

## Small covers over a product of simplices

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**Abstract.** In this paper, we determine the number of equivariant homeomorphism classes of small covers over a product of  $m$  simplices for  $m \leq 3$  or for the dimension of each simplex being greater than 1 and  $m > 3$ . Moreover, we calculate the number of equivariant homeomorphism classes of all orientable small covers over a product of at most three simplices.

### 1. Introduction

The notion of small covers was introduced by Davis and Januszkiewicz [7], where a small cover is a smooth closed manifold  $M^n$  with a locally standard  $(\mathbb{Z}_2)^n$ -action such that its orbit space is a simple convex polytope. For instance, the real projective space  $\mathbb{R}P^n$  with a natural  $(\mathbb{Z}_2)^n$ -action is a small cover over the  $n$ -simplex  $\Delta_n$ . This establishes a direct connection between equivariant topology and combinatorics and makes it possible to study the topology of small covers through the combinatorial structure of quotient spaces.

In [9], Lü and Masuda showed that the equivariant homeomorphism class of a small cover over a simple convex polytope  $P^n$  agrees with the equivalence class of its corresponding  $(\mathbb{Z}_2)^n$ -coloring under the action of automorphism group of face poset of  $P^n$ . This also holds for orientable small covers by the orientability condition in [11] (see Theorem 5.3). But there aren't general formulas to calculate the number of equivariant homeomorphism classes of (orientable) small covers over an arbitrary simple convex polytope.

In recent years, several studies have attempted to enumerate the number of Davis-Januszkiewicz equivalence classes and equivariant homeomorphism classes of small covers over a specific polytope. Cai, Chen and Lü calculated the number of equivariant homeomorphism classes of small covers over 3-dimensional prisms [2]. In 2008, Choi determined the number of Davis-Januszkiewicz equivalence classes of small covers over a product of simplices and the number of equivariant homeomorphism classes of small covers over cubes [4]. There are few results about orientable small covers. Choi calculated the number of Davis-Januszkiewicz equivalence classes of orientable small covers over cubes [5]. Products of simplices are an interesting class of polytopes and more complicated than one might think [13]. And small covers over products of simplices have become an important search object [3, 4, 5, 6, 8, 10]. Motivated by these,

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we determine the number of equivariant homeomorphism classes of small covers over a product of  $m$  simplices for  $m \leq 3$  or for the dimension of each simplex being greater than 1 and  $m > 3$  (see Theorem 2.7 and Theorems 4.1-4.2). Furthermore, we calculate the number of equivariant homeomorphism classes of all orientable small covers over a product of at most three simplices (see Theorems 5.7-5.12).

The paper is organized as follows. In Section 2, we review the basic theory about small covers, calculate the automorphism group of face poset of a product of  $m$  simplices and determine the number of small covers over a product of  $m$  simplices with the dimension of each simplex being greater than 1 up to equivariant homeomorphism. In Section 3 we determine the number of all colorings on  $\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}$  ( $n_2 \geq 1$  and  $n_3 \geq 1$ ) and  $\Delta_1 \times \Delta_{n_2}$  ( $n_2 \geq 1$ ), so that in Section 4 we give a formula to calculate the number of equivariant homeomorphism classes of all small covers over  $\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}$  ( $n_2 \geq 1$  and  $n_3 \geq 1$ ) and  $\Delta_1 \times \Delta_{n_2}$  ( $n_2 \geq 1$ ). In Section 5, similarly we determine the number of equivariant homeomorphism classes of all orientable small covers over a product of at most three simplices.

## 2. Small covers over a product of simplices

A convex polytope  $P^n$  of dimension  $n$  is said to be simple if every vertex of  $P^n$  is the intersection of exactly  $n$  facets (i.e. faces of dimension  $(n - 1)$ ) [13]. An  $n$ -dimensional smooth closed manifold  $M^n$  is said to be a small cover if it admits a smooth  $(\mathbb{Z}_2)^n$ -action such that the action is locally isomorphic to a standard action of  $(\mathbb{Z}_2)^n$  on  $\mathbb{R}^n$  and the orbit space  $M^n/(\mathbb{Z}_2)^n$  is a simple convex polytope of dimension  $n$ .

Let  $P^n$  be a simple convex polytope of dimension  $n$  and  $\mathcal{F}(P^n) = \{F_1, \dots, F_\ell\}$  be the set of facets of  $P^n$ . Suppose that  $\pi : M^n \rightarrow P^n$  is a small cover over  $P^n$ . Then there are  $\ell$  connected submanifolds  $\pi^{-1}(F_1), \dots, \pi^{-1}(F_\ell)$ . Each submanifold  $\pi^{-1}(F_i)$  is fixed pointwise by a  $\mathbb{Z}_2$ -subgroup  $\mathbb{Z}_2(F_i)$  of  $(\mathbb{Z}_2)^n$ , so that each facet  $F_i$  corresponds to the  $\mathbb{Z}_2$ -subgroup  $\mathbb{Z}_2(F_i)$ . Obviously, the  $\mathbb{Z}_2$ -subgroup  $\mathbb{Z}_2(F_i)$  actually agrees with an element  $v_i$  in  $(\mathbb{Z}_2)^n$  as a vector space. For each face  $F$  of codimension  $u$ , since  $P^n$  is simple, there are  $u$  facets  $F_{i_1}, \dots, F_{i_u}$  such that  $F = F_{i_1} \cap \dots \cap F_{i_u}$ . Then, the corresponding submanifolds  $\pi^{-1}(F_{i_1}), \dots, \pi^{-1}(F_{i_u})$  intersect transversally in the  $(n - u)$ -dimensional submanifold  $\pi^{-1}(F)$ , and the isotropy subgroup  $\mathbb{Z}_2(F)$  of  $\pi^{-1}(F)$  is a subgroup of rank  $u$  and is generated by  $\mathbb{Z}_2(F_{i_1}), \dots, \mathbb{Z}_2(F_{i_u})$  (or is determined by  $v_{i_1}, \dots, v_{i_u}$  in  $(\mathbb{Z}_2)^n$ ). Thus, this actually gives a characteristic function [7]

$$\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$$

defined by  $\lambda(F_i) = v_i$  such that whenever the intersection  $F_{i_1} \cap \dots \cap F_{i_u}$  is non-empty,  $\lambda(F_{i_1}), \dots, \lambda(F_{i_u})$  are linearly independent in  $(\mathbb{Z}_2)^n$ . If we regard each nonzero vector of  $(\mathbb{Z}_2)^n$  as being a color, then the characteristic function  $\lambda$  means that each facet is colored by a color. Here we also call  $\lambda$  a  $(\mathbb{Z}_2)^n$ -coloring on  $P^n$ .

In fact, Davis and Januszkiewicz gave a reconstruction process of a small cover by using a  $(\mathbb{Z}_2)^n$ -coloring  $\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$ . Let  $\mathbb{Z}_2(F_i)$  be the subgroup of  $(\mathbb{Z}_2)^n$  generated by  $\lambda(F_i)$ . Given a point  $p \in P^n$ , by  $F(p)$  we denote the minimal face containing  $p$  in its relative interior. Assume  $F(p) = F_{i_1} \cap \dots \cap F_{i_u}$  and  $\mathbb{Z}_2(F(p)) = \bigoplus_{j=1}^u \mathbb{Z}_2(F_{i_j})$ . Note that  $\mathbb{Z}_2(F(p))$  is a subgroup of rank  $u$  in  $(\mathbb{Z}_2)^n$ . Let  $M(\lambda)$  denote  $P^n \times (\mathbb{Z}_2)^n / \sim$ , where  $(p, g) \sim (q, h)$  if  $p = q$  and  $g^{-1}h \in \mathbb{Z}_2(F(p))$ . The free action of  $(\mathbb{Z}_2)^n$  on  $P^n \times (\mathbb{Z}_2)^n$  descends to an action on  $M(\lambda)$  with quotient  $P^n$ . Thus  $M(\lambda)$  is a small cover over  $P^n$  [7].

