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Consistent in invertibility operators and SVEP

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Abstract. If $S(\mathcal{X}) \subset B(\mathcal{X})$, where $B(\mathcal{X})$ denotes the algebra of operators on a Banach space $\mathcal{X}$, then $A \in B(\mathcal{X})$ is $S(\mathcal{X})$ consistent if $AB \in S(\mathcal{X}) \Longleftrightarrow BA \in S(\mathcal{X})$ for every $B \in B(\mathcal{X})$. SVEP is a powerful tool in determining the $S(\mathcal{X})$ consistency of operators A for various choices of the subset $S(\mathcal{X})$.

1. Introduction

Let $B(\mathcal{X})$ denote the algebra of operators (i.e., bounded linear transformations) on a Banach space $\mathcal{X}$, and let $A \in B(\mathcal{X})$. Then the spectra of AB and BA satisfy the equality $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$ for every $B \in B(\mathcal{X})$. This equality extends to a large number of the "more distinguished parts of the spectrum", such as the point spectrum σ_p, the approximate point spectrum σ_a, the Fredholm spectrum σ_e (etc.) [3, 4, 7]. Simple examples, such as U^*U and UU^* where U is the forward unilateral shift, show that the equality can not be extended to include the point 0, and this gives rise to the question of finding conditions, necessary and/or sufficient, under which this equality extends to $\sigma_x(AB) = \sigma_x(BA)$ where σ_x is σ or a distinguished part thereof. Let $S(\mathcal{X})$ denote a subset of $B(\mathcal{X})$. An operator $A \in B(\mathcal{X})$ is said to be $S(\mathcal{X})$ consistent, or consistent in $S(\mathcal{X})$, if $AB \in S(\mathcal{X}) \Longleftrightarrow BA \in S(\mathcal{X})$ for every $B \in B(\mathcal{X})$. In general one considers sets $S(\mathcal{X})$ determined by a regularity: thus $S(\mathcal{X})$ may consist of invertible (left, right or both) or Fredholm (upper, lower or both), or Browder, or Weyl (etc.) elements in $B(\mathcal{X})$. A study of such a "consistency in regularity", extending the work of Gong and Han [6], has been carried out by Djordjević [5].

Recall that $A \in B(\mathcal{X})$ has SVEP, the single-valued extension property, at a point $\lambda_0 \in \mathbb{C}$ if for every open neighbourhood $\mathcal{U}_{\lambda_0}$ of λ_0 the only analytic function $f : \mathcal{U}_{\lambda_0} \longrightarrow \mathcal{X}$ satisfying $(A - \lambda)f(\lambda) = 0$ for every $\lambda \in \mathcal{U}_{\lambda_0}$ is the function $f \equiv 0$. Evidently, A has SVEP at every point of the resolvent set $\rho(A)$ of A.

SVEP provides a simple sufficient condition for determining $S(\mathcal{X})$ consistent operators $A \in B(\mathcal{X})$ for a variety of choices of the subset $S(\mathcal{X})$ of $B(\mathcal{X})$: Sufficient for A to be consistent in invertibility is that either both A and A^* have SVEP, or neither of A and A^* has SVEP, at 0, and sufficient for A to be Fredholm (or Browder, or Weyl) consistent is that either both A and A^* have essential SVEP, or neither of A and A^* has essential SVEP, at 0 (the notion of essential SVEP is defined in the following section). It follows in particular that if an operator $A \in B(\mathcal{X})$ is either decomposable or an invertible isometry or a Riesz or a

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meromorphic operator (more generally, an operator with countable spectrum or such that its spectrum has empty interior) or a Fredholm operator with index 0 or a Drazin invertible operator, then A is consistent in invertibility. We prove in the following that: (i) a necessary and sufficient condition for A to be inconsistent in invertibility is that either A is left invertible and A^* does not have SVEP at 0 or A is right invertible and A does not have SVEP at 0 (equivalently, if and only if A is not invertible and A is either left or right invertible); (ii) A is consistent in left invertibility if and only if the conditions "if A is left invertible then A^* has SVEP at 0 and if AB is left invertible for some $B \in B(\mathcal{X})$ then B^* has SVEP at 0" are satisfied; ; (iii) a necessary and sufficient condition for A to be Fredholm (or Browder, or Weyl) inconsistent is that A is not essentially invertible and A is either left or right essential invertible. Extending these ideas to upper triangular operator matrices $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in B(\mathcal{X} \oplus \mathcal{X})$ it is proved that the consistency spectrum σ_{CI} satisfies $\sigma_{CI}(M_C) \cup \{\sigma_{CI}(A) \cap \sigma_{CI}(B)\} \cup \{\sigma_{CI}(A) \setminus \sigma_{CI}(B) \cup \sigma_{CI}(B) \setminus \sigma_{CI}(A)\} = \sigma_{CI}(A) \cup \sigma_{CI}(B)$.

2. Results

We say that the operator $A \in B(\mathcal{X})$ has SVEP if it has SVEP everywhere. Recall, [1, Corollary 2.24], that a necessary and sufficient condition for a surjective operator $A \in B(\mathcal{X})$ to be invertible is that A has SVEP at 0. Let $\Xi(A) = \{\lambda \in \sigma(A) : A$ does not have SVEP at $\lambda\}$, and let $\Xi(A)^C = \sigma(A) \setminus \Xi(A)$. Let Inv, $\mathrm{Inv}^{\mathrm{l}}$ and $\mathrm{Inv}^{\mathrm{r}}$ denote, respectively, the class of $A \in B(\mathcal{X})$ such that A is invertible, A is left invertible and A is right invertible.

If $S(X) \subset B(\mathcal{X})$ consists of the invertibles and $A \in B(\mathcal{X})$ is $S(\mathcal{X})$ consistent, then we say that A is a "consistent in invertibility operator", or a CI-operator. SVEP provides a simple sufficient condition for an operator to be a CI-operator.

Theorem 2.1. *(i) Sufficient for an operator $A \in B(\mathcal{X})$ to be a CI-operator is that either $0 \in \Xi(A) \cap \Xi(A^*)$ or $0 \in \Xi(A)^C \cap \Xi(A^*)^C$.*

(ii) Necessary and sufficient for $A \in B(\mathcal{X})$ to be inconsistent in invertibility is that one of the following (exclusive) conditions holds:

(a) $A \in Inv^l \cap 0 \in \Xi(A^*)$; (b)$A \in Inv^r \cap 0 \in \Xi(A)$.

Proof. (i). Suppose that both A and A^* have SVEP. Start by assuming that AB is invertible for some $B \in B(\mathcal{X})$.Then there exists an operator $S \in B(\mathcal{X})$ such that $SAB = I = ABS$. The equality $ABS = I$ implies that A is right invertible, hence surjective. Since A has SVEP at 0, A is invertible, and then $ABS = I \Longrightarrow BSA = I$ which implies that B is surjective. Already we have from $SAB = I$ that B is left invertible. Hence B is also invertible. But then BA is invertible, i.e., AB invertible implies BA invertible. Now let BA be invertible. Then there exists an operator $T \in B(\mathcal{X})$ such that $TBA = I = BAT$. Evidently, $TBA = I$ implies A is left invertible. Hence A^* is surjective and has SVEP at 0, and so is invertible. Consequently, $TBA = I \Longrightarrow ATB = I \Longrightarrow B$ is left invertible and $BAT = I \Longrightarrow B$ is surjective, consequently invertible. Hence BA invertible implies AB invertible.

