

Generalized ideal convergence in intuitionistic fuzzy normed linear spaces

Bipan Hazarika^a, Vijay Kumar^b, Bernardo Lafuerza-Guillén^c

^aDepartment of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh-791 112, Arunachal Pradesh, India

^bDepartment of Mathematics, Haryana College of Technology and Management, Kaithal-136027, Haryana, India

^cDepartamento de Estadística y Matemática Aplicada, Universidad de Almería, Almería 04120, Spain.

Abstract. An ideal I is a family of subsets of positive integers \mathbb{N} which is closed under taking finite unions and subsets of its elements. In [19], Kostyrko et al. introduced the concept of ideal convergence as a sequence (x_k) of real numbers is said to be I -convergent to a real number ℓ , if for each $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}$ belongs to I . The aim of this paper is to introduce and study the notion of λ -ideal convergence in intuitionistic fuzzy normed spaces as a variant of the notion of ideal convergence. Also I_λ -limit points and I_λ -cluster points have been defined and the relation between them has been established. Furthermore, Cauchy and I_λ -Cauchy sequences are introduced and studied.

1. Introduction

The concept of ideal convergence as a generalization of statistical convergence, and any concept involving statistical convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modelling, and motion planning in robotics.

The notion of statistical convergence for sequences of real numbers by Steinhaus [40] and Fast [13] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from various points of view. For example, statistical convergence has been investigated in summability theory by (Connor [8], Fridy [14], [15], Šalát [35], Schoenberg [38]), number theory and mathematical analysis by (Buck [2], Mitrinović et al. [28]), topological groups (Çakalli [3, 4]), topological spaces (Di Maio and Kočinac [11]), function spaces (Caserta et al. [5], Caserta and Kočinac [6]), locally convex spaces (Maddox[26]), measure theory (Cheng et al. [7], Connor and Swardson [9], Miller[27]), fuzzy mathematics (Nuray and Savaş [33]). In the recent years, generalization of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions. Mursaleen [32], introduced the λ -statistical convergence for real sequences. In this article, we consider only sequences of real numbers, so that "a sequence" means "a sequence of real numbers".

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Email addresses: bh_rgu@yahoo.co.in (Bipan Hazarika), vjy_kaushik@yahoo.com (Vijay Kumar), b1afuerz@ual.es (Bernardo Lafuerza-Guillén)

Following the introduction of fuzzy set theory by Zadeh [44], there has been extensive research to find applications and fuzzy analogues of the classical theories. The theory of intuitionistic fuzzy sets was introduced by Atanassov [1]. Furthermore, Saadati [34] gave the notion of an intuitionistic fuzzy normed space.

The notion of statistical convergence depends on the density (asymptotic or natural) of subsets of \mathbb{N} . A subset of \mathbb{N} is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in E\}| \text{ exists.}$$

Definition 1.1. A sequence $x = (x_k)$ is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}) = 0.$$

In this case, we write $S\text{-}\lim x = \ell$ or $x_k \rightarrow \ell(S)$ and S denotes the set of all statistically convergent sequences.

The notion of I -convergence was initially introduced by Kostyrko, et. al [19] as a generalization of statistical convergence which is based on the structure of the ideal I of subset of natural numbers \mathbb{N} . Kostyrko, et. al [20] gave some of basic properties of I -convergence and dealt with extremal I -limit points.

Although an ideal is defined as a hereditary and additive family of subsets of a non-empty arbitrary set X , here in our study it suffices to take I as a family of subsets of \mathbb{N} , positive integers, i.e. $I \subset 2^{\mathbb{N}}$, such that $A \cup B \in I$ for each $A, B \in I$, and each subset of an element of I is an element of I .

A non-empty family of sets $F \subset 2^{\mathbb{N}}$ is a filter on \mathbb{N} if and only if $\emptyset \notin F$, $A \cap B \in F$ for each $A, B \in F$, and any subset of an element of F is in F . An ideal I is called *non-trivial* if $I \neq \emptyset$ and $\mathbb{N} \notin I$. Clearly I is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter in \mathbb{N} , called the filter associated with the ideal I . A non-trivial ideal I is called *admissible* if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals can be found in Kostyrko et. al. (see [19]).

Recall that a sequence $x = (x_k)$ of points in \mathbb{R} is said to be I -convergent to a real number ℓ if $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$ (see [19]). In this case we write $I\text{-}\lim x_k = \ell$. The notion was further investigated by Šalát, et.al [36], and others.

Let $A \subset \mathbb{N}$ and $d_n(A) = \frac{1}{s_n} \sum_{k=1}^n \frac{\chi_A(k)}{k}$, for $n \in \mathbb{N}$, where $s_n = \sum_{k=1}^n \frac{1}{k}$. If $\lim_{n \rightarrow \infty} d_n(A)$ exists, then it is called as the *logarithmic density* of A . $I_d = \{A \subset \mathbb{N} : d(A) = 0\}$ is an ideal.

