mathbb{R}^m_+$; suppose that conditions (4),(5) and (6) are satisfied. Then (x, ξ, μ) is feasible for (MVD). Moreover, if the weak duality between (VP) and (MVD) in Theorem 4.1 holds, then (x, ξ, μ) is a weak Pareto solution for (MVD).*

Proof. Since x satisfies conditions (4), (5) and (6), we have that (x, ξ, μ) is feasible for (MVD). By the weak duality theorem Theorem (4.1), it follows that (x, ξ, μ) is a weak Pareto solution for (MVD). We complete the proof. □

Theorem 4.3 (Converse Duality). *Let $(\bar{y}, \bar{\xi}, \bar{\mu})$ be a weak Pareto solution of (MVD). Moreover, we assume that f_i $(i \in K)$ and g_j $(j \in M)$ are (α, ρ_i) and $(\alpha, \bar{\rho}_j)$-right upper-Dini-derivative arcwise connected at y on $D \bigcup pr_X W$ with respect to H, respectively. If*

$$\alpha(x, \bar{y}) \geqslant 0, \text{ for all } x \in D, \tag{20}$$

$$\sum_{i=1}^k \bar{\xi}_i \rho_i(x, \bar{y}) + \sum_{j=1}^m \bar{\mu}_j \bar{\rho}_j(x, \bar{y}) \geq 0, \text{ for all } x \in D, \tag{21}$$

then $\bar{y}$ is a weak Pareto solution of (VP).

Proof. We prove by using contradiction. Assume that $\bar{y}$ is not a weak Pareto solution for (VP), that is there exists $\bar{x} \in D$ such that $f(\bar{x}) < f(\bar{y})$. Hence

$$\bar{\xi}^T f(\bar{x}) + \bar{\mu}^T g(\bar{x}) < \bar{\xi}^T f(\bar{y}) + \bar{\mu}^T g(\bar{y}). \tag{22}$$

Note that f_i $(i \in K)$ and g_j $(j \in M)$ are generalized convex at $\bar{y}$, i.e.,

$$f_i(\bar{x}) - f_i(\bar{y}) \geq \alpha(\bar{x}, \bar{y})(df_i)^+(H_{\bar{y},\bar{x}}(0+)) + \rho_i(\bar{x}, \bar{y}), i \in K,$$
$$g_j(\bar{x}) - g_j(\bar{y}) \geq \alpha(\bar{x}, \bar{y})(dg_j)^+(H_{\bar{y},\bar{x}}(0+)) + \bar{\rho}_j(\bar{x}, \bar{y}), j \in M.$$

One obtains

$$\begin{aligned}&\bar{\xi}^T f(\bar{x}) + \bar{\mu}^T g(\bar{x}) - \left(\bar{\xi}^T f(\bar{y}) + \bar{\mu}^T g(\bar{y})\right)\\ &\quad\geq \alpha(\bar{x}, \bar{y})\left(\sum_{i=1}^k \bar{\xi}_i (df_i)^+(H_{\bar{y},\bar{x}}(0+)) + \sum_{j=1}^m \bar{\mu}_j (dg_j)^+(H_{\bar{y},\bar{x}}(0+))\right) + \left(\sum_{i=1}^k \bar{\xi}_i \rho_i(\bar{x}, \bar{y}) + \sum_{j=1}^m \bar{\mu}_j \bar{\rho}_j(\bar{x}, \bar{y})\right).\end{aligned}$$

This, combining with (20) and (21), can deduce that

$$\bar{\xi}^T f(\bar{x}) + \bar{\mu}^T g(\bar{x}) \geq \bar{\xi}^T f(\bar{y}) + \bar{\mu}^T g(\bar{y}),$$

which is a contradiction to (22). We complete the proof. □

4.2. Mond-Weir Type Duality

Now, in relation to (VP) we consider the following multi-objective dual program which is Mond-Weir type dual in [25]

$$\begin{aligned}
\text{(MWD)} \max_y \ & f(y) \\
& \text{subject to} \\
& \xi^T (df)^+(H_{y,x}(0+)) + \mu^T (dg)^+(H_{y,x}(0+)) \geq 0, \forall x \in D, \\
& \mu^T g(y) \geq 0, \\
& \xi \in R^k_+, \xi^T \mathbf{e} = 1, \mu \in \mathbb{R}^m_+, \ y \in X.
\end{aligned}$$

where $\mathbf{e}$ is k-tuple of 1's. Let W denote the set of all feasible points of (MWD) and denote $pr_X W$ as the projection of the set W on X.

Similar to the proof of Theorems 4.1 to 4.3, we can establish Theorems 4.4 to 4.6. Therefore, we simply state them here.

Theorem 4.4 (Weak Duality)). *Let x and (y, ξ, μ) be feasible points for (VP) and (MWD), respectively. Moreover, we assume that f_i $(i \in K)$ and g_j $(j \in M)$ are (α, ρ_i) and $(\alpha, \bar{\rho}_j)$-right upper-Dini-derivative arcwise connected at y on $D \bigcup pr_X W$ with respect to H, respectively. If*

$$\alpha(x, y) \geq 0,$$
$$\sum_{i=1}^{k} \xi_i \rho_i(x, y) + \sum_{j=1}^{m} \mu_j \bar{\rho}_j(x, y) \geq 0, \textit{ for all } x \in D,$$

then

$$f(x) \nless f(y).$$

Theorem 4.5 (Strong Duality). *Let x be a weak Pareto solution of the multi-objective programming problem (VP) at which for $\mu \in \mathbb{R}^m_+$, the conditions* (4), (5) *and* (6) *are satisfied. Then (x, ξ, μ) is feasible for (MWD). Moreover, if the weak duality between (VP) and (MWD) in Theorem 4.4 holds, then (x, ξ, μ) is a weak Pareto solution for (MWD).*

Theorem 4.6 (Converse Duality). *Let $(\bar{y}, \bar{\xi}, \bar{\mu})$ be a weak Pareto solution of (MWD). Moreover, we assume that f_i $(i \in K)$ and g_j $(j \in M)$ are (α, ρ_i) and $(\alpha, \bar{\rho}_j)$-right upper-Dini-derivative arcwise connected at y on $D \bigcup pr_X W$ with respect to H, respectively. If*

$$\alpha(x, \bar{y}) \geq 0, \textit{ for all } x \in D,$$
$$\sum_{i=1}^{k} \bar{\xi}_i \rho_i(x, \bar{y}) + \sum_{j=1}^{m} \bar{\mu}_j \bar{\rho}_j(x, \bar{y}) \geq 0, \textit{ for all } x \in D,$$

then $\bar{y}$ is a weak Pareto solution of (VP).

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