Suppose next that neither A nor A^* has SVEP at 0. Then A is neither left nor right invertible. This implies that there can not exist operators S and T in $B(\mathcal{X})$ satisfying either of the equalities $ABS = I$ and $TBA = I$. Hence neither of AB and BA is invertible in this case.

(ii) Evidently, $A \notin CI$ if and only if there exists a $B \in B(\mathcal{X})$ such that $AB \in$ Inv (resp., $BA \in$ Inv) and $BA \notin$ Inv (resp., $AB \notin$ Inv), and this happens if and only if there exists a $B \in B(\mathcal{X})$ such that $A \in \mathrm{Inv}^{\mathrm{r}}$, $B \in \mathrm{Inv}^{\mathrm{l}}$, $A \notin \mathrm{Inv}^{\mathrm{l}}$ and $B \notin \mathrm{Inv}^{\mathrm{r}}$ (resp., $A \in \mathrm{Inv}^{\mathrm{l}}$, $B \in \mathrm{Inv}^{\mathrm{r}}$, $A \notin \mathrm{Inv}^{\mathrm{r}}$ and $B \notin \mathrm{Inv}^{\mathrm{l}}$). Observe that if $A \notin$ Inv and $A \in \mathrm{Inv}^{\mathrm{r}}$ (resp., $A \in \mathrm{Inv}^{\mathrm{l}}$), then the right (resp., left) inverse A_r (resp., A_l) of A is the required operator B in the implication above; observe also that if there exists a B satisfying the implication above, then $A \in \mathrm{Inv}^{\mathrm{l}}$ or $A \in \mathrm{Inv}^{\mathrm{r}}$ and $A \notin$ Inv (for the reason that if $A \in$ Inv, then B is invertible and $AB \in \mathrm{Inv} \Longleftrightarrow \mathrm{BA} \in \mathrm{Inv}$). Hence $A \notin CI \Longleftrightarrow A \notin$ Inv, A $\in \mathrm{Inv}^{\mathrm{l}}$ or $A \in \mathrm{Inv}^{\mathrm{r}}$. (Contra positively, $A \in CI \Longleftrightarrow A \in$ Inv or $A \notin \mathrm{Inv}^{\mathrm{r}} \cap \mathrm{Inv}^{\mathrm{l}}$.) Since

$A \in \text{Inv}$ if and only if $A \in \text{Inv}^l$ and $0 \in \Xi(A^*)^C$, or $A \in \text{Inv}^r$ and $0 \in \Xi(A)^C$, we have now that the following two way implications hold:

$$\begin{aligned} A \notin CI &\iff A \notin \text{Inv}, \text{A} \in \text{Inv}^l \ \text{ or } \ \text{A} \in \text{Inv}^r \ \text{ (but not both)} \\ &\iff A \in \text{Inv}^l \cap 0 \in \Xi(\text{A}^*) \ \text{ or } \ \text{A} \in \text{Inv}^r \cap 0 \in \Xi(\text{A}). \end{aligned}$$

This completes the proof. □

Remark 2.2. (i) The condition of Theorem 2.1(i) is not necessary. To see this, let $A \in B(\mathcal{X})$ be a bounded below operator which is not left invertible. (Thus, $A(\mathcal{X})$ is not complemented in $\mathcal{X}$.) Then A has SVEP at 0 and A^* does not have SVEP at 0 (for if A^* were to have SVEP at 0 then the surjective operator A^* would be invertible). Evidently, both AB and BA are not invertible for any $B \in B(\mathcal{X})$. Hence $A \in CI$. Since a bounded below Hilbert space operator is left invertible, this argument fails for Hilbert space operators A.
(ii) Note that $A \in \text{Inv}^l \cap 0 \in \Xi(\text{A}^*) \iff \text{A} \in \text{Inv}^l$ and $A^{*-1}(0) \neq \{0\}$ and $A \in \text{Inv}^r \cap 0 \in \Xi(\text{A}) \iff \text{A} \in \text{Inv}^r$ and $A^{-1}(0) \neq \{0\}$. A more user friendly version of the necessary and sufficient condition of Theorem 2.1(ii), which we shall have occasion to use below, is the following: $A \notin CI \iff A \in \text{Inv}^l \setminus \text{Inv}$ or $A \in \text{Inv}^r \setminus \text{Inv}$.

It is immediate from the above that the following classes of operators are *CI*-operators: Operators $A \in B(\mathcal{X})$ such that $\text{int}\sigma(A) = \emptyset$, normal operators $A \in B(\mathcal{X})$, decomposable operators in $B(\mathcal{X})$, isometries $A \in B(\mathcal{X})$ such that A^* has SVEP (this forces $\|Ax\| = \|x\| = \|A^{-1}x\|$ for all $x \in \mathcal{X}$ and $\|A\| = \|A^{-1}\|$), Riesz operators and meromorphic operators in $B(\mathcal{X})$. Recall that $A \in B(\mathcal{X})$ has a generalized Drazin inverse if and only if $0 \in \text{iso}\sigma(A)$. Thus generalized Drazin invertible operators are *CI*-operators. An operator $A \in B(\mathcal{X})$ is *semi-regular* if $A(\mathcal{X})$ is closed and $A^{-1}(0) \subseteq A^m(\mathcal{X})$ for all positive integers m [1, P7]. For a Banach space operator A such that 0 is an isolated point of $\sigma(A)$, both A and A^* have SVEP at 0; hence such an A is a *CI*-operator. Recall, [1, Theorem 2.49], that a semi-regular operator $A \in B(\mathcal{X})$ has SVEP at 0 (resp., A^* has SVEP at 0) if and only if it is bounded below (resp., surjective), and then $0 \in \text{iso}\sigma_a(A)$ (resp, $0 \in \text{iso}\sigma_a(A^*)$). Thus a semi-regular operator A such that $0 \in \text{iso}\sigma_a(A)$ or $0 \in \text{iso}\sigma_a(A^*)$ implies $0 \in \text{iso}\sigma(A)$ is *CI*. Hyponormal operators $A \in B(\mathcal{H})$ (more, generally, M-hyponormal operators $A \in B(\mathcal{H})$), $AA^* \leq A^*A$ (resp., $(A-\lambda)(A-\lambda)^* \leq M(A-\lambda)^*(A-\lambda)$ for all complex λ and some $M \geq 1$), have SVEP [1, p 170]; hence a hyponormal (resp., M-hyponormal) operator is a *CI*-operator if and only if either A^* has SVEP at 0, or, if A^* does not have SVEP at 0 then A is not bounded below (*cf.* [6, 2.1, 2.2 and 2.3]).

Not every Fredholm operator is a *CI*-operator: consider the unilateral shift above. However, Fredholm operators A such that $\text{ind}(A) = 0$ are *CI*-operators. (Such an operator is referred to as being Weyl at 0.)

