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A note on Putinar's matricial models

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Abstract. In this note we consider the conjecture that every hyponormal Putinar's matricial model of rank two is subnormal. Related to this conjecture, we show that there exists a non rationally cyclic subnormal Putinar's matricial model of rank two and then give a sufficient condition for it to be a subnormal operator.

1. Introducton

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ and $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H},\mathcal{H})$. An operator $T \in L(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *quasinormal* if $T^*T^2 = TT^*T$, *hyponormal* if the self commutator $[T^*, T] = T^*T - TT^* \geq 0$, and *subnormal* if it has a normal extension, i.e., $T = N|_{\mathcal{H}}$, where N is a normal operator on some Hilbert space $\mathcal{K}$ containing $\mathcal{H}$. In general it is quite difficult to determine the subnormality of an operator by definition. An alternative description of subnormality is given by the Bram-Halmos criterion, which states that an operator T is subnormal if and only if

$$\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([2], [4]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1). \tag{1}$$

Condition (1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity (1) for all k. Let $[A, B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

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$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{2}$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (2) is equivalent to the positivity of the $(k+1)\times(k+1)$ operator matrix in (1); the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([15]). The classes of k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality (cf. [5]-[9],[12]-[16],[21]).

In view of the gap theory, it seems to be interesting to consider the following problem:

Which 2-hyponormal operators are subnormal ? (3)

The first inquiry involves the self commutator. Subnormal operators with finite rank self commutators have been extensively studied ([1],[20],[26]-[28]). Particular attention has been paid to hyponormal operators with rank one or rank two self commutators ([17],[22],[24],[25],[26],[29]). In particular, B. Morrel [22] showed that a pure subnormal operator with rank one self commutator (pure means having no normal summand) is unitarily equivalent to a linear function of the unilateral shift. Morrel's theorem can be essentially stated (also see [4, p. 162]) that if

$$\begin{cases} \text{(i) } T \text{ is hyponormal;} \\ \text{(ii) } [T^*, T] \text{ is of rank one; and} \\ \text{(iii) } ker[T^*, T] \text{ is invariant for } T, \end{cases} \tag{4}$$

then $T - \beta$ is quasinormal for some $\beta \in \mathbb{C}$. It would be interesting (in the sense of giving a simple sufficiency for the subnormality) to note that Morrel's theorem gives that

If T satisfies the condition (4), then T is submormal.

On the other hand, it was shown [13, Lemma 2.2] that if T is 2-hyponormal then

$$T(ker[T^*, T]) \subset ker[T^*, T].$$

Therefore by Morrel's theorem, we can see that

every 2-hyponormal operator with rank one self commutator is subnormal.

Recently, S.H. Lee and W.Y. Lee [19] obtained an extension of Morrel's theorem to the case of rank two self commutators:

Theorem 1.1. *([19]) Let $T \in L(\mathcal{H})$. If*
(i) T is a pure hyponormal operator;
(ii) $[T^*, T]$ is of rank two; and
(iii) $ker[T^*, T]$ is invariant for T,
then we have
(1) If $T|_{ran[T^*,T]}$ has a rank one self commutator then T is subnormal;
(2) If $T|_{ran[T^*,T]}$ has a rank two self commutator then T is either a subnormal operator or a "Putinar's martricial model" of rank two; that is, T has the following two diagonal structure, with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots$:

$$T = \begin{pmatrix} B_0 & 0 & 0 & 0 & \cdots \\ A_0 & B_1 & 0 & 0 & \cdots \\ 0 & A_1 & B_2 & 0 & \cdots \\ 0 & 0 & A_2 & B_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5}$$

where

$$\begin{cases} (i)\ dim\mathcal{H}_n = dim\mathcal{H}_{n+1} = 2 \ \ (n \geq 0); \\ (ii)\ [T^*, T] = ([B_0^*, B_0] + A_0^*A_0) \oplus 0_\infty; \\ (iii)\ A_n^*B_{n+1} = B_nA_n^* \ \ (n \geq 0), \end{cases} \tag{6}$$

and if T_n *denotes the compression of* T *to the space* $\mathcal{H}_n \oplus \mathcal{H}_{n+1} \oplus \cdots$ *for* $n \geq 0$, *then*

$$\mathcal{H}_n = ran[T_n^*, T_n] \ \ \textit{for every}\ n \geq 0,$$

and $T_n = m.p.n.e.(T_{n+1})$ $(n \geq 0)$ *(See below for "m.p.n.e"). Here note that under unitary equivalence we may assume that all* A_n *are positive and invertible.*

And they conjectured:

Conjecture 1.1. *The Putinar's matricial model of rank two is subnormal.*

If T is a rationally cyclic subnormal operator of rank two self commutator, then there is a good characterization ([18], [20]). By Morrel's theorem ([22]), they have two diagonal structure. So we can ask the following: If a Putinar's matricial model of rank two is a subnormal operator, is it rationally cyclic ? In this note we will give the negative answer to this question and also give a sufficient condition for a Putinar's matricial model of rank two to be subnormal.