Two small covers  $M_1$  and  $M_2$  over  $P^n$  are said to be weakly equivariantly homeomorphic if there is an automorphism  $\varphi : (\mathbb{Z}_2)^n \rightarrow (\mathbb{Z}_2)^n$  and a homeomorphism  $f : M_1 \rightarrow M_2$  such that  $f(t \cdot x) = \varphi(t) \cdot f(x)$  for every  $t \in (\mathbb{Z}_2)^n$  and  $x \in M_1$ . If  $\varphi$  is an identity, then  $M_1$  and  $M_2$  are equivariantly homeomorphic. Following [7], two small covers  $M_1$  and  $M_2$  over  $P^n$  are said to be Davis-Januszkiewicz equivalent (or simply, D-J equivalent) if there is a weakly equivariant homeomorphism  $f : M_1 \rightarrow M_2$  covering the identity on  $P^n$ .

By  $\Lambda(P^n)$  we denote the set of all  $(\mathbb{Z}_2)^n$ -colorings on  $P^n$ . Then we have

**Theorem 2.1.** (Davis-Januszkiewicz) *All small covers over  $P^n$  are given by  $\{M(\lambda) | \lambda \in \Lambda(P^n)\}$ , i.e. for each small cover  $M^n$  over  $P^n$ , there is a  $(\mathbb{Z}_2)^n$ -coloring  $\lambda$  with an equivariant homeomorphism  $M(\lambda) \rightarrow M^n$  covering the identity on  $P^n$ .*

**Remark 1.** Generally speaking, we can't make sure that there always exist small covers over a simple convex polytope  $P^n$  when  $n \geq 4$ . For example, see [7, Nonexample 1.22]. From [7], we know that there

exists a small cover (i.e. a real projective space) over every simplex. Thus, there exists a small cover over a product of simplices.

There is a natural action of  $GL(n, \mathbb{Z}_2)$  on  $\Lambda(P^n)$  defined by the correspondence  $\lambda \mapsto \sigma \circ \lambda$ , and the action on  $\Lambda(P^n)$  is free. Without loss of generality, we assume that  $F_1, \dots, F_n$  of  $\mathcal{F}(P^n)$  meet at one vertex  $p$  of  $P^n$ . Let  $e_1, \dots, e_n$  be the standard basis of  $(\mathbb{Z}_2)^n$ . Write  $A(P^n) = \{\lambda \in \Lambda(P^n) | \lambda(F_i) = e_i \text{ for } i = 1, \dots, n\}$ . In fact,  $A(P^n)$  is the orbit space of  $\Lambda(P^n)$  under the action of  $GL(n, \mathbb{Z}_2)$ . Then we have

**Lemma 2.2.**  $|\Lambda(P^n)| = |A(P^n)| \times |GL(n, \mathbb{Z}_2)|$ .

Note that  $|GL(n, \mathbb{Z}_2)| = \prod_{k=1}^n (2^n - 2^{k-1})$  [1]. Two small covers  $M(\lambda_1)$  and  $M(\lambda_2)$  over  $P^n$  are D-J equivalent if and only if there is  $\sigma \in GL(n, \mathbb{Z}_2)$  such that  $\lambda_1 = \sigma \circ \lambda_2$ . So the number of D-J equivalence classes of small covers over  $P^n$  is  $|A(P^n)|$ .

Let  $P^n$  be a simple convex polytope of dimension  $n$ . All faces of  $P^n$  form a poset (i.e. a partially ordered set by inclusion). An automorphism of  $\mathcal{F}(P^n)$  is a bijection from  $\mathcal{F}(P^n)$  to itself which preserves the poset structure of all faces of  $P^n$ , and by  $Aut(\mathcal{F}(P^n))$  we denote the group of automorphisms of  $\mathcal{F}(P^n)$ . One can define the right action of  $Aut(\mathcal{F}(P^n))$  on  $\Lambda(P^n)$  by  $\lambda \times h \mapsto \lambda \circ h$ , where  $\lambda \in \Lambda(P^n)$  and  $h \in Aut(\mathcal{F}(P^n))$ . The following theorem is well known [9].

**Theorem 2.3.** *Two small covers over an  $n$ -dimensional simple convex polytope  $P^n$  are equivariantly homeomorphic if and only if there is  $h \in Aut(\mathcal{F}(P^n))$  such that  $\lambda_1 = \lambda_2 \circ h$ , where  $\lambda_1$  and  $\lambda_2$  are their corresponding  $(\mathbb{Z}_2)^n$ -colorings on  $P^n$ .*

So the number of orbits of  $\Lambda(P^n)$  under the action of  $Aut(\mathcal{F}(P^n))$  is just the number of equivariant homeomorphism classes of small covers over  $P^n$ . Thus, we are going to count the orbits. Burnside Lemma is very useful in the enumeration of the number of orbits.

**Burnside Lemma.** *Let  $G$  be a finite group acting on a set  $X$ . Then the number of orbits of  $X$  under the action of  $G$  equals  $\frac{1}{|G|} \sum_{g \in G} |X_g|$ , where  $X_g = \{x \in X | gx = x\}$ .*

Burnside Lemma suggests that we need to understand the structure of  $Aut(\mathcal{F}(P^n))$  in order to determine the number of the orbits of  $\Lambda(P^n)$  under the action of  $Aut(\mathcal{F}(P^n))$ . We shall particularly be concerned with the case in which  $P^n$  is the product of  $m$  simplices.

In the following, let  $P = \Delta_{k_1}^{(1)} \times \dots \times \Delta_{k_1}^{(i_1)} \times \Delta_{k_2}^{(1)} \times \dots \times \Delta_{k_2}^{(i_2)} \times \dots \times \Delta_{k_j}^{(1)} \times \dots \times \Delta_{k_j}^{(i_j)}$ , where  $\Delta_{k_s}^{(u)}$  is a  $k_s$ -simplex with  $1 \leq k_1 < k_2 < \dots < k_j$ ,  $i_1 + \dots + i_j = m$  and  $\sum_{s=1}^j i_s k_s = n$ . Suppose  $P_1 = \Delta_{k_1}^{(1)} \times \dots \times \Delta_{k_1}^{(i_1)}$ ,  $P_2 = \Delta_{k_2}^{(1)} \times \dots \times \Delta_{k_2}^{(i_2)}$ ,  $\dots$ , and  $P_j = \Delta_{k_j}^{(1)} \times \dots \times \Delta_{k_j}^{(i_j)}$ . Then we have  $P = P_1 \times P_2 \times \dots \times P_j$  and  $\mathcal{F}(P) = \bigcup_{k=1}^j P_1 \times \dots \times P_{k-1} \times \mathcal{F}(P_k) \times P_{k+1} \times \dots \times P_j$ .

Let  $S_i$  be the symmetry group of rank  $i$ . Then we arrive at

**Theorem 2.4.** *The automorphism group  $Aut(\mathcal{F}(P))$  is isomorphic to  $(S_{k_1+1})^{i_1} \times S_{i_1} \times (S_{k_2+1})^{i_2} \times S_{i_2} \times \dots \times (S_{k_j+1})^{i_j} \times S_{i_j}$ .*

*Proof.* Let  $\mathcal{F}_{1u_1} = \Delta_{k_1}^{(1)} \times \dots \times \Delta_{k_1}^{(u_1-1)} \times \mathcal{F}(\Delta_{k_1}^{(u_1)}) \times \Delta_{k_1}^{(u_1+1)} \times \dots \times \Delta_{k_1}^{(i_1)}$  for  $1 \leq u_1 \leq i_1$ . Then  $\mathcal{F}(P_1) = \mathcal{F}_{11} \cup \dots \cup \mathcal{F}_{1i_1}$ . Obviously, the automorphism group  $Aut(\mathcal{F}(\Delta_{k_1}^{(u)}))$  is isomorphic to  $S_{k_1+1}$  since there is exactly one automorphism for each permutation of  $k+1$  facets of  $\Delta_{k_1}^{(u)}$ . Thus, the automorphism group  $Aut(\mathcal{F}(P_1))$  contains a group  $(S_{k_1+1})^{i_1}$ , each of which denotes an automorphism under which the facets in  $\mathcal{F}_{11}, \mathcal{F}_{12}, \dots, \mathcal{F}_{1i_1}$  are mapped into  $\mathcal{F}_{11}, \mathcal{F}_{12}, \dots, \mathcal{F}_{1i_1}$  respectively.  $Aut(\mathcal{F}(P_1))$  also contains a group  $S_{i_1}$  because there is one automorphism for each permutation of  $\mathcal{F}_{11}, \mathcal{F}_{12}, \dots, \mathcal{F}_{1i_1}$ . Each automorphism of  $S_{i_1}$  is different from any one of  $(S_{k_1+1})^{i_1}$ . So the automorphism group  $Aut(\mathcal{F}(P_1)) \cong (S_{k_1+1})^{i_1} \times S_{i_1}$ . Similarly we have  $Aut(\mathcal{F}(P_2)) \cong (S_{k_2+1})^{i_2} \times S_{i_2}$ ,  $\dots$ , and  $Aut(\mathcal{F}(P_j)) \cong (S_{k_j+1})^{i_j} \times S_{i_j}$ .