Proposition 2.3. *If an operator $A \in B(\mathcal{X})$ is Fredholm and $ind(A) = 0$, then A is CI.*

Proof. We have two possibilities: either A has SVEP at 0 or A does not have SVEP at 0. If A has SVEP at 0, then A^* has SVEP at 0 (since A Weyl and A has SVEP implies $0 \in \text{iso}\sigma(A)$), which implies that A is *CI*. If, instead, A does not have SVEP at 0, then BA is not left invertible, hence not invertible, for every $B \in B(\mathcal{X})$. Again, if AB is right invertible for some $B \in B(\mathcal{X})$, then A^* has SVEP at 0 (which implies $0 \in \text{iso}\sigma(A)$, which in turn) implies A has SVEP at 0. Thus AB is not right invertible, hence not invertible. □

Remark 2.4. If $A \in \phi$ and $\text{ind}(A) = 0$, then either $\text{dimA}^{-1}(0)(= \dim(\mathcal{X} \setminus \text{A}\mathcal{X}) = 0$ or $\text{dimA}^{-1}(0) > 0$. If $\text{dimA}^{-1}(0) = 0$, then $A \in \text{Inv}$, and if $\text{dimA}^{-1}(0) > 0$, then $A \notin \text{Inv}^l \cap \text{Inv}^r$. This, by Theorem 2.1(ii) (see Remark 2.2(ii)), provides an alternative proof of Proposition 2.3.

The following corollary generalizes a result of Gong and Han [6, 2.5] to Banach spaces.

Corollary 2.5. *If $A \in B(\mathcal{X})$ is invertible, then $A + K$ is CI for every compact operator $K \in B(\mathcal{X})$.*

Proof. $A + K$ is Weyl. □

The following proposition extends our observation on Riesz operators being *CI* to perturbation of Riesz operators by commuting algebraic operators. Recall that $A \in B(\mathcal{X})$ is algebraic if there exists a non-constant polynomial $p(.)$ such that $p(A) = 0$.

Proposition 2.6. *If $A \in B(\mathcal{X})$ is an algebraic operator which commutes with a Riesz operator $R \in B(\mathcal{X})$, then $A + R$ is CI.*

Proof. We prove that $A + R$ and $A^* + R^*$ have SVEP. The operator A being algebraic, $\sigma(A) = \{\lambda_1, ..., \lambda_n\}$ for some integer $n \geq 1$, and

$$A = \oplus_{i=1}^{n} A_i = \oplus_{i=1}^{n} A|_{H_0(A-\lambda_i)} = \oplus_{i=1}^{n} A|_{(A-\lambda_i)^{-t_i}(0)}$$

for some integers $t_i \geq 1$. (Here $H_0(A) = \{x \in \mathcal{X} : \lim_{m\longrightarrow\infty} \|A^m x\|^{\frac{1}{m}} = 0\}$ is the quasi-nilpotent part of A.) The commutativity of A and R implies that

$$R = \oplus_{i=1}^{n} R_i = \oplus_{i=1}^{n} R|_{H_0(A-\lambda_i)},$$

where A_i commutes with R_i for all $1 \leq i \leq n$ and each R_i is a Riesz operator (implies both R_i and R_i^* have SVEP for all $1 \leq i \leq n$). Let $d_{CD} \in B(B(\mathcal{X}))$ denote the generalized derivation $d_{CD}(X) = CX - XD$. Set $R_i + A_i - \lambda_i = T_i$. Observe that the operator $A_i - \lambda_i$ is t_i-nilpotent. Hence

$$d^m_{T_i R_i}(I) = d^m_{R_i T_i}(I) = 0$$

for all $m \geq t_i$, which implies that T_i and R_i are quasinilpotent equivalent operators (see [8, Page 253]). Hence T_i has SVEP if and only R_i has SVEP [8, Proposition 3.4.11]. Thus $R_i + A_i$ has SVEP. Since the same argument works with T_i and R_i replaced by T_i^* and R_i^*, $R_i^* + A_i^*$ also has SVEP. But then $A + R = \oplus_{i=1}^{n} A_i + R_i$ and $A^* + R^* = \oplus_{i=1}^{n} A_i^* + R_i^*$ have SVEP. □

The "*consistent in invertibility spectrum of A*" is the set

$$\sigma_{CI}(A) = \{\lambda \in \sigma(A) : A - \lambda \notin CI\}.$$

Evidently, both A and A^* have SVEP at points in the boundary $\partial\sigma(A)$ of the spectrum of A. Hence $\sigma_{CI}(A) \subseteq \text{int}\sigma(\text{A})$ is an open subset of $\sigma(A)$ which satisfies $\sigma_{CI}(A) \subseteq \sigma_w(A) = \{\lambda \in \sigma(A) : A - \lambda \text{ is not Weyl}\}$ (Proposition 2.3). Furthermore, if we let $S_\ell(A) = \{\lambda \in \sigma(A) : A - \lambda$ is left invertible but A^* does not have SVEP at $\lambda\}$ and $S_r(A) = \{\lambda \in \sigma(A) : A - \lambda$ is right invertible but A does not have SVEP at $\lambda\}$, then it follows from Theorem 2.1 and the implications

$$\begin{aligned} &\lambda \in \sigma_{CI}(A) \iff \text{ either } A - \lambda \text{ is left invertible but not invertible} \\ &\text{or } A - \lambda \text{ is right invertible but not invertible} \\ \iff &\text{ either } A - \lambda \text{ is left invertible but } A^* \text{ does not have SVEP at } \lambda \\ &\text{or } A - \lambda \text{ is right invertible but } A \text{ does not have SVEP at } \lambda \end{aligned}$$

that

$$\begin{aligned} \sigma_{CI}(A) &= S_\ell(A) \cup S_r(A) \subseteq \{\lambda \in \sigma(A) : (\Xi(A) \cup \Xi(A^*)) \setminus (\Xi(A) \cap \Xi(A^*))\} \\ &= \{\lambda \in \sigma(A) : (\Xi(A) \cap \Xi(A^*)^C) \cup (\Xi(A)^C \cap \Xi(A^*))\}. \end{aligned}$$

Recall, [1, Theorem 2.39], that if $f : \mathcal{U} \to \mathbb{C}$ is an analytic function from an open neighbourhood $\mathcal{U}$ of $\sigma(A)$ such that f is non-constant on connected components of $\mathcal{U}$, $f \in H_c(\sigma(A))$, then $f(A)$ has SVEP at $\lambda \in \mathbb{C}$ if and only if A has SVEP at every $\mu \in \sigma(A)$ for which $f(\mu) = \lambda$. Furthermore, $\sigma_{CI}(f(A)) = S_\ell(f(A)) \cup S_r(f(A))$ (exclusive or) $\subseteq f(S_\ell(A)) \cup f(S_r(A)) = f(S_\ell(A) \cup S_r(A)) = f(\sigma_{CI}(A))$. Thus, $\sigma_{CI}(f(A)) \subseteq f(\sigma_{CI}(A))$ for every $f \in H_c(\sigma(A))$. The reverse inclusion fails, as follows upon considering a polynomial $p(.)$ and an operator A such that $0 \in p(\sigma_{CI}(A))$ and $0 \notin \sigma_{CI}(p(A))$ [6]. If $A = A_1 \oplus A_2 \in B(\mathcal{X} \oplus \mathcal{X})$, then A has SVEP at λ if and only if A_1 and A_2 have SVEP at λ, and A is left (resp., right) invertible if and only if A_1 and A_2 are left (resp., right) invertible. Hence

$$\sigma_{CI}(A_1 \oplus A_2) = \sigma_{CI}(A) = S_\ell(A) \cup S_r(A) \subseteq \sigma_{CI}(A_1) \cup \sigma_{CI}(A_2).$$

The inclusion can be proper: consider $A = U \oplus U^* \in B(\ell^2 \oplus \ell^2)$, where U is the forward unilateral shift, when it is seen that $\emptyset = \sigma_{CI}(A) \subset \sigma_{CI}(U) \cup \sigma_{CI}(U^*) =$ the interior of the unit disc $\mathbb{D}$.