2. The main result

We first review a few essential facts concerning weak subnormality that we will need to begin with. An operator $T \in L(\mathcal{H})$ is said to be *weakly subnormal* if there exist operator $A \in L(\mathcal{H}', \mathcal{H})$ and $B \in L(\mathcal{H}')$ such that the following conditions hold:

$$[T^*, T] = AA^* \text{ and } A^*T = BA^*,$$

or equivalently there is an extension $\widehat{T} := \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ of T such that

$$\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f \text{ for all } f \in \mathcal{H}.$$

The operator $\widehat{T}$ is called a *partially normal extension* (briefly, p.n.e.) of T. We also say that $\widehat{T}$ in $L(\mathcal{K})$ is a *minimal partially normal extension* (briefly, m.p.n.e.) of T if $\mathcal{K}$ has no proper subspace containing $\mathcal{H}$ to which the restriction of $\widehat{T}$ is also a partially normal extension of T. For convenience, if $\widehat{T}$ = m.p.n.e.(T) is also weakly subnormal then we write $\widehat{T}^{(2)} := \widehat{\widehat{T}}$ and more generally,

$$\widehat{T}^{(n)} := \widehat{\widehat{T}^{(n-1)}},$$

which will be called the *n-th minimal partially normal extension* of T. It was ([10], [11], [13]) shown that

$$\text{2-hyponormal} \Longrightarrow \text{weakly subnormal} \Longrightarrow \text{hyponormal}$$

and the converses of both implications are not true in general. It was [13] known that

$$T \text{ is weakly subnormal} \Longrightarrow T(ker[T^*, T]) \subset ker[T^*, T]$$

and it was [11] known that if $\widehat{T}$:= m.p.n.e.(T) then for any $k \geq 1$,

$$T \text{ is } (k+1)\text{-hyponormal} \Longleftrightarrow T \text{ is weakly subnormal and } \widehat{T} \text{ is } k\text{-hyponormal}.$$

So, in particular, one can see that

$$\text{If } T \text{ is subnormal, then } \widehat{T} \text{ is subnormal.}$$

It is worth to noticing that Morrel's theorem gives that

every weakly subnormal operator with rank one self commutator is subnormal.

Now we will show that there exists a Putinar's matricial model of rank two which is subnormal but not rationally cyclic.

Theorem 2.1. *There is a non rationally cyclic subnormal Putinar's matricial model of rank two.*

Proof. Let $B_0 = \begin{pmatrix} 0 & \sqrt{\frac{\alpha}{2+2\alpha}} \\ \sqrt{\frac{\alpha}{2+2\alpha}} & 0 \end{pmatrix} (0 < \alpha < 1)$ and $A_0 = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & 1 \end{pmatrix}$. Then using the relations

$$[B_{n+1}^*, B_{n+1}] + A_{n+1}^2 = A_n^2 \text{ and } A_n B_{n+1} = B_n A_n \text{ for } n \geq 0,$$

we can successively define A_n's and B_n's . A straightforward calculation shows that $A_{n+6} = A_n$ and $B_{n+6} = B_n$. Hence we have the following operator

$$T = \begin{pmatrix} B_0 & 0 & 0 & 0 & \cdots \\ A_0 & B_1 & 0 & 0 & \cdots \\ 0 & A_1 & B_2 & 0 & \cdots \\ 0 & 0 & A_2 & B_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots .$$

Since $[T^*, T] = C \oplus 0_\infty$, where $C = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$, T is hyponormal and has a rank two self commutator. So it is a pure hyponormal operator. By [19, Theorem 2], it is subnormal. On the other hand, if $\Lambda := B_0$, then $\{\Lambda, C\}$ are complete unitary invariants, so that

$$\Lambda := (T^*|_{ran[T^*,T]})^* \text{ and } C := [T^*, T]|_{ran[T^*,T]}.$$

Recall ([20], [18]) that the characterization of rationally cyclic subnormal operators with rank two self commutators as follows: If S is a pure rationally cyclic subnormal operator with rank two self commutator, then S is unitarily equivalent to a dilation and shift of one of the following operators:

(a) $S = S_1 \oplus S_2$, where $S_j = \alpha_j U + \beta_j$ with $\beta_j \in \mathbb{C}$ and $\alpha_j > 0$ for $j = 1, 2$,

(b) $S = U_\lambda$,

(c) $S = U(U + \alpha)$ for some $\alpha \in \mathbb{C} \backslash \{0\}$,

(d) $S = \alpha U + \delta U(1 - \delta U)^{-1}$, where $\alpha \in \mathbb{C} \backslash \{0\}$ and $0 < |\delta| < 1$,

where if we let $\mathbb{T}_a = \partial\mathbb{D} \cup \{a\}$ $(0 < a < 1)$ and λ be a measure on $\mathbb{T}_a$ such that $d\lambda(e^{i\theta}) = \frac{d\theta}{2\pi}$ and $\lambda(\{a\}) = \nu$, then U_λ is the multiplication by z on $P^2(\lambda)$ which is the closure of the polynomials under the inner product

$$\langle f, g \rangle = \int_{\mathbb{T}_a} f(z)\overline{g(z)} d\lambda(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} f(z)\overline{g(z)} \frac{dz}{z} + \nu f(a)\overline{g(a)},$$

and U is the unilateral shift. Then Λ and C are given by:

(1) For case (a), $\Lambda = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}$ and $C = \begin{pmatrix} \alpha_1^2 & 0 \\ 0 & \alpha_2^2 \end{pmatrix}$.

(2) For case (b),
$a = 0 : \Lambda = \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{1+\nu}} & 0 \end{pmatrix}$ and $C = \begin{pmatrix} \frac{1}{1+\nu} & 0 \\ 0 & \frac{1}{1+\nu} \end{pmatrix}$.
$a \neq 0 \ (|a| < 1) : \Lambda = \begin{pmatrix} 0 & 0 \\ \frac{a(1-|a|^2)}{|a|\sqrt{1+\nu-|a|^2}} & a \end{pmatrix}$ and

$$C = \begin{pmatrix} 1 - \nu(1-|a|^2)^3\left(\frac{1+\nu|a|^2-|a|^2}{1+\nu-|a|^2}\right) & -\frac{\nu|a|(1-|a|^2)}{(1+\nu|a|^2-|a|^2)\sqrt{1+\nu-|a|^2}} \\ -\frac{\nu|a|(1-|a|^2)}{(1+\nu|a|^2-|a|^2)\sqrt{1+\nu-|a|^2}} & \nu(1-|a|^2)^3\left(\frac{1+\nu|a|^2-|a|^2}{1+\nu-|a|^2}\right) \end{pmatrix}.$$

(3) For case (c), $\Lambda = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1+|\alpha|^2 & \alpha \\ \overline{\alpha} & 1 \end{pmatrix}$.
(4) For case (d), $\Lambda = \begin{pmatrix} 0 & 0 \\ (\alpha\overline{\delta}+\rho^2)\rho & \alpha\overline{\delta}+\rho^2 \end{pmatrix}$ and

$$C = \begin{pmatrix} |\alpha|^2 + \alpha\overline{\delta} + \overline{\alpha}\delta + \rho^2 & (\alpha\overline{\delta}+\rho^2)\rho \\ (\overline{\alpha}\delta+\rho^2)\rho & \rho^4 \end{pmatrix} \quad \left(\text{with } \rho = \sqrt{\frac{|\delta|^2}{1-|\delta|^2}}\ \right).$$