Let $T = (t_{nk})$ be a *regular* non-negative matrix. For $A \subset \mathbb{N}$, define $d_T^{(n)}(A) = \sum_{k=1}^{\infty} t_{nk} \chi_A(k)$, for all $n \in \mathbb{N}$. If $\lim_{n \rightarrow \infty} d_T^{(n)}(A) = d_T(A)$ exists, then $d_T(A)$ is called as T -density of A . Clearly $I_{d_T} = \{A \subset \mathbb{N} : d_T(A) = 0\}$ is an ideal.

Note 1. I_δ and I_d are particular cases of I_{d_T} .

(i) Asymptotic density, for

$$t_{nk} = \begin{cases} \frac{1}{n}, & \text{if } n \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) Logarithmic density, for

$$t_{nk} = \begin{cases} \frac{k^{-1}}{s_n}, & \text{if } n \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

If we take $I = I_f = \{A \subseteq \mathbb{N} : A \text{ is a finite subset}\}$, then I_f is a non-trivial admissible ideal of \mathbb{N} and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$, where $\delta(A)$ denote the asymptotic density of the set A , then I_δ is a non-trivial admissible ideal of \mathbb{N} and the corresponding convergence coincides with the statistical convergence.

The existing literature on ideal convergence and its generalizations appears to have been restricted to real or complex sequences, but in recent years these ideas have been also extended to the sequences in fuzzy normed spaces [22] and intuitionistic fuzzy normed spaces ([18, 21, 29–31]). Recently, in [12], Esi and Hazarika introduced and studied the concept of λ -ideal convergence in intuitionistic fuzzy 2-normed space. Further details on ideal convergence, we refer to [10, 16, 17, 23, 24, 41–43] and many others.

Now we recall some notations and basic definitions that we are going to use in this paper.

Definition 1.2. ([39]) A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a *continuous t -norm* if the following conditions are satisfied

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0, 1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 1.3. ([39]) A binary operation o : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a *continuous t -conorm* if the following conditions are satisfied

1. o is associative and commutative,
2. o is continuous,
3. $a o 0 = a$ for all $a \in [0, 1]$,
4. $a o b \leq c o d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 1.4. ([34]) The five-tuple $(X, \mu, \nu, *, o)$ is said to be an *intuitionistic fuzzy normed linear space* (for short, IFNLS) if X is a linear space, $*$ is a continuous t -norm, o is a continuous t -conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

1. $\mu(x, t) + \nu(x, t) \leq 1$,
2. $\mu(x, t) > 0$,
3. $\mu(x, t) = 1$ if and only if $x = 0$,
4. $\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$,
5. $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$,
6. $\mu(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,
7. $\lim_{t \rightarrow \infty} \mu(x, t) = 1$ and $\lim_{t \rightarrow 0} \mu(x, t) = 0$,
8. $\nu(x, t) < 1$,
9. $\nu(x, t) = 0$ if and only if $x = 0$,
10. $\nu(\alpha x, t) = \nu\left(x, \frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$,
11. $\nu(x, t) o \nu(y, s) \geq \nu(x + y, t + s)$,
12. $\nu(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,
13. $\lim_{t \rightarrow \infty} \nu(x, t) = 0$ and $\lim_{t \rightarrow 0} \nu(x, t) = 1$,

In this case (μ, ν) is called an *intuitionistic fuzzy norm* on X and we denote it by (μ, ν) .

Definition 1.5. Let $(X, \mu, \nu, *, o)$ be an intuitionistic fuzzy normed linear space, and let $r \in (0, 1)$ and $x \in X$. The set

$$B(x, r, t) = \{y \in X : \mu(y - x, t) > 1 - r \text{ and } \nu(y - x, t) < r, \text{ for } t > 0\}$$

is called *open ball* with center x and radius r with respect to t .

Definition 1.6. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. The *generalized de la Vallée-Poussin mean* is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $J_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be (V, λ) -summable to a number L (see [25]) if $t_n(x) \rightarrow L$ as $n \rightarrow \infty$. If $\lambda_n = n$, then (V, λ) -summability reduces to $(C, 1)$ -summability.

Mursaleen [32] defined λ -statistically convergent sequence as follows: A sequence $x = (x_k)$ is said to be λ -statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in J_n : |x_k - L| \geq \varepsilon\}| = 0.$$

Let S_λ denotes the set of all λ -statistically convergent sequences. If $\lambda_n = n$, then S_λ is the same as S .