The *Weyl essential approximate point spectrum* $\sigma_{aw}(A)$ of A,

$$\sigma_{aw}(A) = \{\lambda \in \sigma(A) : A - \lambda \text{ is not upper semi Fredholm or } \operatorname{ind}(A - \lambda) > 0\},$$

is a subset of (the Weyl spectrum $\sigma_w(A)$ and) the *Browder essential approximate point spectrum*

$$\sigma_{ab}(A) = \{\lambda \in \sigma(A) : A - \lambda \text{ is not upper semi Fredholm or } \operatorname{asc}(A - \lambda) = \infty\}$$

of A.

Theorem 2.7. $\sigma_{ab}(A) \cap \sigma_{CI}(A) \subseteq \sigma_{aw}(A) \subseteq \sigma_{ab}(A)$ *for every* $A \in B(\mathcal{X})$.

Proof. If $\lambda \notin \sigma_{aw}(A)$, then $A - \lambda$ is upper semi Fredholm and $\operatorname{ind}(A - \lambda) \leq 0$. Here either A^* has SVEP at λ or A^* does not have SVEP at λ. If A^* has SVEP at λ, then $\operatorname{ind}(A - \lambda) \geq 0$. This forces $A - \lambda$ to be Weyl and such that both A and A^* have SVEP at λ. Hence $\lambda \notin \sigma_{CI}(A)$ in this case. If, instead, A^* does not have SVEP at λ, then either A has SVEP at λ or A does not have SVEP at λ. Evidently, if both A and A^* do not have SVEP at λ, then $\lambda \notin \sigma_{CI}(A)$. If, on the other hand, A has SVEP (but A^* does not have SVEP) at λ, then $A - \lambda$ is upper semi Fredholm with $\operatorname{asc}(A - \lambda) < \infty$, i.e., $\lambda \notin \sigma_{ab}(A)$. Thus $\lambda \notin \sigma_{aw}(A) \Longrightarrow \lambda \notin (\sigma_{ab}(A) \cap \sigma_{CI}(A))$. The remaining inclusion being well known [1], the proof is complete. □

Theorem 2.7 implies that if $\sigma_{ab}(A) \subseteq \sigma_{CI}(A)$ for an operator $A \in B(\mathcal{X})$, then $\sigma_{ab}(A) = \sigma_{aw}(A)$: operators A satisfying the equality $\sigma_{ab}(A) = \sigma_{aw}(A)$ have been described as satisfying a-Browder's theorem in the literature [1].

Left, right multiplication. Let L_A and $R_A \in B(B(\mathcal{X}))$ denote the operators $L_A(X) = AX$ and $R_A(X) = XA$, respectively. SVEP transfers both ways from A to L_A and R_{A^*}. Recall that $\sigma(L_A) = \sigma(R_A) = \sigma(A)$.

Lemma 2.8. *For an operator* $A \in B(\mathcal{X})$, *A (resp., A^*) has SVEP at μ if and only if L_A (resp., R_A) has SVEP at μ.*

Proof. We start by considering the left multiplication operator L_A. Suppose that A has SVEP at μ. Let $\mathcal{U}$ be an open neighbourhood of μ, and let $F(\lambda) : \mathcal{U} \longrightarrow B(\mathcal{X})$ be an analytic function such that $(L_A - \lambda)F(\lambda) = L_{A-\lambda}F(\lambda) = 0$ for all $\lambda \in \mathcal{U}$. The function $F(\lambda)x : \mathcal{U} \longrightarrow \mathcal{X}$ is analytic for every $x \in \mathcal{X}$ and satisfies $(A - \lambda)F(\lambda)x = 0$. This, if A has SVEP at μ, implies that $F(\lambda)x = 0$ for all $\lambda \in \mathcal{U}$; since this is true for all x, we must have $F(\lambda) \equiv 0$ in $\mathcal{U}$. Conversely, assume that L_A has SVEP at μ. For $\varphi \in \mathcal{X}^*$ and $y \in \mathcal{X}$, define the operator $\varphi \otimes y \in B(\mathcal{X})$ by setting $(\varphi \otimes y)(x) = \varphi(x)y$ for all $x \in \mathcal{X}$. Let $f; \mathcal{U} \longrightarrow \mathcal{X}$, $\mathcal{U}$ as above, be an analytic function such that $(A - \lambda)f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$. Then

$$(L_A - \lambda)(\varphi \otimes f(\lambda) = \varphi \otimes (A - \lambda)f(\lambda) = 0,$$

which (if L_A has SVEP at μ) implies that $\varphi \otimes f(\lambda) = 0$ on $\mathcal{U}$ for all $\varphi \in \mathcal{X}^*$. Hence $f(\lambda) \equiv 0$ on $\mathcal{U}$, i.e., A has SVEP at μ.

For the right multiplication operator R_A, we argue as follows. Clearly, $R_A - \mu = R_{A-\mu}$. Let $J : B(\mathcal{X}) \to B(\mathcal{X}^*)$ denote the isometric isomorphism defined by setting $J(T) = T^*$ for all $T \in B(\mathcal{X})$. Then J establishes the similarity $JR_{A-\mu} = L_{A^*-\mu I^*}J$. Since similarities preserve SVEP, $R_{A-\mu}$ has SVEP at 0 if and only if $L_{A^*-\mu I^*}$ has SVEP at 0, if and only if A^* has SVEP at μ. □

It is easily seen that L_A is left (resp., right) invertible if and only if A is left (resp., right) invertible, and R_A is left (rep., right) invertible if and only if A^* is left (resp., right) invertible. Evidently, $\sigma_{CI}(A) = \sigma_{CI}(A^*)$. Hence:

Theorem 2.9. $\sigma_{CI}(A) = \sigma_{CI}(L_A) = \sigma_{CI}(R_A)$ *for every* $A \in B(\mathcal{X})$.

Proof. In view of Lemma 2.8 and the observations above,

$$\sigma_{CI}(L_A) = S_l(L_A) \cup S_r(A) = S_l(A) \cup S_r(A) = \sigma_{CI}(A),$$

and

$$\sigma_{CI}(R_A) = S_l(R_A) \cup S_r(R_A) = S_l(A^*) \cup S_r(A^*) = \sigma_{CI}(A^*).$$