Since  $k_1 < k_2 < \dots < k_j$ , under the automorphisms of  $Aut(\mathcal{F}(P))$  the facets of  $P_1 \times \dots \times P_{k-1} \times \mathcal{F}(P_k) \times P_{k+1} \times \dots \times P_j$  are mapped to  $P_1 \times \dots \times P_{k-1} \times \mathcal{F}(P_k) \times P_{k+1} \times \dots \times P_j$  for  $1 \leq k \leq j$ . Thus we have that  $Aut(\mathcal{F}(P))$  is isomorphic to  $(S_{k_1+1})^{i_1} \times S_{i_1} \times (S_{k_2+1})^{i_2} \times S_{i_2} \times \dots \times (S_{k_j+1})^{i_j} \times S_{i_j}$ .  $\square$

Let  $\Delta_{n_1}, \Delta_{n_2}$  and  $\Delta_{n_3}$  be  $n_1$ -simplex,  $n_2$ -simplex and  $n_3$ -simplex respectively. From Theorem 2.4, we have

**Corollary 2.5.** *The automorphism group  $Aut(\mathcal{F}(\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3}))$  is isomorphic to*

$$\begin{cases} S_{n_1+1} \times S_{n_2+1} \times S_{n_3+1} & \text{for } n_1 < n_2 < n_3, \\ S_{n_1+1} \times S_{n_2+1} \times S_{n_3+1} \times S_2 & \text{for } n_1 < n_2 = n_3, \\ S_{n_1+1} \times S_{n_2+1} \times S_2 \times S_{n_3+1} & \text{for } n_1 = n_2 < n_3, \\ S_{n_1+1} \times S_{n_2+1} \times S_{n_3+1} \times S_3 & \text{for } n_1 = n_2 = n_3, \end{cases}$$

and the automorphism group  $Aut(\mathcal{F}(\Delta_{n_1} \times \Delta_{n_2}))$  is isomorphic to

$$\begin{cases} S_{n_1+1} \times S_{n_2+1} & \text{for } n_1 < n_2, \\ S_{n_1+1} \times S_{n_2+1} \times S_2 & \text{for } n_1 = n_2. \end{cases}$$

Let us recall some basic definitions about acyclic digraphs [12]. A “digraph” means a graph with at most one edge directed from any vertex  $v_i$  to another vertex  $v_j$ . An “acyclic” means there is no cycle of any length. The outdegree of a vertex  $v$ ,  $outdeg(v)$ , is the number of edges of the digraph with initial vertex  $v$ .

Let  $P = \Delta_{k_1}^{(1)} \times \dots \times \Delta_{k_1}^{(i_1)} \times \Delta_{k_2}^{(1)} \times \dots \times \Delta_{k_2}^{(i_2)} \times \dots \times \Delta_{k_j}^{(1)} \times \dots \times \Delta_{k_j}^{(i_j)}$ , where  $\Delta_{k_s}^{(u)}$  is a  $k_s$ -simplex with  $1 \leq k_1 < k_2 < \dots < k_j$ ,  $i_1 + \dots + i_j = m$  and  $\sum_{s=1}^j i_s k_s = n$ . From [4, Theorem 2.8], we know that the number of

D-J equivalence classes of small covers over  $P$  is  $|A(P)| = \sum_{G \in \mathcal{G}_m} \prod_{s=1}^j (2^{k_s} - 1)^{outdeg(v_{i_1+\dots+i_{s-1}+1})+\dots+outdeg(v_{i_1+\dots+i_s})}$ , where  $\mathcal{G}_m$  is the set of acyclic digraphs with labeled  $m$  nodes and  $V(G) = \{v_1, \dots, v_m\}$  is the labeled vertex set of  $G$ . By using Lemma 2.2, we have

**Lemma 2.6.** *Let  $P = \Delta_{k_1}^{(1)} \times \dots \times \Delta_{k_1}^{(i_1)} \times \Delta_{k_2}^{(1)} \times \dots \times \Delta_{k_2}^{(i_2)} \times \dots \times \Delta_{k_j}^{(1)} \times \dots \times \Delta_{k_j}^{(i_j)}$ . Then the number of all colorings on  $P$  is*

$$|\Lambda(P)| = \prod_{t=1}^n (2^n - 2^{t-1}) \sum_{G \in \mathcal{G}_m} \prod_{s=1}^j (2^{k_s} - 1)^{outdeg(v_{i_1+\dots+i_{s-1}+1})+\dots+outdeg(v_{i_1+\dots+i_s})},$$

where  $\mathcal{G}_m$  and  $V(G)$  are as above.

Below we determine the number of equivariant homeomorphism classes of small covers over  $P$  if  $2 \leq k_1 < k_2 < \dots < k_j$ .

**Theorem 2.7.** *Let  $P = \Delta_{k_1}^{(1)} \times \dots \times \Delta_{k_1}^{(i_1)} \times \Delta_{k_2}^{(1)} \times \dots \times \Delta_{k_2}^{(i_2)} \times \dots \times \Delta_{k_j}^{(1)} \times \dots \times \Delta_{k_j}^{(i_j)}$ . By  $E(P)$  we denote the number of equivariant homeomorphism classes of small covers over  $P$ . If  $2 \leq k_1 < k_2 < \dots < k_j$ , then*

$$E(P) = \frac{\prod_{t=1}^n (2^n - 2^{t-1}) \sum_{G \in \mathcal{G}_m} \prod_{s=1}^j (2^{k_s} - 1)^{outdeg(v_{i_1+\dots+i_{s-1}+1})+\dots+outdeg(v_{i_1+\dots+i_s})}}{[(k_1+1)!]^{i_1} \times i_1! \times [(k_2+1)!]^{i_2} \times i_2! \times \dots \times [(k_j+1)!]^{i_j} \times i_j!},$$

where  $\mathcal{G}_m$  and  $V(G)$  are as above.

*Proof.* From Theorem 2.3, Burnside Lemma and Theorem 2.4, we have that

$$E(P) = \frac{\sum_{g \in Aut(\mathcal{F}(P))} |\Lambda_g|}{[(k_1+1)!]^{i_1} \times i_1! \times [(k_2+1)!]^{i_2} \times i_2! \times \dots \times [(k_j+1)!]^{i_j} \times i_j!}$$

where  $\Lambda_g = \{\lambda \in \Lambda(P) | \lambda = \lambda \circ g\}$ .