We next find the rank of $[\Lambda^*, \Lambda]$ for each case:
(1) $[\Lambda^*, \Lambda] = 0$, and so $rank[\Lambda^*, \Lambda] = 0$,
(2) $a = 0 : [\Lambda^*, \Lambda] = \begin{pmatrix} \frac{1}{1+\nu} & 0 \\ 0 & -\frac{1}{1+\nu} \end{pmatrix}$, and so $rank[\Lambda^*, \Lambda] = 2$
$a \neq 0 : [\Lambda^*, \Lambda] = \begin{pmatrix} |\alpha|^2 & \overline{\alpha}a \\ \alpha\overline{a} & -|\alpha|^2 \end{pmatrix}$, where $\alpha = \frac{a(1-|a|^2)}{|a|\sqrt{1+\nu-|a|^2}}$. Since $det[\Lambda^*, \Lambda] = -|\alpha|^2(|\alpha|^2 + |a|^2) \neq 0$, $rank[\Lambda^*, \Lambda] = 2$.
(3) $[\Lambda^*, \Lambda] = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & -|\alpha|^2 \end{pmatrix}$, and so $rank[\Lambda^*, \Lambda] = 2$,
(4) $[\Lambda^*, \Lambda] = |\alpha\overline{\delta} + \rho^2|^2 \begin{pmatrix} \rho^2 & \rho \\ \rho & -\rho^2 \end{pmatrix}$. Since $det\begin{pmatrix} \rho^2 & \rho \\ \rho & -\rho^2 \end{pmatrix} = -\rho^2(1+\rho^2) \neq 0$, $rank[\Lambda^*, \Lambda] = 2$.

Assume that T is rationally cyclic. Then T must be unitarily equivalent to a dilation and shift of one of the above operators. Since $rank[\Lambda^*, \Lambda]$ is invariant under dilation and shift, and $rank[B_0^*, B_0] = 0$, T is not unitarily equivalent to a dilation and shift of one of the cases (b), (c), and (d). Hence T must be the case (a). Since , for C and Λ for T,

$$\begin{pmatrix} 0 & \sqrt{\frac{\alpha}{2+2\alpha}} \\ \alpha\sqrt{\frac{\alpha}{2+2\alpha}} & 0 \end{pmatrix} = C\Lambda \neq \Lambda C = \begin{pmatrix} 0 & \alpha\sqrt{\frac{\alpha}{2+2\alpha}} \\ \sqrt{\frac{\alpha}{2+2\alpha}} & 0 \end{pmatrix},$$

C and Λ for T are not simultaneously diagonalizable. Since the simultaneous diagonalization is invariant under a dilation and shift, T is not unitarily equivalent to the case (a) and so T is not rationally cyclic. Hence T is the desired operator. □

Corollary 2.2. *Let T be a Putinar matricial model of rank two. If $A_{-1}^2 (:= [B_0^*, B_0] + A_0^2)$ and B_0 are simultaneously diagonalizable, then T is a subnormal operator.*

Proof. Since A_{-1}^2 and B are simultaneously diagonalizable, we may write that

$$A_{-1}^2 = \begin{pmatrix} \alpha_1^2 & 0 \\ 0 & \alpha_2^2 \end{pmatrix}, \ B_0 = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}.$$

Now we can find a rationally cyclic subnormal operator S with its complete unitary invariants such as (see the proof of Theorem 2.1.):

$$\Lambda = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \alpha_1^2 & 0 \\ 0 & \alpha_2^2 \end{pmatrix}.$$

By Morrel's Theorem [22], S has two diagonal structure as in (5). Since T is a Putinar's matricial model of rank two, it also satisfies the relation (6). So S and T must be the same and T is subnormal. □

If a Putinar's matricial model of rank two T is rationally cyclic, then we can get the following:

Corollary 2.3. *Let T be a rationally cyclic operator represented by a Putinar matricial model of rank two. If $rank[B_0^*, B_0] = 1$, then T can not be a subnormal operator.*

Proof. If T is subnormal, then it is a dilation and shift of one of the four cases in the proof of Theorem 2.1. For them, $rank[\Lambda^*, \Lambda] = 0$ or 2. Since $rank[\Lambda^*, \Lambda]$ is invariant under dilation and shift and B_0 is Λ for T, $rank[B_0^*, B]$ can not be 1. Hence T can not be subnormal. □

Now we will give a sufficient condition for a Putinar's matricial model of rank two to be subnormal.