This completes the proof. □

Fredholm consistent operators. We say that an operator $A \in B(\mathcal{X})$ is Fredholm consistent, or ϕ-consistent, if for every $B \in B(\mathcal{X})$ either AB and BA are both Fredholm or neither of AB and BA is Fredholm. Fredholm consistent operators have been considered in [5]. Here our objective is to provide a characterization of Fredholm consistent operators which is similar in spirit to our characterization of consistent in invertibility operators. We start by introducing a construction, known as the Sadovskii/Buoni, Harte, Wickstead construction [9, Page 159], which leads to a representation of the Calkin algebra $B(\mathcal{X})/\mathcal{K}(\mathcal{X})$ as an algebra of operators on a suitable Banach space. Let $\ell^\infty(\mathcal{X})$ denote the Banach space of all bounded sequences $x = (x_n)_{n=1}^\infty$ of elements of $\mathcal{X}$ endowed with the norm $\|x\|_\infty := \sup_{n\in\mathbb{N}} \|x_n\|$, and write T_∞, $T_\infty x := (Tx_n)_{n=1}^\infty$ for all $x = (x_n)_{n=1}^\infty$, for the operator induced by T on $\ell^\infty(\mathcal{X})$. The set $m(\mathcal{X})$ of all precompact sequences of elements of $\mathcal{X}$ is a closed subspace of $\ell^\infty(\mathcal{X})$ which is invariant for T_∞. Let $\mathcal{X}_q := \ell^\infty \mathcal{X})/m(\mathcal{X})$, and denote by T_q the operator T_∞ on $\mathcal{X}_q$. The mapping $T \mapsto T_q$ is then a unital homomorphism from $B(\mathcal{X}) \to B(\mathcal{X}_q)$ with kernel $\mathcal{K}(\mathcal{X})$ which induces a norm decreasing monomorphism from $B(\mathcal{X})/\mathcal{K}(\mathcal{X})$ to $B(\mathcal{X}_q)$ with the following properties (see [9, Section 17] for details):

(*i*) T is upper semi-Fredholm, $T \in \phi_+$, if and only if T_q is injective, if and only if T_q is bounded below;

(*ii*) T is lower semi-Fredholm, $T \in \phi_-$, if and only if T_q is surjective;

(*iii*) T is Fredholm, $T \in \phi$, if and only if T_q is invertible.

The definition of SVEP obviously extends to the algebra $B(\mathcal{X}_q)$: we say in the following that $T \in B(\mathcal{X})$ has *essential SVEP* at a point if T_q has SVEP at the point. Observe that SVEP for T at a point neither implies, nor is implied, by essential SVEP for T at the point [2, Page 291].

Call an operator *essentially invertible (essentially left, respectively essentially right, invertible)* if $A_q \in \mathrm{Inv}$ ($A_q \in \mathrm{Inv}^l$, resp. $A_q \in \mathrm{Inv}^r$).

Theorem 2.10. *Let $A \in B(\mathcal{X})$.*

(i) Sufficient condition for A to be ϕ-consistent is that either both A and A^* have essential SVEP, or neither A nor A^* has essential SVEP, at 0.

(ii) Necessary and sufficient for A to be ϕ-inconsistent is that either A is right essentially invertible and $\dim(A^{-1}(0)) = \infty$ *or A is left essentially invertible and* $\dim(A^{*-1}(0)) = \infty$.

Proof. The proof of the theorem is similar to that of Theorem 2.1.

(*i*) If $AB \in \phi(\mathcal{X})$, then $(AB)_q = A_qB_q$ is invertible. This implies that if A_q has SVEP at 0, then A_q and B_q are invertible. In turn this implies that $B_qA_q = (BA)_q$ is invertible; hence $BA \in \phi(\mathcal{X})$. Similarly, if $BA \in \phi(\mathcal{X})$ and A_q^* has SVEP at 0, then $AB \in \phi(\mathcal{X})$. Observe that if neither of A_q and A_q^* has SVEP at 0, then A_q is neither left nor right invertible. Consequently, neither of A_qB_q and B_qA_q is invertible; hence neither of AB and BA is in $\phi(\mathcal{X})$.

(*ii*) Evidently, A is not ϕ-consistent if and only if there exists $B \in B(\mathcal{X})$ such that $AB \in \phi(\mathcal{X})$ (resp., $BA \in \phi(\mathcal{X})$) and $BA \notin \phi(\mathcal{X})$ (resp., $AB \notin \phi(\mathcal{X})$), i.e., if and only if there exists $B_q \in B(\mathcal{X}_q)$ such that $A_qB_q \in \mathrm{Inv}$ (resp., $B_qA_q \in \mathrm{Inv}$) and $B_qA_q \notin \mathrm{Inv}$ (resp., $A_qB_q \notin \mathrm{Inv}$); equivalently, A is not ϕ-consistent if and only if $A_q \notin CI$. Hence the following two way implications hold:

$$\begin{aligned} A \text{ is not } \phi-\text{consistent} &\iff A_q \in \mathrm{Inv}^l \setminus \mathrm{Inv} \text{ or } A_q \in \mathrm{Inv}^r \setminus \mathrm{Inv} \\ &\iff A \text{ is left essentially invertible and } \dim(A^{*-1}(0)) = \infty, \\ &\quad\text{or}\quad A \text{ is right essentially invertible and } \dim(A^{-1}(0)) = \infty. \end{aligned}$$

This completes the proof. □

Remark 2.11. (i) Taking the contrapositive of

$$A \text{ is not } \phi-\text{consistent} \iff A_q \in \mathrm{Inv}^l \setminus \mathrm{Inv} \text{ or } A_q \in \mathrm{Inv}^r \setminus \mathrm{Inv}$$

we have: A is ϕ-consistent if and only if either A is essentially invertible (equivalently, Fredholm), or, A is neither left nor right essentially invertible.

(ii) If we let $\sigma_\phi(A) = \{\lambda \in \sigma(A); A - \lambda \text{ is not } \phi\text{-consistent}\}$, then it follows from the argument above that $\sigma_\phi(A) = \sigma_{CI}(A_q)$. Hence $\sigma_\phi(A) \subseteq \{\lambda \in \sigma(A_q) : (\Xi(A_q) \cup \Xi(A_q^*)) \setminus (\Xi(A_q) \cap \Xi(A_q^*))\} = \{\lambda \in \sigma(A_q) : (\Xi(A_q) \cap \Xi(A_q^*)^C) \cup (\Xi(A_q)^C \cap \Xi(A_q^*))\}$.

$A \in B(\mathcal{X})$ is Browder if $A \in \Phi$ and asc(A) = dsc(A) $< \infty$; A is said to be Browder consistent, denoted $A \in (BC)$, if for every $B \in B(\mathcal{X})$ either both of AB and BA are Browder or neither of AB and BA is Browder. Fredholm consistency determines Browder coonsistency.

Theorem 2.12. *$A \in (BC) \Longleftrightarrow A$ is ϕ-consistent.*

Proof. Start by recalling that an operator $T \in B(\mathcal{X})$ such that asc(T) = dsc(T) $< \infty$ is said to be Drazin invertible, and that if AB is Drazin invertible then 0 is at worst in the resolvent of BA (i.e., either 0 is in the resolvent of BA or BA is Drazin invertible) [10, Theorem 3]. Thus asc(AB) = dsc(AB) $< \infty \Longleftrightarrow$ asc(BA) = dsc(BA) $< \infty$. The necessity is now obvious. To prove the sufficiency, observe that if neither of AB and BA is Fredholm for some B, then $A \in (BC)$; if both AB and BA are Fredholm, then (since by the argument above AB has finite ascent and descent if and only if BA has finite ascent and descent) again $A \in (BC)$. □

$A \in B(\mathcal{X})$ is *Weyl consistent*, $A \in (WC)$, if AB and BA are either both Weyl or neither is Weyl for every $B \in B(\mathcal{X})$. Observe that if AB and BA are Fredholm for all B, then $A, B \in \phi$ and ind(AB) = ind(BA); hence $A \in (WC) \Longleftrightarrow A$ is ϕ-consistent. A more revealing result is the following.

