Filomat 27:5 (2013), 761–766
DOI 10.2298/FIL1305761L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Products of composition and n-th differentiation operators from α-Bloch space to Q_p space

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Abstract. Let φ be an analytic self-map of the open unit disk D on the complex plane and $\alpha > 0, p \geq 0, n \in \mathbb{N}$. In this paper, the boundedness and compactness of the products of composition operators and *nth* differentiation operators $C_\varphi D^n$ from α-Bloch space B^α and B_0^α to Q_p space are investigated.

1. Introduction and preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary, $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$ and $\mathbb{N}$ the set of all nonnegative integers. For α is any positive real number, then the generalized Bloch space B^α of the unit disk $\mathbb{D}$ consists of analytic functions $f : \mathbb{D} \to \mathbb{C}$, such that

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha |f'(z)| < \infty.$$

For $f \in B^\alpha(\mathbb{D})$, define

$$\|f\|_{B^\alpha} = |f(0)| + \sup_{z\in D}(1 - |z|^2)^\alpha |f'(z)|.$$

Under the norm, $B^\alpha(\mathbb{D})$ is a Banach space. Note that $B^1(\mathbb{D})$ is the usual Bloch space, which was first considered by Arazy [1]. The little Bloch space B_0^α consists of all $f \in B^\alpha$

$$\lim_{|z|\to 1}(1 - |z|^2)^\alpha |f'(z)| = 0.$$

The characterizations of generalized Bloch space were studied by many researchers. For more details about B^α see [5, 15, 17].

For $a \in \mathbb{D}$, $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$ is the Green's function in $\mathbb{D}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Mӧbius map of $\mathbb{D}$ interchanging the points zero and a. The space Q_p is defined as follows

$$Q_p = \left\{ f \in H(\mathbb{D}) : \|f\|_{Q_p} = (\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g^p(z, a) dA(z))^{\frac{1}{2}} < \infty, p \geq 0 \right\}.$$

2010 *Mathematics Subject Classification*. Primary 47B38
Keywords. analytic self-map, composition operator, *nth* differentiation operator, α-Bloch space, Q_p space, boundedness, compactness.
Received: 14 January 2012; Accepted: 15 Jun 2012
Communicated by Dragan S. Djordjević
Research supported by the NNSF of China (No. 11201127; 11271112), the Young Core Teachers Program of Henan Province (No. 2011GGJS-062), the Natural Science Foundation of Henan Province(No. 122300410110; 2010A110009) and the Foster Foundation of Henan Normal University (No. 2010PL01).
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Where $dA(z)$ denotes the normalized Lebesgue area measure, so that $A(\mathbb{D}) = 1$. $f \in H(\mathbb{D})$ belongs to $Q_{p,0}$, which is the little subspace of Q_p,

$$Q_{p,0} = \left\{ f \in H(\mathbb{D}) : \|f\|_{Q_{p,0}} = \lim_{|a|\to 1} \sup \int_{\mathbb{D}} |f'(z)|^2 g^p(z,a) dA(z) = 0, p \geq 0 \right\}.$$

Spaces Q_p and $Q_{p,0}$ have attracted a lot of attention in recent years, see examples [12, 13, 16]. Moreover motivated by the theory of Q_p, $Q_K(p,q)$ is recently introduced in [11–13].

Let φ denote a nonconstant analytic self-map of $\mathbb{D}$. Associated with φ, a linear composition operator $C_\varphi f = f \circ \varphi$ is induced for $f \in H(\mathbb{D})$. Thus, lots of attentions have been attracted to solve the problem of characterizing the boundedness and compactness of composition operators on many Banach spaces of analytic functions, see examples [2, 7].

Let D be the differentiation operator $Df = f'$ and n be a nonnegative integer, we have $D^n f = f^{(n)}$, $f \in H(\mathbb{D})$. The differentiation operator is typically unbounded on many analytic function spaces. The products of composition operator and *nth* differentiation operator are defined as following:

$$C_\varphi D^n f = f^{(n)} \circ \varphi, f \in H(\mathbb{D}).$$

If $n = 0$, we get the linear composition operator. If $n = 1$, we get $C_\varphi D$, which was studied in [4, 8].

In [12], J. Xiao studied the composition operator mapping B^α to Q_s. In [6], Li considered composition operator mapping generally weighted Bloch space and $Q^q_{\log}$. In [14], Yang, Xu and Marko Kotilainen introduced the boundedness and compactness of composition operator between Bloch type spaces to Q_K type spaces.

Motivated by [4] and the definition of the weighted differentiation composition operator, denoted by $D^n_{\varphi,u} f(z) = u(z) f^{(n)}(\varphi(z))$ in [10]. The purpose of this paper is to characterize the products of composition operator and *nth* differentiation operator from B^α to Q_p, that is $C_\varphi D^n : B^\alpha \to Q_p$, where $n \in \mathbb{N}$. The sufficient and necessary conditions for the boundedness and the compactness of $C_\varphi D^n$ are given.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leq A \leq CB$.