By the linear independence condition of characteristic functions, we have that  $\Lambda_g$  is empty if  $g$  isn't unit element of the automorphism group  $Aut(\mathcal{F}(P))$ . Thus,

$$E(P) = \frac{|\Lambda(P)|}{[(k_1+1)!]^{i_1} \times i_1! \times [(k_2+1)!]^{i_2} \times i_2! \times \dots \times [(k_j+1)!]^{i_j} \times i_j!}$$

The theorem is proved with Lemma 2.6.  $\square$

**Corollary 2.8.** *Let  $2 \leq n_1 \leq n_2 \leq n_3$ . The number of equivariant homeomorphism classes of small covers over  $\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3}$  is  $E(\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3}) =$*

- (1)  $\frac{\prod_{t=1}^{n_1+n_2+n_3} (2^{n_1+n_2+n_3-2^{t-1}})}{(n_1+1)!(n_2+1)!(n_3+1)!} (2^{2n_1+n_2} + 2^{n_1+2n_2} + 2^{2n_1+n_3} + 2^{n_1+2n_3} + 2^{2n_2+n_3} + 2^{n_2+2n_3} - 2^{2n_1} - 2^{2n_2} - 2^{2n_3} - 2^{n_1+n_2} - 2^{n_1+n_3} - 2^{n_2+n_3} + 1)$  for  $n_1 < n_2 < n_3$ ,
- (2)  $\frac{\prod_{t=1}^{n_1+2n_2} (2^{n_1+2n_2-2^{t-1}})}{2 \cdot (n_1+1)!(n_2+1)!^2} (2^{2n_1+n_2+1} + 2^{n_1+2n_2+1} + 2^{3n_2+1} - 2^{2n_1} - 3 \cdot 2^{2n_2} + 1 - 2^{n_1+n_2+1})$  for  $n_1 < n_2 = n_3$ ,
- (3)  $\frac{\prod_{t=1}^{2n_2+n_3} (2^{2n_2+n_3-2^{t-1}})}{2 \cdot [(n_2+1)!]^2 (n_3+1)!} (2^{2n_2+n_3+1} + 2^{n_2+2n_3+1} + 2^{3n_2+1} - 2^{2n_3} - 3 \cdot 2^{2n_2} + 1 - 2^{n_2+n_3+1})$  for  $n_1 = n_2 < n_3$ ,
- (4)  $\frac{\prod_{t=1}^{3n_2} (2^{3n_2-2^{t-1}})}{6 \cdot [(n_2+1)!]^3} (6 \cdot 2^{3n_2} - 6 \cdot 2^{2n_2} + 1)$  for  $n_1 = n_2 = n_3$ .

**Corollary 2.9.** *Let  $2 \leq n_1 \leq n_2$ . The number of equivariant homeomorphism classes of small covers over  $\Delta_{n_1} \times \Delta_{n_2}$  is  $E(\Delta_{n_1} \times \Delta_{n_2}) =$*

$$\begin{cases} \frac{(2^{n_1+2n_2-1} \prod_{t=1}^{n_1+n_2} (2^{n_1+n_2-2^{t-1}}))}{(n_1+1)!(n_2+1)!} & \text{for } 2 \leq n_1 < n_2, \\ \frac{(2^{n_1+1-1} \prod_{t=1}^{2n_1} (2^{2n_1-2^{t-1}}))}{2 \cdot [(n_1+1)!]^2} & \text{for } 2 \leq n_1 = n_2. \end{cases}$$

In the following, we consider the case  $n_1 = 1$  and  $m \leq 3$ .

### 3. Colorings on $\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3}$ and $\Delta_1 \times \Delta_{n_2}$

In order to determine the number of equivariant homeomorphism classes of small covers over  $\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}$  and  $\Delta_1 \times \Delta_{n_2}$ , we calculate the number of colorings on  $\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}$  and  $\Delta_1 \times \Delta_{n_2}$ .

Let  $F'_1, F'_2$  be two vertices of 1-simplex  $\Delta_1$ . By  $F'_3, \dots, F'_{n_2+3}$  we denote all facets of  $\Delta_{n_2}$ , and by  $F'_{n_2+4}, \dots, F'_{n_2+n_3+4}$  we denote all facets of  $\Delta_{n_3}$ . Set  $\mathcal{F}' = \{F_i = F'_i \times \Delta_{n_2} \times \Delta_{n_3} | 1 \leq i \leq 2\}$ ,  $\mathcal{F}'' = \{F_i = \Delta_1 \times F'_i \times \Delta_{n_3} | 3 \leq i \leq n_2 + 3\}$  and  $\mathcal{F}''' = \{F_i = \Delta_1 \times \Delta_{n_2} \times F'_i | n_2 + 4 \leq i \leq n_2 + n_3 + 4\}$ . Then  $\mathcal{F}(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) = \mathcal{F}' \cup \mathcal{F}'' \cup \mathcal{F}'''$ . We have

**Theorem 3.1.** *The number of  $(\mathbb{Z}_2)^{n_2+n_3+1}$ -colorings on  $\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}$  is*

$$|\Lambda(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})| = (2^{2n_2+n_3} + 2^{n_2+2n_3} + 2^{2n_2} + 2^{2n_3} + 2^{n_2+1} + 2^{n_3+1} - 2^{n_2+n_3} - 3) \prod_{t=1}^{n_2+n_3+1} (2^{n_2+n_3+1} - 2^{t-1}).$$

*Proof.* Let  $e_1, e_2, \dots, e_{n_2+n_3+1}$  be the standard basis of  $(\mathbb{Z}_2)^{n_2+n_3+1}$ . We choose  $F_1$  from  $\mathcal{F}'$ , choose  $F_3, \dots, F_{n_2+2}$  from  $\mathcal{F}''$  and choose  $F_{n_2+4}, \dots, F_{n_2+n_3+3}$  from  $\mathcal{F}'''$  such that they meet at one vertex of  $\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}$ . Then

$$A(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) = \{\lambda \in \Lambda(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) \mid \lambda(F_1) = e_1, \lambda(F_i) = e_{i-1} \text{ for } 3 \leq i \leq n_2 + 2; \lambda(F_i) = e_{i-2} \text{ for } n_2 + 4 \leq i \leq n_2 + n_3 + 3\}.$$

By Lemma 2.2, we have that

$$\begin{aligned} |\Lambda(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})| &= |A(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})| \times |GL(n_2 + n_3 + 1, \mathbb{Z}_2)| \\ &= \prod_{t=1}^{n_2+n_3+1} (2^{n_2+n_3+1} - 2^{t-1}) |A(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})|. \end{aligned}$$

Write

$$A_0(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) = \{\lambda \in A(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) \mid \lambda(F_2) = e_1\},$$

$$A_1(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) = \{\lambda \in A(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) \mid \exists k_1, \dots, k_i \text{ such that } \lambda(F_2) = e_1 + e_{k_1} + \dots + e_{k_i}, \text{ where } 2 \leq k_1 < \dots < k_i \leq n_2 + 1 \text{ and } 1 \leq i \leq n_2\},$$

$$A_2(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) = \{\lambda \in A(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) \mid \exists t_1, \dots, t_j \text{ such that } \lambda(F_2) = e_1 + e_{t_1} + \dots + e_{t_j}, \text{ where } n_2 + 2 \leq t_1 < \dots < t_j \leq n_2 + n_3 + 1 \text{ and } 1 \leq j \leq n_3\},$$

$$A_3(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) = \{\lambda \in A(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) \mid \exists k_1, \dots, k_i, t_1, \dots, t_j \text{ such that } \lambda(F_2) = e_1 + e_{k_1} + \dots + e_{k_i} + e_{t_1} + \dots + e_{t_j}, \text{ where } 2 \leq k_1 < \dots < k_i \leq n_2 + 1, n_2 + 2 \leq t_1 < \dots < t_j \leq n_2 + n_3 + 1, 1 \leq i \leq n_2 \text{ and } 1 \leq j \leq n_3\}.$$

By the definition of  $(\mathbb{Z}_2)^{n_2+n_3+1}$ -colorings, we have  $|A(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})| = \sum_{i=0}^3 |A_i(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})|$ . Then, our argument is divided into the following cases.