Theorem 2.13. *A sufficient condition for an operator $A \in B(\mathcal{X})$ to be (WC) is that $\alpha(A) = \infty \Longleftrightarrow \beta(A) = \infty$. Furthermore, if $\mathcal{X} = \mathcal{H}$ is a Hilbert space and $A(\mathcal{H})$ is closed, then this condition is necessary too.*

Proof. Start by observing that if $A(\mathcal{X})$ is not closed, then neither of A, AB and BA is Weyl (thus $A \in (WC)$); hence we may assume that $A(\mathcal{X})$ is closed. We have two possibilities: either $A(\mathcal{X}) = \mathcal{X}$ or $A(\mathcal{X}) \subset \mathcal{X}$. Let $A(\mathcal{X}) = \mathcal{X}$. If A has SVEP (at 0), then A is invertible, hence $A \in (WC)$. Assume therefore that A does not have SVEP. Since $\beta(A) = 0$ and A^* has SVEP, $\alpha(A) > \beta(A) = 0$. The hypothesis $\alpha(A) = \infty \Longleftrightarrow \beta(A) = \infty$ implies that $\alpha(A) < \infty$; hence A is Fredholm, which then forces AB and BA to be either Weyl or non-Weyl together. Hence $A \in (WC)$ in this case also. Assume now that $A(\mathcal{X}) \neq \mathcal{X}$. Since $\alpha(A) = \infty \Longleftrightarrow \beta(A) = \infty$, $A \in (WC)$ if either of $\alpha(A)$ or $\beta(A)$ is infinite; if neither of $\alpha(A)$ and $\beta(A)$ is infinite, then A is Fredholm, hence once again in (WC). This completes the proof of the sufficiency. To see the necessity, let $\mathcal{X} = \mathcal{H}$ be a Hilbert space and let $A(\mathcal{H}) = \overline{A(\mathcal{H})}$. If $\alpha(A) = \infty$ and $\beta(A) < \infty$ (the hypothesis $A(\mathcal{H})$ is closed is redundant in this case), then A is lower semi-Fredholm (but not Fredholm). Evidently, the operator A^*A is Weyl but the operator AA^* is not Weyl (not even Fredholm). Again, if $\alpha(A) < \infty$ and $\beta(A) = \infty$, then AA^* is Weyl but A^*A is not Weyl. Thus in either case $A \notin (WC)$. □

The inclusion $\sigma_{CI}(A) \subseteq \sigma_{aw}(A)$ fails. For example, if $A \in B(\mathcal{H})$ is the forward unilateral shift, then $0 \notin \sigma_{aw}(A)$ and $0 \in \sigma_{CI}(A)$. Observe that the forward unilateral shift has SVEP: if A does not have SVEP on $\sigma(A) \setminus \sigma_{aw}(A)$, then the inclusion does hold.

Theorem 2.14. *If $\sigma(A) \setminus \sigma_{aw}(A) \subseteq \Xi(A)$ for some $A \in B(\mathcal{X})$, then $\sigma_{CI}(A) \subseteq \sigma_{aw}(A)$.*

Proof. Take a point $\lambda \in \sigma(A)$ such that $\lambda \notin \sigma_{aw}(A)$. Then $\lambda \in \phi_+(A)$ and ind$(A - \lambda) \leq 0$. We have two cases: either A^* has SVEP at λ or A^* does not have SVEP at λ. The first of these cases is not possible: if A^* has SVEP at λ, then $\lambda \in \phi(A)$, ind$(A - \lambda) = 0$ and A^* has SVEP at λ, which implies that (0 is an isolated point of the spectrum of A, and so) A has SVEP at λ – a contradiction. If A^* does not have SVEP at λ, then neither of A and A^* has SVEP at λ, which by Theorem 2.1 implies that $\lambda \notin \sigma_{CI}(A)$. Hence $\sigma_{CI}(A) \subseteq \sigma_{aw}(A)$. □

Consistent in left (resp., right) invertibility, *CLI* (resp., *CRI*), operators. $A \in B(\mathcal{X})$ is a *CLI* (resp., *CRI*) operator if, for every $B \in B(\mathcal{X})$, either both AB and BA are left (resp., right) invertible or neither of them is left (resp., right) invertible. Choosing $B = I$, the left invertibility of A is a necessary condition for AB and BA to be left invertible for all $B \in B(\mathcal{X})$; again, choosing A to be left invertible and $B = A^*$, we see that A^* has SVEP is a necessary condition for AB and BA to be left invertible for all $B \in B(\mathcal{X})$. (Observe that if A is left invertible and A^* has SVEP, then A is invertible; consequently, $BA = A^*A$ is invertible.) The following theorem, *cf.* [5, Theorem 2.6], proves that these conditions are almost necessary and sufficient.

Theorem 2.15. *$A \in B(\mathcal{X})$ is a CLI operator if and only if the following conditions are satisfied:*
(a) $A \notin \mathrm{Inv}^l$ implies $AB \notin \mathrm{Inv}^l$ for every $B \in B(\mathcal{X})$.
(b) A left invertible implies A^* has SVEP at 0.
(c) AB left invertible for some $B \in B(\mathcal{X})$ implies B^* has SVEP at 0.

Proof. Sufficiency. Evidently, if (a) holds, then AB and BA are not left invertible for every $B \in B(\mathcal{X})$, and hence A is a CLI operator. If (b) holds, then $A \in \mathrm{Inv}$. Hence if $BA \in \mathrm{Inv}^l$ for some operator B, then $B \in \mathrm{Inv}^l$ and this forces AB to be left invertible; if, instead, $AB \in \mathrm{Inv}^l$, then $B \in \mathrm{Inv}^l$, implying thereby that $BA \in \mathrm{Inv}^l$. Finally, if (c) holds, then $B \in \mathrm{Inv}$; hence $A \in \mathrm{Inv}^l$, which then forces BA to be left invertible.

Necessity. Given an $A \in B(\mathcal{X})$, either $A \notin \mathrm{Inv}^l$ or $A \in \mathrm{Inv}^l \cap 0 \in \Xi(A^*)$ or $A \in \mathrm{Inv}^l \cap 0 \notin \Xi(A^*)$. Suppose that $A \notin CLI$. Then $A \in \mathrm{Inv}^l \cap 0 \notin \Xi(A^*)$ is not possible for the reason that then $A \in \mathrm{Inv}$ and hence $AB \in \mathrm{Inv}^l \Longleftrightarrow BA \in \mathrm{Inv}^l$. If $A \notin \mathrm{Inv}^l$, then $BA \notin \mathrm{Inv}^l$ for all $B \in B(\mathcal{X})$. Hence if $A \notin CLI$, then there exists a $B_0 \in B(\mathcal{X})$ such that $AB_0 \in \mathrm{Inv}^l$. Since $0 \notin \Xi(B^*)$ and $AB_0 \in \mathrm{Inv}^l$ implies $B_0 \in \mathrm{Inv} \Longrightarrow A \in \mathrm{Inv}^l \cap B_0 \in \mathrm{Inv} \Longrightarrow B_0A \in \mathrm{Inv}^l$, if $A \notin CLI$ then $0 \in \Xi(B^*)$ for every B such that $AB \in \mathrm{Inv}^l$. If, instead, $A \in \mathrm{Inv}^l$, then there exists an operator $B \in B(\mathcal{X})$ such that $B \notin \mathrm{Inv}^l$ and $BA \in \mathrm{Inv}^l$. Consequently, $A \notin CLI$ only if $AB \notin \mathrm{Inv}^l$, and this happens only if $0 \in \Xi(A^*)$. □

A duality arguments proves that $A \in CRI$ if and only if the following conditions are satisfied:
(a)′ $A \notin \mathrm{Inv}^r$ implies $BA \notin \mathrm{Inv}^r$ for every $B \in B(\mathcal{X})$.
(b)′ A right invertible implies A has SVEP at 0.
(c)′ BA right invertible for some $B \in B(\mathcal{X})$ implies B has SVEP at 0.

