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On double sequences of continuous functions having continuous P-limits II

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Abstract. The goal of this paper is to relax the conditions of the following theorem: Let A be a compact closed set; let the double sequence of function

$$\begin{array}{cccc} s_{1,1}(x), & s_{1,2}(x) & s_{1,3}(x) & \ldots \\ s_{2,1}(x), & s_{2,2}(x) & s_{2,3}(x) & \ldots \\ s_{3,1}(x), & s_{3,2}(x) & s_{3,3}(x) & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

have the following properties:

1. for each (m, n) $s_{m,n}(x)$ is continuous in A;
2. for each x in A we have $\mathrm{P}-\lim_{m,n} s_{m,n}(x) = s(x)$;
3. $s(x)$ is continuous in A;
4. there exists M such that for all (m, n) and all x in A $|s_{m,n}(x)| \leq M$.

Then there exists a $\mathcal{T}$-transformation such that

$$\mathrm{P}-\lim_{m,n} \sigma_{m,n}(x) = s(x) \text{ uniformly in } A$$

and to that end we obtain the following. In order that the transformation be such that

$$\mathrm{P}-\lim_{s\to s_0(S);t\to t_0(T)} \sigma(s;t;x) = 0$$

uniformly with respect x for every double sequence of continuous functions $(s_{m,n}(x))$ define over A such that $s_{m,n}(x)$ is bounded over A and for all (m, n) and $\mathrm{P}-\lim_{m,n} s_{m,n}(x) = 0$ over A it is necessary and sufficient that

$$\mathrm{P}-\lim_{s\to s_0(S);t\to t_0(T)} \sum_{k,l=1,1}^{\infty,\infty} |a_{k,l}(s,t)| = 0.$$

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1. Introduction

In a previous paper [5] we provided an answer to the following question: Is it necessarily the case that if $s_{m,n}(x)$ is a bounded for all (m, n) and x with continuous elements and P-converges to a continuous function there exists an RH-regular matrix transformation that maps $(s_{m,n}(x))$ into a uniformly P-convergent double sequence? The answer was granted by the following theorem.

Theorem 1.1. *Let A be a compact closed set; let the double sequence of function*

$$\begin{array}{cccc} s_{1,1}(x), & s_{1,2}(x) & s_{1,3}(x) & \ldots \\ s_{2,1}(x), & s_{2,2}(x) & s_{2,3}(x) & \ldots \\ s_{3,1}(x), & s_{3,2}(x) & s_{3,3}(x) & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

have the following properties:

1. *for each* (m, n) $s_{m,n}(x)$ *is continuous in* A;
2. *for each* x *in* A *we have* $P - \lim_{m,n} s_{m,n}(x) = s(x)$;
3. $s(x)$ *is continuous in* A;
4. *there exists* M *such that for all* (m, n) *and all* x *in* A $|s_{m,n}(x)| \leq M$.

Then there exists a $\mathcal{T}$*-transformation such that*

$$P - \lim_{m,n} \sigma_{m,n}(x) = s(x) \text{ uniformly in } A.$$

The goal of this paper is to provide an answer to the question of how far if at all, these conditions can be relaxed and remains an answer to the question above.

2. Definitions, notations and preliminary results

Definition 2.1. [Pringsheim, 1900] A double sequence $x = [X_{k,l}]$ has *Pringsheim limit* L (denoted by P-$\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $\left|X_{k,l} - L\right| < \epsilon$ whenever $k, l > N$. Such an x is describe more briefly as "P-convergent".

Definition 2.2. [Patterson, 2000] The double sequence y is a *double subsequence* of x provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that if $x_j = x_{n_j,k_j}$, then y is formed by

$$\begin{array}{cccc} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & -. \end{array}$$

In [7] Robison presented the following notion of conservative four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of such notion.

Definition 2.3. The four-dimensional matrix $\mathcal{A}$ is said to be *RH-regular* if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

Theorem 2.4. ([3, 7]) *The four dimensional matrix* $\mathcal{A}$ *is RH-regular if and only if*

RH_1: $P\text{-}\lim_{m,n} a_{m,n,k,l} = 0$ *for each* k *and* l;
RH_2: $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;
RH_3: $P\text{-}\lim_{m,n} \sum_{k=1}^{\infty} \left|a_{m,n,k,l}\right| = 0$ *for each* l;
RH_4: $P\text{-}\lim_{m,n} \sum_{l=1}^{\infty} \left|a_{m,n,k,l}\right| = 0$ *for each* k;
RH_5: $\sum_{k,l=1,1}^{\infty,\infty} \left|a_{m,n,k,l}\right|$ *is P-convergent*;
RH_6: *there exist finite positive integers* Δ *and* Γ *such that*
$\sum_{k,l>\Gamma} \left|a_{m,n,k,l}\right| < \Delta$.