2. Main results

Based on a result from [9], in [3] the authors proved the following result.

Lemma 2.1. Suppose that $\alpha \in (0, \infty)$. Then there exist two holomorphic functions $f, g \in B^\alpha$ such that

$$|f'(z)| + |g'(z)| \geq \frac{C}{(1 - |z|^2)^\alpha},$$

for all $z \in \mathbb{D}$.

Lemma 2.2. Suppose that $\alpha \in (0, \infty)$, for any positive integer n, there exist two functions $f, g \in B^\alpha$ such that

$$|f^{(n)}(z)| + |g^{(n)}(z)| \geq \frac{C}{(1 - |z|^2)^{\alpha+n-1}}.$$

Proof. From Lemma 2.1, there exist two functions $f, g \in B^\alpha$ such that

$$|f'(z)| + |g'(z)| \geq \frac{C}{(1 - |z|^2)^\alpha}.$$

By the following well-known characterization for B^α, see Proposition 8 of [17],

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| \asymp \sum_{j=0}^{n-1} |f^{(j)}(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^{\alpha+n-1} |f^{(n)}(z)|.$$

We know that

$$|f'(z)| \leq C(1-|z|^2)^{n-1}|f^{(n)}(z)|,$$
$$|g'(z)| \leq C(1-|z|^2)^{n-1}|g^{(n)}(z)|.$$

It follows that

$$|f^{(n)}(z)| + |g^{(n)}(z)| \geq \frac{C}{(1-|z|^2)^{\alpha+n-1}}.$$

Lemma 2.3. Let n be a nonnegative integer. Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic, $\alpha > 0, p \geq 0$. Then $C_\varphi D^n : B^\alpha \to Q_p$ is compact if and only if $C_\varphi D^n : B^\alpha \to Q_p$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in B^α which converges to zero uniformly on compact subsets of $\mathbb{D}$, $\|C_\varphi D^n f_k\|_{Q_p} \to 0$ as $k \to \infty$.

Proof. This characterization of compactness can be proved in a standard way, see [2], so we omit the proof.

Theorem 2.4. Let n be a nonnegative integer. Assume that $p \geq 0$, $\alpha > 0$, $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic, then the following statements are equivalent:

(1) $C_\varphi D^n : B^\alpha \to Q_p$ is bounded;

(2) $C_\varphi D^n : B_0^\alpha \to Q_p$ is bounded;

(3) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(\alpha+n)}} g^p(z,a)dA(z) < \infty$.

Proof. (1) $\Rightarrow$ (2) is clearly true, since $B_0^\alpha \in B^\alpha$.

Suppose (2) holds, then there exists a constant C such that for all $f \in B_0^\alpha$, $\|C_\varphi D^n f\|_{Q_p} \leq C\|f\|_{B_0^\alpha}$.

Given $f \in B^\alpha, g \in B^\alpha$, the function $f_t(z) = f(tz) \in B_0^\alpha, g_t(z) = g(tz) \in B_0^\alpha$, where $0 < t < 1$, since the property $\|f_t\|_{B^\alpha} \leq \|f\|_{B^\alpha}, \|g_t\|_{B^\alpha} \leq \|g\|_{B^\alpha}$. By Lemma 2.2,

$$\begin{aligned}
\infty &> \sup_{a\in\mathbb{D}} 2\int_{\mathbb{D}} [|(f^{(n)}\circ\varphi)'(z)|^2 + |(g^{(n)}\circ\varphi)'(z)|^2] g^p(z,a)dA(z) \\
&\geq \sup_{a\in\mathbb{D}} 2\int_{\mathbb{D}} [|(f_t^{(n)}\circ\varphi)'(z)|^2 + |(g_t^{(n)}\circ\varphi)'(z)|^2] g^p(z,a)dA(z) \\
&\geq \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} [|(f_t^{(n)}\circ\varphi)'(z)| + |(g_t^{(n)}\circ\varphi)'(z)|]^2 g^p(z,a)dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} [|(f_t^{(n+1)}\circ\varphi)(z)| + |(g_t^{(n+1)}\circ\varphi)(z)|]^2 |\varphi'(z)|^2 g^p(z,a)dA(z) \\
&\geq C\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|t\varphi'(z)|^2}{(1-|\varphi(tz)|^2)^{2(\alpha+n)}} g^p(z,a)dA(z).
\end{aligned}$$

The above estimate together with the Fatou's lemma, then (3) holds.