**Case 1.** Calculation of  $|A_0(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})|$ .

By the linear independence condition of characteristic functions, we have  $\lambda(F_{n_2+3}) = e_2 + \dots + e_{n_2+1}, e_2 + \dots + e_{n_2+1} + e_1, e_2 + \dots + e_{n_2+1} + e_{g_1} + \dots + e_{g_h}$  or  $e_2 + \dots + e_{n_2+1} + e_1 + e_{g_1} + \dots + e_{g_h}$ , where  $n_2 + 2 \leq g_1 < \dots < g_h \leq n_2 + n_3 + 1$  and  $1 \leq h \leq n_3$ . When  $\lambda(F_{n_2+3}) = e_2 + \dots + e_{n_2+1}$  or  $e_2 + \dots + e_{n_2+1} + e_1$ , by the linear independence condition of characteristic functions,  $\lambda(F_{n_2+n_3+4}) = e_{n_2+2} + \dots + e_{n_2+n_3+1} + e_{a_1} + \dots + e_{a_s}$ , where  $1 \leq a_1 < \dots < a_s \leq n_2 + 1$  and  $0 \leq s \leq n_2 + 1$ . When  $\lambda(F_{n_2+3}) = e_2 + \dots + e_{n_2+1} + e_{g_1} + \dots + e_{g_h}$  or  $e_2 + \dots + e_{n_2+1} + e_1 + e_{g_1} + \dots + e_{g_h}$ , where  $n_2 + 2 \leq g_1 < \dots < g_h \leq n_2 + n_3 + 1$  and  $1 \leq h \leq n_3$ , similarly we have  $\lambda(F_{n_2+n_3+4}) = e_{n_2+2} + \dots + e_{n_2+n_3+1}$  or  $e_{n_2+2} + \dots + e_{n_2+n_3+1} + e_1$ .

$$\text{Thus, we have } |A_0(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})| = 2^{n_2+2} + 2^{n_3+2} - 4.$$

**Case 2.** Calculation of  $|A_1(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})|$ .

By the linear independence condition of characteristic functions, no matter which value of  $\lambda(F_2)$  is chosen, we have  $\lambda(F_{n_2+3}) = e_2 + \dots + e_{n_2+1}$  or  $e_2 + \dots + e_{n_2+1} + e_{g_1} + \dots + e_{g_h}$ , where  $n_2 + 2 \leq g_1 < \dots < g_h \leq n_2 + n_3 + 1$  and  $1 \leq h \leq n_3$ . When  $\lambda(F_{n_2+3}) = e_2 + \dots + e_{n_2+1}$ , by the linear independence condition of characteristic functions,  $\lambda(F_{n_2+n_3+4}) = e_{n_2+2} + \dots + e_{n_2+n_3+1} + e_{a_1} + \dots + e_{a_s}$ , where  $1 \leq a_1 < \dots < a_s \leq n_2 + 1$  and  $0 \leq s \leq n_2 + 1$ . When  $\lambda(F_{n_2+3}) = e_2 + \dots + e_{n_2+1} + e_{g_1} + \dots + e_{g_h}$ , where  $n_2 + 2 \leq g_1 < \dots < g_h \leq n_2 + n_3 + 1$  and  $1 \leq h \leq n_3$ , similarly we have  $\lambda(F_{n_2+n_3+4}) = e_{n_2+2} + \dots + e_{n_2+n_3+1}$ .

$$\text{Thus, we have } |A_1(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})| = (2^{n_2} - 1)(2^{n_2+1} + 2^{n_3} - 1).$$

**Case 3.** Calculation of  $|A_2(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})|$ .

By the linear independence condition of characteristic functions, no matter which value of  $\lambda(F_2)$  is chosen, we have  $\lambda(F_{n_2+3}) = e_2 + \dots + e_{n_2+1}, e_2 + \dots + e_{n_2+1} + e_1, e_2 + \dots + e_{n_2+1} + e_{g_1} + \dots + e_{g_h}$  or  $e_2 + \dots + e_{n_2+1} + e_1 + e_{g_1} + \dots + e_{g_h}$ , where  $n_2 + 2 \leq g_1 < \dots < g_h \leq n_2 + n_3 + 1$  and  $1 \leq h \leq n_3$ . When  $\lambda(F_{n_2+3}) = e_2 + \dots + e_{n_2+1}$ , we have  $\lambda(F_{n_2+n_3+4}) = e_{n_2+2} + \dots + e_{n_2+n_3+1} + e_{a_1} + \dots + e_{a_s}$ , where  $2 \leq a_1 < \dots < a_s \leq n_2 + 1$  and  $0 \leq s \leq n_2$ . When  $\lambda(F_{n_2+3}) = e_2 + \dots + e_{n_2+1} + e_1, e_2 + \dots + e_{n_2+1} + e_{g_1} + \dots + e_{g_h}$  or  $e_2 + \dots + e_{n_2+1} + e_1 + e_{g_1} + \dots + e_{g_h}$ , where  $n_2 + 2 \leq g_1 < \dots < g_h \leq n_2 + n_3 + 1$  and  $1 \leq h \leq n_3$ , we have  $\lambda(F_{n_2+n_3+4}) = e_{n_2+2} + \dots + e_{n_2+n_3+1}$ .

$$\text{Thus, we have } |A_2(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})| = (2^{n_3} - 1)(2^{n_2} + 2^{n_3+1} - 1).$$

**Case 4.** Calculation of  $|A_3(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})|$ .

By the linear independence condition of characteristic functions, no matter which value of  $\lambda(F_2)$  is chosen,

we have  $\lambda(F_{n_2+3}) = e_2 + \dots + e_{n_2+1}$  or  $e_2 + \dots + e_{n_2+1} + e_{g_1} + \dots + e_{g_h}$ , where  $n_2 + 2 \leq g_1 < \dots < g_h \leq n_2 + n_3 + 1$  and  $1 \leq h \leq n_3$ . When  $\lambda(F_{n_2+3}) = e_2 + \dots + e_{n_2+1}$ , we have  $\lambda(F_{n_2+n_3+4}) = e_{n_2+2} + \dots + e_{n_2+n_3+1} + e_{a_1} + \dots + e_{a_s}$ , where  $2 \leq a_1 < \dots < a_s \leq n_2 + 1$  and  $0 \leq s \leq n_2$ . When  $\lambda(F_{n_2+3}) = e_2 + \dots + e_{n_2+1} + e_{g_1} + \dots + e_{g_h}$ , where  $n_2 + 2 \leq g_1 < \dots < g_h \leq n_2 + n_3 + 1$  and  $1 \leq h \leq n_3$ , we have  $\lambda(F_{n_2+n_3+4}) = e_{n_2+2} + \dots + e_{n_2+n_3+1}$ .

Thus, we have  $|A_3(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})| = (2^{n_2} - 1)(2^{n_3} - 1)(2^{n_2} + 2^{n_3} - 1)$ .

The proof is completed.  $\square$

In the similar way, we can prove the following

**Theorem 3.2.** *The number of  $(\mathbb{Z}_2)^{n_2+1}$ -colorings over  $\Delta_1 \times \Delta_{n_2}$  is*

$$|\Lambda(\Delta_1 \times \Delta_{n_2})| = (2^{n_2} + 1) \prod_{t=1}^{n_2+1} (2^{n_2+1} - 2^{t-1}).$$

In fact, Choi obtained the above two theorems in [4, Example 2.9].

**4. The number of small covers up to equivariant homeomorphism**

In this section, we determine the number of equivariant homeomorphism classes of small covers over  $\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}$  and  $\Delta_1 \times \Delta_{n_2}$ .

**Theorem 4.1.** *The number of equivariant homeomorphism classes of small covers over  $\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}$  is  $E(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) =$*

- (1)  $\frac{\prod_{t=1}^{n_2+n_3+1} (2^{n_2+n_3+1} - 2^{t-1})}{2 \cdot (n_2+1)! \cdot (n_3+1)!} (2^{2n_2+n_3} + 2^{n_2+2n_3} + 2^{2n_2} + 2^{2n_3} + 3 \cdot 2^{n_2+1} + 3 \cdot 2^{n_3+1} - 2^{n_2+n_3} - 7)$  for  $1 < n_2 < n_3$ ,
- (2)  $\frac{\prod_{t=1}^{n_2+n_3+1} (2^{n_2+n_3+1} - 2^{t-1})}{4 \cdot (n_2+1)! \cdot (n_3+1)!} (2^{2n_2+n_3} + 2^{n_2+2n_3} + 2^{2n_2} + 2^{2n_3} + 3 \cdot 2^{n_2+1} + 3 \cdot 2^{n_3+1} - 2^{n_2+n_3} - 7)$  for  $1 < n_2 = n_3$ ,
- (3)  $\frac{\prod_{t=1}^{n_3+2} (2^{n_3+2} - 2^{t-1})}{8 \cdot (n_3+1)!} (3 \cdot 2^{2n_3} + 3 \cdot 2^{n_3+2} + 17)$  for  $1 = n_2 < n_3$ ,
- (4) 259 for  $1 = n_2 = n_3$ .