For an operator $T \in B(\mathcal{X})$, let $\sigma_{CLI}(T) = \{\lambda \in \sigma(T) : T - \lambda \notin CLI\}$ and $\sigma_{CRI}(T) = \{\lambda \in \sigma(T) : T - \lambda \notin CRI\}$ denote, respectively, the *consistent in left invertibility* and the *consistent in right invertibility* spectrum of T. Evidently, a sufficient condition for $0 \in \sigma_{CLI}(A)$ (resp., $0 \in \sigma_{CRI}(A)$) is that $A \in \mathrm{Inv}^l \setminus \mathrm{Inv}^r$ (resp., $A \in \mathrm{Inv}^r \setminus \mathrm{Inv}^l$.

Proposition 2.16. *$\sigma_{CI}(A) \subseteq \sigma_{CLI}(A) \cup \sigma_{CRI}(A)$ for every $A \in B(\mathcal{X})$. The reverse inclusion fails.*

Proof. Start by observing that to prove the inclusion it suffices to prove $0 \in \sigma_{CI}(A) \Longrightarrow 0 \in \sigma_{CLI}(A) \cup \sigma_{CRI}(A)$. Recall from the proof of Theorem 2.1(ii) that $0 \in \sigma_{CI}(A)$ if and only if $A \in (\mathrm{Inv}^l \cup \mathrm{Inv}^r) \setminus \mathrm{Inv} = (\mathrm{Inv}^l \setminus \mathrm{Inv}) \cup (\mathrm{Inv}^r \setminus \mathrm{Inv}) = (\mathrm{Inv}^l \setminus \mathrm{Inv}^r) \cup (\mathrm{Inv}^r \setminus \mathrm{Inv}^l)$. Hence $0 \in \sigma_{CLI}(A) \cup \sigma_{CRI}(R)$.

To see that the reverse inclusion fails, let $A = U \oplus U^*$, where $U \in B(\mathcal{H})$ is the forward unilateral shift. Then both A and A^* fail to have SVEP at 0. Hence $0 \notin \sigma_{CI}(A)$ by Theorem 2.1(i). Now let $B_1 = I \oplus U$ and $B_2 = U^* \oplus I$. Then $AB_1 \in \mathrm{Inv}^l$, $B_2A \in \mathrm{Inv}^r$, $B_1A \notin \mathrm{Inv}^l$ and $AB_2 \notin \mathrm{Inv}^r$. Hence $0 \in \sigma_{CLI}(A) \cap \sigma_{CRI}(A)$. □

The following corollary is immediate from Theorem 2.15.

Corollary 2.17. *$A \in B(\mathcal{X})$ is upper semi–Fredholm consistent, $A \in UFC$, if and only if the following conditions are satisfied:*
(a) A not upper semi-Fredholm implies AB not upper semi-Fredhom for every $B \in B(\mathcal{X})$.
(b) A upper semi-Fredholm implies A_q^* has SVEP at 0.
(c) AB upper semi-Fredholm for some $B \in B(\mathcal{X})$ implies B_q^* has SVEP at 0.

Similarly, A is lower semi-Fredholm consistent, $A \in LFC$, if and only if the following conditions are satisfied:
(a)′ A not lower semi-Fredholm implies BA not lower semi-Fredhom for every $B \in B(\mathcal{X})$.
(b) A lower semi-Fredholm implies A_q has SVEP at 0.
(c) BA lower semi-Fredholm for some $B \in B(\mathcal{X})$ implies B_q has SVEP at 0.

3. Application to upper triangular operator matrices.

If $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{B}(\mathcal{X} \oplus \mathcal{X})$ is an upper triangular operator matrix, then the spectra of A, B and M_C satisfy the following well known inclusion

$$\sigma(M_C) \subseteq \sigma(A) \cup \sigma(B).$$

This phenomenon persists for the consistency spectrum. Recall, that the operator $M_0 = A \oplus B \in \mathcal{B}(\mathcal{X} \oplus \mathcal{X})$ has SVEP at 0 if and only if A and B have SVEP at 0; M_C (resp., M_C^*) has SVEP at 0 implies A (resp., B^*) has SVEP at 0, and if A, B (resp., A^*, B^*) have SVEP at 0 then M_C (resp., M_C^*) has SVEP at 0. Observe that

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix},$$

where the operator $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ is invertible. Hence, the left (resp., right) invertibility of M_C implies the left (resp., the right) invertibility of A (resp., B).

The following proposition is immediate from Theorem 2.1(i) and the above.

Proposition 3.1. *A sufficient condition for M_C to be a CI-operator is that* $0 \in \{\Xi(A) \cap \Xi(B)\} \cap \{\Xi(A^*) \cap \Xi(B^*)\}$, *or,* $0 \in \{\Xi(A)^C \cap \Xi(B)^C\} \cap \{\Xi(A^*)^C \cap \Xi(B^*)^C\}$.

An equality similar to the well known spectral equality

$$\sigma(M_C) \cup \{\sigma(A) \cap \sigma(B)\} = \sigma(A) \cup \sigma(B)$$

does not hold for the consistent in invertibility spectrum $\sigma_{CI}(M_C)$.

Example 3.2. Let $A \in B(\mathcal{H})$ be the forward unilateral shift, $Q \in B(\mathcal{H})$ be a quasinilpotent operator, and let $M_0 = A \oplus B$. Then M_0 is neither left nor right invertible ($\Longrightarrow 0 \notin \sigma_{CI}(M_0)$), $\sigma_{CI}(B) = \emptyset$ and $0 \in \sigma_{CI}(A)$. Hence $0 \notin \sigma_{CI}(M_0) \cup \{\sigma_{CI}(A) \cap \sigma_{CI}(B)\}$ and $0 \in \sigma_{CI}(A) \cup \sigma_{CI}(B)$.

The following theorem shows that if we augment $\sigma_{CI}(M_0) \cup \{\sigma_{CI}(A) \cap \sigma_{CI}(B)\}$ by $\{\sigma_{CI}(A) \setminus \sigma_{CI}(B) \cup \sigma_{CI}(B) \setminus \sigma_{CI}(A)\}$, then we obtain $\sigma_{CI}(A) \cup \sigma_{CI}(B)$. But before that we prove the inclusion:

Proposition 3.3. $\sigma_{CI}(M_C) \subseteq \sigma_{CI}(A) \cup \sigma_{CI}(B)$.