Next, we show that (3) implies (1), let $f \in B^\alpha$, then using the following equation

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| \asymp \sum_{j=0}^{n-1}|f^{(j)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+n-1}|f^{(n)}(z)|.$$

We can get

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f^{(n)}(\varphi(z)))'|^2 g^p(z,a)dA(z) \\
=\ &\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n+1)}(\varphi(z))|^2|\varphi'(z)|^2 g^p(z,a)dA(z) \\
\leq\ &\|f\|_{B^\alpha}^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(\alpha+n)}} g^p(z,a)dA(z).
\end{aligned}$$

Hence, for $f \in B^\alpha$ (1) follows by (3).

Theorem 2.5. Let n be a nonnegative integer. Assume that $p \geq 0$, $\alpha > 0$, $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic, then the following statements are equivalent:
(1) $C_\varphi D^n : B^\alpha \to Q_p$ is compact;
(2) $C_\varphi D^n : B_0^\alpha \to Q_p$ is compact;
(3) $\varphi \in Q_p$ and

$$\lim_{r\to 1} \sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>r\}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(\alpha+n)}} g^p(z,a)dA(z) = 0. \tag{2.1}$$

Proof. Obviously (1) implies (2). Suppose (2) holds, then the operator is bounded, from this and since

$$g_n(z) = \frac{z^{n+1}}{(n+1)!} \in B_0^\alpha.$$

It follows that

$$\|\varphi\|_{Q_p} = \|C_\varphi D^n(g_n)\|_{Q_p} \leq \|C_\varphi D^n\|_{B_0^\alpha \to Q_p} \|g_n\|_{Q_p}.$$

That is $\varphi \in Q_p$. Set

$$f_k(z) = \frac{z^{k+n}}{((k+n)!/(k-1)!)(1-(1-1/k)^2)^\alpha}.$$

From the definition of B_0^α, we know $f_k \in B_0^\alpha$, where $k \in \mathbb{N}$. Moreover there is a positive constant C such that $\|f_k\|_{B_0^\alpha} \leq C$ and $f_k(z) \to 0$ locally uniformly on the unit disk as $k \to \infty$. Then by the compactness of $C_\varphi D^n$, $\|C_\varphi D^n f_k\|_{Q_p} \to 0$, $k \to \infty$. This means that for $\forall\ \varepsilon > 0$, $\exists\ k_0 \in \mathbb{N}$ for all $k \geq k_0$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi^{k-1}(z)|^2}{(1-(1-1/k)^2)^{2\alpha}} |\varphi'(z)|^2 g^p(z,a)dA(z) < \varepsilon.$$

Thus for $0 < r < 1$

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \frac{1}{(1-(1-1/k_0)^2)^{2\alpha}} \int_{\mathbb{D}} |\varphi^{k_0-1}(z)|^2 |\varphi'(z)|^2 g^p(z,a)dA(z) \\
\geq\ & \frac{r^{2(k_0-1)}}{(1-(1-1/k_0)^2)^{2\alpha}} \sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>r\}} |\varphi'(z)|^2 g^p(z,a)dA(z).
\end{aligned}$$

Taking $\frac{r^{2(k_0-1)}}{(1-(1-1/k_0)^2)^{2\alpha}} > 1$, we get

$$\sup_{a\in\mathbb{D}} \int_{\{|\varphi(z)|>r\}} |\varphi'(z)|^2 g^p(z,a)dA(z) < \varepsilon. \tag{2.2}$$

Now let f with $\|f\|_{B_0^\alpha} \leq 1$, we consider the functions $f_t(z) = f(tz)$, $t \in (0,1)$. Thus $f_t \in \mathbb{B}_{B_0^\alpha}$, $f_t \to f$ locally uniformly on $\mathbb{D}$ as $t \to 1$. By the compactness of $C_\varphi D^n$, $\|C_\varphi D^n f - C_\varphi D^n f_t\|_{Q_p} \to 0$ as $t \to 1$. Then $\forall\ \varepsilon > 0$, $\exists\ t_0 \in (0,1)$, $\forall\ t > t_0$

$$\sup_{\alpha\in\mathbb{D}} \int_{\mathbb{D}} |(f^{(n)} \circ \varphi)'(z) - (f_t^{(n)} \circ \varphi)'(z)|^2 g^p(z,a)dA(z) < \varepsilon.$$