We shall consider four dimensional transformation in the following setting, that is

$$\sigma(s,t;x) = \sum_{k,l=1,1}^{\infty,\infty} a_{k,l}(s,t)s_{k,l}(x) \tag{1}$$

P-converges for all $s \in S$ and $t \in T$ and $x \in A$

$$\mathrm{P} - \lim_{s\to s_0(S);t\to t_0(T)} \sigma(s,t;x) = \sigma(x)$$

3. Main results

Theorem 3.1. *In order that* (T) *may be such that*

$$P - \lim_{s\to s_0(S);t\to t_0(T)} \sigma(s;t;x) = 0 \tag{2}$$

uniformly with respect x *for every double sequence of continuous functions* $(s_{m,n}(x))$ *define over* A *such that* $s_{m,n}(x)$ *is bounded over* A *and for all* (m,n) *and* $P - \lim_{m,n} s_{m,n}(x) = 0$ *over* A *it is necessary and sufficient that*

$$P - \lim_{s\to s_0(S);t\to t_0(T)} \sum_{k,l=1,1}^{\infty,\infty} |a_{k,l}(s,t)| = 0. \tag{3}$$

Proof. Let us establish the necessary part. Observe that for $(s_{m,n}(x))$ define as

$$s_{m,n}(x) = \begin{cases} 1 & \text{if} \quad m = n = k = l \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

Thus (2.1) must satisfy the following condition for each (k,l)

$$\mathrm{P} - \lim_{s\to s_0(S);t\to t_0(T)} a_{k,l}(s,t) = 0. \tag{4}$$

We show that (3.2) is a necessary condition by supposing that (2.1) satisfies (3.3) but not (3.2) and defining an admissible double sequence $(s_{m,n}(x))$ for which (3.1) fails. Note (3.2) is not true. That is,

$$\mathrm{P} - \lim_{s\to s_0(S);t\to t_0(T)} \sum_{k,l=1,1}^{\infty,\infty} |a_{k,l}(s,t)| \neq 0.$$

Therefore

$$\mathrm{P} - \lim_{s\to s_0(S);t\to t_0(T)} \sum_{k,l=1,1}^{\infty,\infty} |a_{k,l}(s,t)| > \phi \text{ where } \phi > 0. \tag{5}$$

Therefore there exists a double sequence (s_m, t_n) of points (S,T) with P-limit (s_0, t_0) such that

$$\mathrm{P} - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{k,l}(s_m, t_n)| > \phi.$$

We can choose (m_1, n_1) such that

$$\sum_{k,l=1,1}^{\infty,\infty} |a_{k,l}(s_{m_1}, t_{n_1})| > \phi$$

and choose $M_1 > m_1$ and $N_1 > n_1$ such that

$$\sum_{k,l=1,1}^{M_1,N_1} |a_{k,l}(s_{m_1}, t_{n_1})| > \phi.$$

We can to choose $m_2 > M_2$ and $n_2 > N_2$ by (3.2) and (3.3) such that

$$\sum_{k,l=1,1}^{M_1,N_1} |a_{k,l}(s_\alpha, t_\beta)| > \frac{\phi}{2},$$

for $\alpha \geq m_2, \beta \geq n_2$ and

$$\sum_{k,l=1,1}^{\infty,\infty} |a_{k,l}(s_{m_2}, t_{n_2})| > \phi$$

and choose $M_2 > m_2$ and $N_2 > n_2$ such that

$$\sum_{k,l=1,1}^{M_2,N_2} |a_{k,l}(s_{m_2}, t_{n_2})| > \phi.$$

We can to choose $m_3 > M_3$ and $n_3 > N_3$ by (3.2) and (3.3) such that

$$\sum_{k,l=1,1}^{M_2,N_2} |a_{k,l}(s_\alpha, t_\beta)| > \frac{\phi}{2}.$$

for $\alpha \geq m_3, \beta \geq n_3$ and

$$\sum_{k,l=1,1}^{M_2,N_2} |a_{k,l}(s_{m_2}, t_{n_2})| > \phi$$

and choose $M_3 > m_3$ and $N_3 > n_3$ such that

$$\sum_{\{(k,l):M_2<k\leq M_3 \text{ or } N_2<l\leq N_3\}} |a_{k,l}(s_{m_3}, t_{n_3})| > \frac{\phi}{2}.$$