*Proof.* First we consider the case  $1 < n_2 < n_3$ . From Theorem 2.3, Burnside Lemma and Corollary 2.5, we have that when  $1 < n_2 < n_3$ ,

$$E(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) = \frac{1}{2 \cdot (n_2+1)! \cdot (n_3+1)!} \sum_{g \in \text{Aut}(\mathcal{F}(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}))} |\Lambda_g|,$$

where  $\Lambda_g = \{\lambda \in \Lambda(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) \mid \lambda = \lambda \circ g\}$ .

From Corollary 2.5, when  $1 < n_2 < n_3$ , the automorphism group  $\text{Aut}(\mathcal{F}(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})) \cong S_2 \times S_{n_2+1} \times S_{n_3+1}$ . If  $g$  is the generator of  $S_2$ -subgroup of  $\text{Aut}(\mathcal{F}(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}))$  and  $\lambda \in \Lambda_g$ , then  $\lambda(F_1) = \lambda(F_2)$ . By the argument of Case 1 in Theorem 3.1, we have  $|\Lambda_g| = (2^{n_2+2} + 2^{n_3+2} - 4) \prod_{t=1}^{n_2+n_3+1} (2^{n_2+n_3+1} - 2^{t-1})$ . If  $g$  isn't the generator of the  $S_2$ -subgroup and isn't unit element of  $\text{Aut}(\mathcal{F}(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}))$ , then by the linear independence condition of characteristic functions we know that  $\Lambda_g$  is empty. From Theorem 3.1 we have that when  $1 < n_2 < n_3$ ,

$$E(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}) = \frac{\prod_{t=1}^{n_2+n_3+1} (2^{n_2+n_3+1} - 2^{t-1})}{2 \cdot (n_2+1)! \cdot (n_3+1)!} (2^{2n_2+n_3} + 2^{n_2+2n_3} + 2^{2n_2} + 2^{2n_3} + 3 \cdot 2^{n_2+1} + 3 \cdot 2^{n_3+1} - 2^{n_2+n_3} - 7).$$

When  $1 < n_2 = n_3$ , similarly we determine the number  $E(\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3})$ .

Next we consider the case  $1 = n_2 < n_3$ . From Theorem 2.3, Burnside Lemma and Corollary 2.5, we have that when  $1 = n_2 < n_3$ ,

$$E(\Delta_1 \times \Delta_1 \times \Delta_{n_3}) = \frac{1}{8 \cdot (n_3+1)!} \sum_{g \in \text{Aut}(\mathcal{F}(\Delta_1 \times \Delta_1 \times \Delta_{n_3}))} |\Lambda_g|,$$

where  $\Lambda_g = \{\lambda \in \Lambda(\Delta_1 \times \Delta_1 \times \Delta_{n_3}) \mid \lambda = \lambda \circ g\}$ .

From Corollary 2.5, when  $1 = n_2 < n_3$ , the automorphism group  $\text{Aut}(\mathcal{F}(\Delta_1 \times \Delta_1 \times \Delta_{n_3})) \cong S_2 \times S_2 \times S_2 \times S_{n_3+1}$ . If  $g$  is the generator  $a$  of the first  $S_2$ -subgroup of  $\text{Aut}(\mathcal{F}(\Delta_1 \times \Delta_1 \times \Delta_{n_3}))$  and  $\lambda \in \Lambda_g$ , then  $\lambda(F_1) = \lambda(F_2)$ . By the argument of Case 1 in Theorem 3.1, we have  $|\Lambda_g| = (2^{n_3+2} + 4) \prod_{t=1}^{n_3+2} (2^{n_3+2} - 2^{t-1})$ . If  $g$  is the generator  $b$  of the second  $S_2$ -subgroup of  $\text{Aut}(\mathcal{F}(\Delta_1 \times \Delta_1 \times \Delta_{n_3}))$  and  $\lambda \in \Lambda_g$ , then  $\lambda(F_3) = \lambda(F_4)$ . By the argument of Theorem 3.1, we also have  $|\Lambda_g| = (2^{n_3+2} + 4) \prod_{t=1}^{n_3+2} (2^{n_3+2} - 2^{t-1})$ . If  $g$  is the automorphism  $ab$  and  $\lambda \in \Lambda_g$ , then  $\lambda(F_1) = \lambda(F_2)$  and  $\lambda(F_3) = \lambda(F_4)$ . Similarly we have  $|\Lambda_g| = 4 \prod_{t=1}^{n_3+2} (2^{n_3+2} - 2^{t-1})$ . If  $g$  is other automorphism and isn't unit element of  $\text{Aut}(\mathcal{F}(\Delta_1 \times \Delta_1 \times \Delta_{n_3}))$ , by the linear independence condition of characteristic functions, we have that  $\Lambda_g$  is empty. From Theorem 3.1, we have that when  $1 = n_2 < n_3$ ,

$$E(\Delta_1 \times \Delta_1 \times \Delta_{n_3}) = \frac{\prod_{t=1}^{n_3+2} (2^{n_3+2} - 2^{t-1})}{8 \cdot (n_3+1)!} (3 \cdot 2^{2n_3} + 3 \cdot 2^{n_3+2} + 17).$$

When  $1 = n_2 = n_3$ ,  $\Delta_1 \times \Delta_{n_2} \times \Delta_{n_3}$  is a 3-cube  $I^3$  and the automorphism group  $\text{Aut}(\mathcal{F}(I^3)) \cong S_2^3 \times S_3$ . From [4], we know that there are 259 equivariant homeomorphism classes of small covers over  $I^3$ .

The proof is completed.  $\square$

**Remark 2.** When  $2 \leq n_1 \leq n_2 \leq n_3$ , by calculating the number of colorings we can also determine the number of equivariant homeomorphism classes of small covers over  $\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3}$  and the result is the same as Corollary 2.8.

Similarly, we have

**Theorem 4.2.** *The number of equivariant homeomorphism classes of small covers over  $\Delta_1 \times \Delta_{n_2}$  is*

$$E(\Delta_1 \times \Delta_{n_2}) = \begin{cases} \frac{(2^{n_2+3}) \prod_{t=1}^{n_2+1} (2^{n_2+1} - 2^{t-1})}{2 \cdot (n_2+1)!} & \text{for } n_2 > 1, \\ 6 & \text{for } n_2 = 1. \end{cases}$$

### 5. Orientable small covers on the product of simplices

Nakayama and Nishimura found an orientability condition for a small cover [11].

**Theorem 5.1.** *For a basis  $\{e_1, \dots, e_n\}$  of  $(\mathbb{Z}_2)^n$ , a homomorphism  $\varepsilon : (\mathbb{Z}_2)^n \rightarrow \mathbb{Z}_2 = \{0, 1\}$  is defined by  $\varepsilon(e_i) = 1 (i = 1, \dots, n)$ . A small cover  $M(\lambda)$  over a simple convex polytope  $P^n$  is orientable if and only if there exists a basis  $\{e_1, \dots, e_n\}$  of  $(\mathbb{Z}_2)^n$  such that the image of  $\varepsilon\lambda$  is  $\{1\}$ .*

We call a  $(\mathbb{Z}_2)^n$ -coloring which satisfies the orientability condition in Theorem 5.1 an orientable coloring of  $P^n$ . We know that there exists an orientable small cover over every simple convex 3-polytope [11]. Similarly we can know the existence of orientable small covers over the product of at most three simplices by existence of orientable colorings and determine the number of equivariant homeomorphism classes.