Then we fix t, the triangle inequality and (2.2) give,

$$\begin{aligned} &\sup_{a\in\mathbb{D}}\int_{\{|\varphi(z)|>r\}}|(f^{(n)}\circ\varphi)'(z)|^2 g^p(z,a)dA(z)\\ \leq\ & \sup_{a\in\mathbb{D}}\int_{\{|\varphi(z)|>r\}}|(f^{(n)}\circ\varphi)'(z)-(f_t^{(n)}\circ\varphi)'(z)|^2 g^p(z,a)dA(z)\\ &+\sup_{a\in\mathbb{D}}\int_{\{|\varphi(z)|>r\}}|(f_t^{(n)}\circ\varphi)'(z)|^2 g^p(z,a)dA(z)\\ \leq\ & \varepsilon+\|f_t^{(n+1)}\|_{H^\infty}^2\sup_{a\in\mathbb{D}}\int_{\{|\varphi(z)|>r\}}|\varphi'(z)|^2 g^p(z,a)dA(z)\\ \leq\ & \varepsilon(1+\|f_t^{(n+1)}\|_{H^\infty}^2). \end{aligned} \tag{2.3}$$

Having in mind (2.2) and (2.3) we conclude that for each $f(z)\in\mathbb{B}_{B_0^\alpha}$ and $\varepsilon>0$, there is δ depending on f,ε, such that for $r\in[\delta,1)$,

$$\sup_{a\in\mathbb{D}}\int_{\{|\varphi(z)|>r\}}|(f^{(n)}\circ\varphi)'(z)|^2 g^p(z,a)dA(z)<\varepsilon. \tag{2.4}$$

Since $C_\varphi D^n$ is compact, it maps the unit ball of B_0^α to a relatively compact subset of Q_p. Thus for each $\varepsilon>0$ there exists a finite collection of functions $f_1,f_2,...,f_k$ in the unit ball of B_0^α, such that for each $\|f\|_{B_0^\alpha}\leq 1$ there is a $k_0\in\{1,2,...,k\}$ with

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(f^{(n)}\circ\varphi)'(z)-(f_{k_0}^{(n)}\circ\varphi)'(z)|^2 g^p(z,a)dA(z)<\varepsilon.$$

By (2.4), we get that for $\delta=\max_{1\leq j\leq k}\delta(f_j,\varepsilon)$ and $r\in[\delta,1)$,

$$\sup_{a\in\mathbb{D}}\int_{\{|\varphi(z)|>r\}}|(f_k^{(n)}\circ\varphi)'(z)|^2 g^p(z,a)dA(z)<\varepsilon.$$

Thus we get that

$$\sup_{a\in\mathbb{D}}\int_{\{|\varphi(z)|>r\}}|(f^{(n)}\circ\varphi)'(z)|^2 g^p(z,a)dA(z)<2\varepsilon.$$

So we can shown that for any $\varepsilon>0$, there exists $\delta\in[0,1)$ such that for all f in the unit ball of B_0^α

$$\sup_{a\in\mathbb{D}}\int_{\{|\varphi(z)|>r\}}|(f^{(n)}\circ\varphi)'(z)|^2 g^p(z,a)dA(z)<2\varepsilon.$$

By Lemma 2.2 and the Fatou's Lemma, (2.1) holds.

To prove (3) $\Rightarrow$ (1), we assume that $\varphi\in Q_p$ and (2.1) holds. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions in the unit ball of B^α, such that $\sup_{k\in\mathbb{N}}\|f_k\|_{B^\alpha}<\infty$ and $f_k\to 0$ as $k\to\infty$, uniformly on the compact subsets of the unit disk.

Let $r\in(0,1)$, then

$$\begin{aligned} \|C_\varphi D^n f_k\|_{Q_p}^2 &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(f_k^{(n)}\circ\varphi)'(z)|^2 g^p(z,a)dA(z)\\ &= \sup_{a\in\mathbb{D}}\int_{\{|\varphi(z)|\leq r\}}|(f_k^{(n)}\circ\varphi)'(z)|^2 g^p(z,a)dA(z)\\ &+ \sup_{a\in\mathbb{D}}\int_{\{|\varphi(z)|>r\}}|(f_k^{(n)}\circ\varphi)'(z)|^2 g^p(z,a)dA(z)\\ &= I_1+I_2. \end{aligned}$$

Since $f_k \to 0$ as $k \to \infty$, uniformly on $\mathbb{D}$ on the compact subsets of the unit disk along with Cauchy's estimate gives that $f_k^{(n+1)} \to 0$ on compact subsets of $\mathbb{D}$ as $k \to \infty$. Letting $k \to \infty$ in I_1, using the fact that ε is an arbitrary positive number and by the assumption $\varphi \in Q_p$, we obtain that $I_1 \leq \varepsilon \|\varphi\|^2_{Q_p}$.

On the other hand,

$$I_2 \leq \|f_k\|^2_{B^\alpha} \int_{\{|\varphi(z)|>r\}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(\alpha+n)}} g^p(z,a)dA(z).$$

By (2.1), it follows that $I_2 \leq \varepsilon$. From the above proof, then we get $\|C_\varphi D^n f_k\|_{Q_p} \to 0$ as $k \to \infty$. so $C_\varphi D^n : B^\alpha \to Q_p$ is compact. The proof is completed.

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