Inductively we can choose

$$m_1 < M_1 < m_2 < M_2 < m_3 < M_3 < \cdots,$$

and

$$n_1 < N_1 < n_2 < N_2 < n_3 < N_3 < \cdots$$

such that for $p, q = 2, 3, 4, 5, \ldots$

$$\sum_{k,l=1,1}^{M_{p-1},N_{q-1}} |a_{k,l}(s_\alpha, t_\beta)| > \frac{\phi}{2},$$

for $\alpha \geq m_p, \beta \geq n_q$ and

$$\sum_{k,l=1,1}^{M_p,N_q} |a_{k,l}(s_\alpha, t_\beta)| > \phi$$

then

$$\sum_{\{(k,l):M_{p-1}+1\leq k\leq M_p \text{ or } N_{q-1}+1\leq l\leq N_q\}} |a_{k,l}(s_{m_p}, t_{n_q})| > \frac{\phi}{2};\ p, q = 1, 2, 3, \ldots. \tag{6}$$

where $M_0 = N_0 = 0$. Let $(x_{\alpha,\beta})$ be a double sequence of distinct points of A such that no point is a P-limit point of $(x_{\alpha,\beta})$. Then corresponding to each point $x_{p,q}$ of $(x_{p,q})$ there is a positive number $r_{p,q}$ such that $r(x_{p,q}, x_{i,j}) > 2r_{p,q}$ $p \neq i$ and $q \neq j$ where $r(x_{p,q}, x_{i,j})$ denote the distance between $x_{p,q}$ and $x_{i,j}$. Let the set of points x of A for which $r(x_{p,q}, x) < r_{p,q}$ be denoted by $A_{p,q}$. Therefore the double sequence of set are mutually exclusive subset of A. Now define the double sequence $(s_{p,q}(x))$ over A as follows for each (k,l) such that $M_{p-1} < k \leq M_p$ and $N_{q-1} < l \leq N_q$ for $p,q = 1,2,3,\ldots$

$$s_{k,l}(x) = \begin{cases} \operatorname{sqn}(a_{k,l}(s_{m_p}, t_{n_q})\left[1 - \frac{r(x_{p,q},x)}{r_{p,q}}\right], & \text{over} \quad A_{p,q}; \\ 0, & \text{over} \quad A - A_{p,q}. \end{cases}$$

Note

1. $s_{m,n}(x)$ is continuous over A for each (m,n);
2. $s_{m,n}(x)$ is bounded over A for each (m,n);
3. $\mathrm{P} - \lim_{m,n} s_{m,n}(x) = 0$ over A.

Therefore $(s_{m,n}(x))$ is an admissible double sequence. However

$$s_{k,l}(x_{p,q}) = \begin{cases} \operatorname{sqn}(a_{k,l}(s_{m_p}, t_{n_q}), & \text{for} \quad M_{p-1} < k \leq M_p \ \& \ N_{q-1} < l \leq N_q; \\ 0, & \text{for} \quad k \leq M_{p-1} \text{ and } k > M_p; \\ & \quad l \leq N_{q-1} \text{ and } l > N_q; \end{cases}$$

for each (p,q).

$$\begin{aligned} \sigma(s_{m_p}, t_{n_q}; x_{p,q}) &= \textstyle\sum_{M_{p-1}<k\leq M_p \ \& \ N_{q-1}<l\leq N_q} a_{k,l}(s_{m_p}, t_{n_q}) \operatorname{sgn} a_{k,l}(s_{m_p}, t_{n_q}; x_{p,q}) \\ &= \textstyle\sum_{M_{p-1}+1\leq k\leq M_p \ \& \ N_{q-1}+1\leq l\leq N_q} |a_{k,l}(s_{m_p}, t_{n_q})| \end{aligned}$$

Therefore by (3.5)

$$\mathrm{P} - \limsup_{p,q} \sigma(s_{m_p}, t_{n_q}; x_{p,q}) \geq \frac{\phi}{2}$$

Therefore (3.1) fails thus (3.2) holds. The sufficient part will follows from the following theorem. □

Theorem 3.2. *In order that (2.1) may be such that (3.1) is implied for every bounded double sequence of functions (3.2) is sufficient.*

Proof. Let (G) satisfy (3.2) and let M be a constant such that $|S_{m,n}(x)| < M$ over N for all (m,n) then

$$\begin{aligned} |\sigma(s,t;x)| &\leq \textstyle\sum_{k,l=1,1}^{\infty,\infty} |a_{k,l}(s,t)||s_{k,l}(x)| \\ &\leq M \textstyle\sum_{k,l=1,1}^{\infty,\infty} |a_{k,l}(s,t)|) \end{aligned}$$

and (3.1) follows. □

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