Proof. It would suffice to prove that $0 \in \sigma_{CI}(M_C) \Longrightarrow 0 \in \sigma_{CI}(A) \cup \sigma_{CI}(B)$. Clearly, Theorem 2.1(ii),

$$\begin{aligned} 0 \in \sigma_{CI}(M_C) &\iff M_C \in \mathrm{Inv}^{\mathrm{l}} \setminus \mathrm{Inv} \;\; \cup \;\; \mathrm{Inv}^{\mathrm{r}} \setminus \mathrm{Inv} \\ &\Longrightarrow A \in \mathrm{Inv}^{\mathrm{l}} \setminus \mathrm{Inv} \;\text{ or }\; \mathrm{B} \in \mathrm{Inv}^{\mathrm{r}} \setminus \mathrm{Inv} \\ &\Longrightarrow 0 \in \sigma_{CI}(A) \cup \sigma_{CI}(B), \end{aligned}$$

which completes the proof. □

Theorem 3.4. $\sigma_{CI}(M_C) \cup \{\sigma_{CI}(A) \cap \sigma_{CI}(B)\} \cup \{\sigma_{CI}(A) \setminus \sigma_{CI}(B) \cup \sigma_{CI}(B) \setminus \sigma_{CI}(A)\} = \sigma_{CI}(A) \cup \sigma_{CI}(B)$.

Proof. In view of Proposition 3.3, to prove the equality it would suffice to prove that $0 \in \sigma_{CI}(A) \cup \sigma_{CI}(B)$ implies $0 \in \sigma_{CI}(M_C) \cup \{\sigma_{CI}(A) \cap \sigma_{CI}(B)\} \cup \{\sigma_{CI}(A) \setminus \sigma_{CI}(B) \cup \sigma_{CI}(B) \setminus \sigma_{CI}(A)\}$. We start by assuming $0 \in \sigma_{CI}(A)$. Then either (a) $A \in \mathrm{Inv}^{\mathrm{l}} \setminus \mathrm{Inv}$ or (b) $A \in \mathrm{Inv}^{\mathrm{r}} \setminus \mathrm{Inv}$. If (a) holds, then either (a_1) $B \in \mathrm{Inv}^{\mathrm{l}} \setminus \mathrm{Inv}$, or, (a_2) $B \in \mathrm{Inv}$, or, (a_3) $B \notin \mathrm{Inv}^{\mathrm{l}} \cap \mathrm{Inv}^{\mathrm{r}}$, or, (a_4) $B \in \mathrm{Inv}^{\mathrm{r}} \setminus \mathrm{Inv}$.

If (a) and (a_1) hold, then $M_C \in \mathrm{Inv}^{\mathrm{l}} \setminus \mathrm{Inv} \Longrightarrow 0 \in \sigma_{\mathrm{CI}}(\mathrm{M_C})$. If (a) and (a_2) hold, then $M_C \in \mathrm{Inv}^{\mathrm{l}}$. We claim that $M_C \notin \mathrm{Inv}^{\mathrm{r}}$: for if $M_C \in \mathrm{Inv}^{\mathrm{r}}$, then $M_C \in \mathrm{Inv} \Longrightarrow \mathrm{A} \in \mathrm{Inv}$ (since $B \in \mathrm{Inv}$), which contradicts $A \notin \mathrm{Inv}^{\mathrm{r}}$. Hence $0 \in \sigma_{CI}(M_C)$ in this case also. Suppose next that (a) and (a_3) are satisfied. Then B is neither left nor right invertible; hence $0 \notin \sigma_{CI}(B)$ and $0 \in \sigma_{CI}(A)$, equivalently, $0 \in \sigma_{CI}(A) \setminus \sigma_{CI}(B)$. Finally, if (a_4) is satisfied, then $B \in \text{Inv}^{\text{r}} \setminus \text{Inv}$ implies $0 \in \sigma_{CI}(B)$. Hence, because of (a), $0 \in \sigma_{CI}(A) \cap \sigma_{CI}(B)$ in this case.

Arguing similarly for the case in which (b) holds, and either (b_1) $B \in \text{Inv}^{\text{r}} \setminus \text{Inv}$ or (b_2) $B \in \text{Inv}$ or (b_3) $B \notin \text{Inv}$ or (b_4) $B \in \text{Inv}^{\text{l}} \setminus \text{Inv}$, it is seen that $0 \in \sigma_{CI}(M_C) \cup \{\sigma_{CI}(A) \cap \sigma_{CI}(B)\} \cup \sigma_{CI}(A) \setminus \sigma_{CI}(B)$.

Finally, to complete the proof, we observe that a similar argument works in the case in which $0 \in \sigma_{CI}(B)$ to prove that $0 \in \sigma_{CI}(M_C) \cup \{\sigma_{CI}(A) \cap \sigma_{CI}(B)\} \cup \sigma_{CI}(B) \setminus \sigma_{CI}(A)$. □

Fredholm consistency spectrum $\sigma_{FC}(M_C)$**.** Let $M_C(q)$ denote the image of M_C in the algebra $\mathcal{B}(\mathcal{X}_q \oplus \mathcal{X}_q)$, $A(q) = (A \oplus I)_q$, $B(q) = (I \oplus B)_q$ and $C(q) = \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}_q$. Then $M_C(q) = B(q)C(q)A(q)$, the operator $C(q)$ is invertible, $A(q) = A_q \oplus I_q$, $B(q) = I_q \oplus B_q$, $M_C(q)$ has SVEP at 0 (equivalently, M_C has essential SVEP at 0) implies A_q has SVEP at 0 and $M_C(q)^*$ has SVEP at 0 implies B_q^* has SVEP at 0. Evidently, Theorem 2.10, M_C is ϕ-consistent (i.e., $0 \notin \sigma_{FC}(M_C)$) if both $M_C(q)$ and $M_C(q)^*$ have, or do not have, SVEP at 0; furthermore, a necessary and sufficient condition for M_C to be ϕ-consistent is that either $(M_C)_q \in CI$. The following corollary is the analogue of Theorem 3.4 for $\sigma_{FC}(M_C)$.

Corollary 3.5. $\sigma_{FC}(M_C) \cup \{\sigma_{FC}(A) \cap \sigma_{FC}(B)\} \cup \{\sigma_{FC}(A) \setminus \sigma_{FC}(B) \cup \sigma_{FC}(B) \setminus \sigma_{FC}(A)\} = \sigma_{FC}(A) \cup \sigma_{FC}(B)$.

Proof. Proposition 3.3 implies the inclusion $\sigma_{FC}(M_C) \subseteq \sigma_{FC}(A) \cup \sigma_{FC}(B)$ (and hence the forward inclusion "$\subseteq$" in the equality of the statement), and Theorem 3.4 implies the backward inclusion "$\supseteq$" in the equality of the statement. □

Let $\sigma_{BC}(T) = \{\lambda \in \sigma(T) : T - \lambda \notin (BC)\}$ and $\sigma_{WC}(T) = \{\lambda \in \sigma(T) : T - \lambda \notin (WC)\}$ denote, respectively, the *Browder consistency* and the *Weyl consistency* spectrum of T. Then $\sigma_{FC}(T) = \sigma_{BC}(T) = \sigma_{WC}(T)$ for every $T \in B(\mathcal{X})$ (this follows from the results of the earlier section). The following corollary is immediate from this observation and the corollary above.

Corollary 3.6. $\sigma_x(M_C) \cup \{\sigma_x(A) \cap \sigma_x(B)\} \cup \{\sigma_x(A) \setminus \sigma_x(B) \cup \sigma_x(B) \setminus \sigma_x(A)\} = \sigma_x(A) \cup \sigma_x(B)$, *where* $\sigma_x = \sigma_{BC}$ *or* σ_{WC}.

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