By  $O(P^n)$  we denote the set of all orientable colorings on  $P^n$ . There is a natural action of  $GL(n, \mathbb{Z}_2)$  on  $O(P^n)$  defined by the correspondence  $\lambda \mapsto \sigma \circ \lambda$ , and the action on  $O(P^n)$  is free. Assume that  $F_1, \dots, F_n$  of  $\mathcal{F}(P^n)$  meet at one vertex  $p$  of  $P^n$ . Let  $e_1, \dots, e_n$  be the standard basis of  $(\mathbb{Z}_2)^n$ . Write  $B(P^n) = \{\lambda \in O(P^n) \mid \lambda(F_i) = e_i \text{ for } i = 1, \dots, n\}$ . It is easy to check that  $B(P^n)$  is the orbit space of  $O(P^n)$  under the action of  $GL(n, \mathbb{Z}_2)$ .

**Remark 3.** In fact, we have  $B(P^n) = \{\lambda \in O(P^n) \mid \lambda(F_i) = e_i \text{ for } i = 1, \dots, n, \text{ and for } n+1 \leq j \leq \ell, \lambda(F_j) = e_{j_1} + e_{j_2} + \dots + e_{j_{2h_j+1}}, \text{ where } 1 \leq j_1 < j_2 < \dots < j_{2h_j+1} \leq n\}$ . Below we show that  $\lambda(F_j) = e_{j_1} + e_{j_2} + \dots + e_{j_{2h_j+1}}$  for  $n+1 \leq j \leq \ell$ .

If  $\lambda \in O(P^n)$ , there exists a basis  $\{e'_1, \dots, e'_n\}$  of  $(\mathbb{Z}_2)^n$  such that for  $1 \leq i \leq \ell$ ,  $\lambda(F_i) = e'_{i_1} + \dots + e'_{i_{2f_i+1}}$ , where  $1 \leq i_1 < \dots < i_{2f_i+1} \leq n$ . Since  $\lambda(F_i) = e_i$  for  $i = 1, \dots, n$ , we have  $e_i = e'_{i_1} + \dots + e'_{i_{2f_i+1}}$ . So we obtain that for  $n + 1 \leq j \leq \ell$ , there aren't  $j_1, \dots, j_{2k}$  such that  $\lambda(F_j) = e_{j_1} + \dots + e_{j_{2k}}$ , where  $1 \leq j_1 < \dots < j_{2k} \leq n$ .

Since  $B(P^n)$  is the orbit space of  $O(P^n)$ , we have

**Lemma 5.2.**  $|O(P^n)| = |B(P^n)| \times |GL(n, \mathbb{Z}_2)|$ .

Two orientable small covers  $M(\lambda_1)$  and  $M(\lambda_2)$  over  $P^n$  are D-J equivalent if and only if there is  $\sigma \in GL(n, \mathbb{Z}_2)$  such that  $\lambda_1 = \sigma \circ \lambda_2$ . Thus, the number of D-J equivalence classes of orientable small covers over  $P^n$  is  $|B(P^n)|$ .

One can define the right action of  $Aut(\mathcal{F}(P^n))$  on  $O(P^n)$  by  $\lambda \times h \mapsto \lambda \circ h$ , where  $\lambda \in O(P^n)$  and  $h \in Aut(\mathcal{F}(P^n))$ . By improving the classifying result on small covers in [9], we have

**Theorem 5.3.** *Two orientable small covers over an  $n$ -dimensional simple convex polytope  $P^n$  are equivariantly homeomorphic if and only if there is  $h \in Aut(\mathcal{F}(P^n))$  such that  $\lambda_1 = \lambda_2 \circ h$ , where  $\lambda_1$  and  $\lambda_2$  are their corresponding orientable colorings on  $P^n$ .*

*Proof.* We know Theorem 5.3 is true by combining Lemma 5.4 in [9] with Theorem 5.1.  $\square$

By Theorem 5.3, the number of orbits of  $O(P^n)$  under the action of  $Aut(\mathcal{F}(P^n))$  is just the number of equivariant homeomorphism classes of orientable small covers over  $P^n$ . So we also are going to count the orbits.

In the similar way, we calculate the number of all orientable colorings on the product of  $m$  simplices for  $m \leq 3$  by Theorem 5.1, Remark 3 and Lemma 5.2.

**Theorem 5.4.** *By  $|O(\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3})|$  we denote the number of all orientable colorings on  $\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3}$ . Then we have*

$$(1) |O(\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3})| = (2^{4u_1+2u_2-5} + 2^{2u_1+4u_2-5} + 2^{4u_1+2u_3-5} + 2^{2u_1+4u_3-5} + 2^{4u_2+2u_3-5} + 2^{2u_2+4u_3-5} - 2^{4u_1-4} - 2^{4u_2-4} - 2^{4u_3-4} - 2^{2u_1+2u_2-3} - 2^{2u_1+2u_3-3} - 2^{2u_2+2u_3-3} + 1) \prod_{t=1}^{2u_1+2u_2+2u_3-3} (2^{2u_1+2u_2+2u_3-3} - 2^{t-1}) \text{ for } n_1 = 2u_1 - 1, n_2 = 2u_2 - 1 \text{ and } n_3 = 2u_3 - 1,$$

$$(2) |O(\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3})| = (2^{4u_1+2u_2-5} + 2^{2u_1+4u_2-5} + 2^{4u_1+2u_3-4} + 2^{4u_2+2u_3-4} - 2^{4u_1-4} - 2^{4u_2-4} - 2^{2u_1+2u_2-3}) \prod_{t=1}^{2u_1+2u_2+2u_3-2} (2^{2u_1+2u_2+2u_3-2} - 2^{t-1}) \text{ for } n_1 = 2u_1 - 1, n_2 = 2u_2 - 1 \text{ and } n_3 = 2u_3,$$

$$(3) |O(\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3})| = (2^{4u_1+2u_2-4} + 2^{4u_1+2u_3-4} - 2^{4u_1-4}) \prod_{t=1}^{2u_1+2u_2+2u_3-1} (2^{2u_1+2u_2+2u_3-1} - 2^{t-1}) \text{ for } n_1 = 2u_1 - 1, n_2 = 2u_2 \text{ and } n_3 = 2u_3,$$

(4) *There exist no orientable colorings over  $\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3}$  for  $n_1 = 2u_1, n_2 = 2u_2$  and  $n_3 = 2u_3$ .*

**Theorem 5.5.** *The number of all orientable colorings on  $\Delta_{n_1} \times \Delta_{n_2}$  is  $|O(\Delta_{n_1} \times \Delta_{n_2})| =$*

$$\begin{cases} (2^{2u_1-2} + 2^{2u_2-2} - 1) \prod_{t=1}^{2u_1+2u_2-2} (2^{2u_1+2u_2-2} - 2^{t-1}) & \text{for } n_1 = 2u_1 - 1 \text{ and } n_2 = 2u_2 - 1, \\ 2^{2u_1-2} \prod_{t=1}^{2u_1+2u_2-1} (2^{2u_1+2u_2-1} - 2^{t-1}) & \text{for } n_1 = 2u_1 - 1 \text{ and } n_2 = 2u_2, \\ 0 & \text{for } n_1 = 2u_1 \text{ and } n_2 = 2u_2. \end{cases}$$

**Theorem 5.6.** *The number of all orientable colorings on  $\Delta_{n_1}$  is*

$$|O(\Delta_{n_1})| = \begin{cases} \prod_{t=1}^{2u_1-1} (2^{2u_1-1} - 2^{t-1}) & \text{for } n_1 = 2u_1 - 1, \\ 0 & \text{for } n_1 = 2u_1. \end{cases}$$

Similarly, we determine the number of equivariant homeomorphism classes of all orientable small covers over the product of at most three simplices by Corollary 2.5, Burnside Lemma and Theorems 5.3-5.6.