Theorem 2.4. *Let T be a Putinar's matricial model of rank two as in (5). If $B_n = B_{n+1}^*$ for some $n \geq 0$, then T is a subnormal operator.*

Proof. Consider the operator

$$T_{n+1} = \begin{pmatrix} B_{n+1} & 0 & 0 & 0 & \cdots \\ A_{n+1} & B_{n+2} & 0 & 0 & \cdots \\ 0 & A_{n+2} & B_{n+3} & 0 & \cdots \\ 0 & 0 & A_{n+3} & B_{n+4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ on } \widetilde{\mathcal{H}}_{n+1} = \mathcal{H}_{n+1} \oplus \mathcal{H}_{n+2} \oplus \cdots .$$

Since T is a Putinar's matricial model of rank two, A_n is positive and invertible and so we can find $\sqrt{A_n}$ and $\sqrt{A_n}^{-1}$. Let $\widetilde{B} = \sqrt{A_n} B_{n+1} \sqrt{A_n}^{-1}$ and define operators A and B on $\widetilde{\mathcal{H}}_{n+1}$.

$$A = \begin{pmatrix} \sqrt{A_n} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} \widetilde{B} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$BA = \begin{pmatrix} \widetilde{B} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \sqrt{A_n} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \widetilde{B}\sqrt{A_n} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$AT_{n+1} = \begin{pmatrix} \sqrt{A_n} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} B_{n+1} & 0 & 0 & 0 & \cdots \\ A_{n+1} & B_{n+2} & 0 & 0 & \cdots \\ 0 & A_{n+2} & B_{n+3} & 0 & \cdots \\ 0 & 0 & A_{n+3} & B_{n+4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \sqrt{A_n}B_{n+1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

So $BA = AT_{n+1}$. On the other hand,

$$T_{n+1}^* A = \begin{pmatrix} B_{n+1}^* & A_{n+1}^* & 0 & 0 & \cdots \\ 0 & B_{n+2}^* & A_{n+2}^* & 0 & \cdots \\ 0 & 0 & B_{n+3}^* & A_{n+3}^* & \cdots \\ 0 & 0 & 0 & B_{n+4}^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \sqrt{A_n} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} B_{n+1}^*\sqrt{A_n} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$AB = \begin{pmatrix} \sqrt{A_n}\widetilde{B} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since $A_nB_{n+1} = B_nA_n$,

$$\sqrt{A_n}\widetilde{B} = \sqrt{A_n}\,\sqrt{A_n}B_{n+1}\,\sqrt{A_n}^{-1} = A_nB_{n+1}\,\sqrt{A_n}^{-1} = B_nA_n\,\sqrt{A_n}^{-1} = B^*_{n+1}\,\sqrt{A_n}.$$

So $AB = T^*_{n+1}A$.

Now let N be an operator defined on $\widetilde{\mathcal{H}}_{n+1} \bigoplus \widetilde{\mathcal{H}}_{n+1} \bigoplus \widetilde{\mathcal{H}}_{n+1}$.

$$N = \begin{pmatrix} T^*_{n+1} & 0 & 0 \\ A & B & 0 \\ 0 & A & T_{n+1} \end{pmatrix}.$$

Then

$$N^*N = \begin{pmatrix} T_{n+1} & A & 0 \\ 0 & B & A \\ 0 & 0 & T^*_{n+1} \end{pmatrix}\begin{pmatrix} T^*_{n+1} & 0 & 0 \\ A & B & 0 \\ 0 & A & T_{n+1} \end{pmatrix} = \begin{pmatrix} T_{n+1}T^*_{n+1} + A^2 & AB & 0 \\ BA & B^2 + A^2 & AT_{n+1} \\ 0 & T^*_{n+1}A & T^*_{n+1}T_{n+1} \end{pmatrix}$$

and

$$NN^* = \begin{pmatrix} T^*_{n+1} & 0 & 0 \\ A & B & 0 \\ 0 & A & T_{n+1} \end{pmatrix}\begin{pmatrix} T_{n+1} & A & 0 \\ 0 & B & A \\ 0 & 0 & T^*_{n+1} \end{pmatrix} = \begin{pmatrix} T^*_{n+1}T_{n+1} & T^*_{n+1}A & 0 \\ AT_{n+1} & A^2 + B^2 & BA \\ 0 & AB & A^2 + T_{n+1}T^*_{n+1} \end{pmatrix}.$$

Since T is a Putinar's matricial model of rank two, $[T^*_{n+1}, T_{n+1}] = ([B^*_{n+1}, B_{n+1}] + A^2_{n+1}) \oplus 0_\infty = A^2_n \oplus 0_\infty = A^2$. Hence, by the previous calculations, we have $N^*N = NN^*$, i.e., it is normal. Since N is clearly a normal extension of T_{n+1}, T_{n+1} is subnormal. Since T is the $(n+1)$-th minimal partially normal extension of T_{n+1}, T should be subnormal. □

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