**Theorem 5.7.** By  $E_0(\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3})$  we denote the number of equivariant homeomorphism classes of orientable small covers over  $\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3}$ . If  $n_1 = 2u_1 - 1, n_2 = 2u_2 - 1$  and  $n_3 = 2u_3 - 1$ , then  $E_0(\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3}) =$

- (1)  $\frac{|O(\Delta_{2u_1-1} \times \Delta_{2u_2-1} \times \Delta_{2u_3-1})|}{(2u_1)!(2u_2)!(2u_3)!}$  for  $u_1 \neq u_2, u_1 \neq u_3, u_2 \neq u_3$  and  $u_1, u_2, u_3 > 1$ ,
- (2)  $\frac{|O(\Delta_{2u_1-1} \times \Delta_{2u_2-1} \times \Delta_{2u_3-1})|}{2 \cdot (2u_1)!(2u_2)!(2u_3)!}$  for  $u_1 = u_2, u_1 \neq u_3$  and  $u_1, u_3 > 1$ ,
- (3)  $\frac{|O(\Delta_{2u_1-1} \times \Delta_{2u_2-1} \times \Delta_{2u_3-1})|}{6 \cdot (2u_1)!(2u_2)!(2u_3)!}$  for  $u_1 = u_2 = u_3 > 1$ ,
- (4)  $(2^{4u_2+2u_3-5} + 2^{2u_2+4u_3-5} + 2^{4u_2-4} + 2^{4u_3-4} + 2^{2u_2-1} + 2^{2u_3-1} - 2^{2u_2+2u_3-3} - 1) \frac{\prod_{t=1}^{2u_2+2u_3-1} (2^{2u_2+2u_3-1-2^{t-1}})}{2 \cdot (2u_2)!(2u_3)!}$  for  $u_1 = 1, u_2 \neq u_3$  and  $u_2, u_3 > 1$ ,
- (5)  $(2^{4u_2+2u_3-5} + 2^{2u_2+4u_3-5} + 2^{4u_2-4} + 2^{4u_3-4} + 2^{2u_2-1} + 2^{2u_3-1} - 2^{2u_2+2u_3-3} - 1) \frac{\prod_{t=1}^{2u_2+2u_3-1} (2^{2u_2+2u_3-1-2^{t-1}})}{4 \cdot (2u_2)!(2u_3)!}$  for  $u_1 = 1$  and  $u_2 = u_3 > 1$ ,
- (6)  $(3 \cdot 2^{4u_3-4} + 2^{2u_3} + 5) \frac{\prod_{t=1}^{2u_3+1} (2^{2u_3+1-2^{t-1}})}{8 \cdot (2u_3)!}$  for  $u_1 = u_2 = 1$  and  $u_3 > 1$ ,
- (7) 70 for  $u_1 = u_2 = u_3 = 1$ .

**Theorem 5.8.** If  $n_1 = 2u_1 - 1, n_2 = 2u_2 - 1$  and  $n_3 = 2u_3$ , then  $E_0(\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3}) =$

- (1)  $\frac{|O(\Delta_{2u_1-1} \times \Delta_{2u_2-1} \times \Delta_{2u_3})|}{(2u_1)!(2u_2)!(2u_3+1)!}$  for  $u_1 \neq u_2$  and  $u_1, u_2 > 1$ ,
- (2)  $\frac{|O(\Delta_{2u_1-1} \times \Delta_{2u_2-1} \times \Delta_{2u_3})|}{2 \cdot (2u_1)!(2u_2)!(2u_3+1)!}$  for  $u_1 = u_2 > 1$ ,
- (3)  $(2^{4u_2+2u_3-4} + 2^{4u_2-4} + 2^{2u_2-1} + 2^{2u_3+1} - 2) \frac{\prod_{t=1}^{2u_2+2u_3} (2^{2u_2+2u_3-2^{t-1}})}{2 \cdot (2u_2)!(2u_3+1)!}$  for  $u_1 = 1$  and  $u_2 > 1$ ,
- (4)  $(2^{2u_3+2} + 4) \frac{\prod_{t=1}^{2u_3+2} (2^{2u_3+2-2^{t-1}})}{8 \cdot (2u_3+1)!}$  for  $u_1 = u_2 = 1$ .

**Theorem 5.9.** If  $n_1 = 2u_1 - 1, n_2 = 2u_2$  and  $n_3 = 2u_3$ , then  $E_0(\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3}) =$

- (1)  $\frac{|O(\Delta_{2u_1-1} \times \Delta_{2u_2} \times \Delta_{2u_3})|}{(2u_1)!(2u_2+1)!(2u_3+1)!}$  for  $u_1 > 1$  and  $u_2 \neq u_3$ ,
- (2)  $\frac{|O(\Delta_{2u_1-1} \times \Delta_{2u_2} \times \Delta_{2u_3})|}{2 \cdot (2u_1)!(2u_2+1)!(2u_3+1)!}$  for  $u_1 > 1$  and  $u_2 = u_3$ ,
- (3)  $(2^{2u_2+1} + 2^{2u_3+1} - 2) \frac{\prod_{t=1}^{2u_2+2u_3+1} (2^{2u_2+2u_3+1-2^{t-1}})}{2 \cdot (2u_2+1)!(2u_3+1)!}$  for  $u_1 = 1$  and  $u_2 \neq u_3$ ,
- (4)  $(2^{2u_2+1} + 2^{2u_3+1} - 2) \frac{\prod_{t=1}^{2u_2+2u_3+1} (2^{2u_2+2u_3+1-2^{t-1}})}{4 \cdot (2u_2+1)!(2u_3+1)!}$  for  $u_1 = 1$  and  $u_2 = u_3$ .

**Theorem 5.10.** If  $n_1 = 2u_1, n_2 = 2u_2$  and  $n_3 = 2u_3$ , there exist no orientable small covers over  $\Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3}$ .

**Theorem 5.11.** By  $E_0(\Delta_{n_1} \times \Delta_{n_2})$  we denote the number of equivariant homeomorphism classes of orientable small covers over  $\Delta_{n_1} \times \Delta_{n_2}$ . Then  $E_0(\Delta_{n_1} \times \Delta_{n_2}) =$

$$\left\{ \begin{array}{ll} \frac{|O(\Delta_{2u_1-1} \times \Delta_{2u_2-1})|}{(2u_1)!(2u_2)!} & \text{for } n_1 = 2u_1 - 1 > 1, n_2 = 2u_2 - 1 > 1 \text{ and } n_1 \neq n_2, \\ \frac{|O(\Delta_{2u_1-1} \times \Delta_{2u_2-1})|}{2 \cdot (2u_1)!(2u_2)!} & \text{for } n_1 = 2u_1 - 1 > 1, n_2 = 2u_2 - 1 > 1 \text{ and } n_1 = n_2, \\ (2^{2u_2-2} + 1) \frac{\prod_{t=1}^{2u_2} (2^{2u_2} - 2^{t-1})}{2 \cdot (2u_2)!} & \text{for } n_1 = 1 \text{ and } n_2 = 2u_2 - 1 > 1, \\ 3 & \text{for } n_1 = n_2 = 1, \\ \frac{|O(\Delta_{2u_1-1} \times \Delta_{2u_2})|}{(2u_1)!(2u_2+1)!} & \text{for } n_1 = 2u_1 - 1 > 1 \text{ and } n_2 = 2u_2, \\ \frac{\prod_{t=1}^{2u_2+1} (2^{2u_2+1} - 2^{t-1})}{(2u_2+1)!} & \text{for } n_1 = 1 \text{ and } n_2 = 2u_2, \\ 0 & \text{for } n_1 = 2u_1 \text{ and } n_2 = 2u_2. \end{array} \right.$$

**Theorem 5.12.** By  $E_o(\Delta_{n_1})$  we denote the number of equivariant homeomorphism classes of orientable small covers over  $\Delta_{n_1}$ . Then  $E_o(\Delta_{n_1}) =$

$$\left\{ \begin{array}{ll} \frac{\prod_{t=1}^{2u_1-1} (2^{2u_1-1} - 2^{t-1})}{(2u_1)!} & \text{for } n_1 = 2u_1 - 1 > 1, \\ 1 & \text{for } n_1 = 1, \\ 0 & \text{for } n_1 = 2u_1. \end{array} \right.$$

**Remark 4.** Actually all small covers over  $\Delta_{2u_1-1}$  are orientable, and the number of equivariant homeomorphism classes of orientable small covers over  $\Delta_{2u_1-1}$  is just the number of equivariant homeomorphism classes of small covers over  $\Delta_{2u_1-1}$ .

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