Definition 1.7. ([37]) Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal. A sequence $x = (x_k)$ is said to be I - $[V, \lambda]$ -summable to a number L if, for every $\varepsilon > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} |x_k - L| \geq \varepsilon \right\} \in I.$$

Throughout the paper, we denote by I an admissible ideal of subsets of \mathbb{N} , and $\lambda = (\lambda_n)$ is a sequence as in Definition 1.6, unless otherwise stated.

2. Results

We now give our main results.

Definition 2.1. Let $I \subset 2^{\mathbb{N}}$ and $(X, \mu, \nu, *, o)$ be an IFNLS. A sequence $x = (x_k)$ in X is said to be I_λ -convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0, 1)$ and $t > 0$, the set

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L, t) \geq \varepsilon \right\} \in I.$$

L is called the I_λ -limit of the sequence $x = (x_k)$ and we write $I_\lambda^{(\mu, \nu)} - \lim x = L$.

Example 2.2. Let $(\mathbb{R}, | \cdot |)$ denote the space of all real numbers with the usual norm, and let $a * b = ab$ and $aob = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. For all $x \in \mathbb{R}$ and every $t > 0$, consider $\mu(x, t) = \frac{t}{t+|x|}$ and $\nu(x, t) = \frac{|x|}{t+|x|}$. Then $(\mathbb{R}, \mu, \nu, *, o)$ is an IFNLS. If we take $I = \{A \subset \mathbb{N} : \delta(A) = 0\}$, where $\delta(A)$ denotes the natural density of the set A , then I is a non-trivial admissible ideal. Define a sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} 1, & \text{if } k = i^2, i \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Then for every $\varepsilon \in (0, 1)$ and for any $t > 0$, the set

$$K(\varepsilon, t) = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k, t) \leq 1 - \varepsilon \text{ or } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k, t) \geq \varepsilon \right\}$$

will be a finite set. Hence, $\delta(K(\varepsilon, t)) = 0$ and consequently $K(\varepsilon, t) \in I$, i.e., $I_\lambda^{(\mu, \nu)} - \lim x = 0$.

The proof of the following results are straightforward, thus omitted.

Lemma 2.3. Let $(X, \mu, \nu, *, o)$ be an IFNLS and $x = (x_k)$ be a sequence in X . Then, for every $\varepsilon > 0$ and $t > 0$ the following statements are equivalent:

- (i) $I_\lambda^{(\mu, \nu)} - \lim x = L$,
- (ii) $\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L, t) \leq 1 - \varepsilon\} \in I$ and $\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L, t) \geq \varepsilon\} \in I$,
- (iii) $\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L, t) > 1 - \varepsilon$ and $\frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L, t) < \varepsilon\} \in F(I)$,
- (iv) $\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L, t) > 1 - \varepsilon\} \in F(I)$ and $\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L, t) < \varepsilon\} \in F(I)$,
- (v) $I_\lambda - \lim \mu(x_k - L, t) = 1$ and $I_\lambda - \lim \nu(x_k - L, t) = 0$.

Theorem 2.4. Let $(X, \mu, \nu, *, o)$ be an IFNLS and if a sequence $x = (x_k)$ in X is I_λ -convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) , then $I_\lambda^{(\mu, \nu)} - \lim x$ is unique.

Here, we introduce the notion of λ -convergence in an IFNLS and discuss some properties.

Definition 2.5. Let $(X, \mu, \nu, *, o)$ be an IFNLS. A sequence $x = (x_k)$ in X is λ -convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for $\varepsilon \in (0, 1)$ and every $t > 0$, there exists $n_o \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L, t) > 1 - \varepsilon \text{ and } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L, t) < \varepsilon$$

for all $n \geq n_o$. In this case, we write $(\mu, \nu)^\lambda - \lim x = L$.

Theorem 2.6. Let $(X, \mu, \nu, *, o)$ be an IFNLS and let $x = (x_k)$ in X . If $x = (x_k)$ is λ -convergent with respect to the intuitionistic fuzzy norm (μ, ν) , then $(\mu, \nu)^\lambda - \lim x$ is unique.

Proof. Suppose that $(\mu, \nu)^\lambda - \lim x = L_1$ and $(\mu, \nu)^\lambda - \lim x = L_2$ ($L_1 \neq L_2$). Given $\varepsilon \in (0, 1)$ and choose $\beta \in (0, 1)$ such that $(1 - \beta) * (1 - \beta) > 1 - \varepsilon$ and $\beta o \beta < \varepsilon$. Then for any $t > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L_1, t) > 1 - \varepsilon \text{ and } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L_1, t) < \varepsilon$$

for all $n \geq n_1$. Also, there exists $n_2 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L_2, t) > 1 - \varepsilon \text{ and } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L_2, t) < \varepsilon$$

for all $n \geq n_2$. Now, consider $n_o = \max\{n_1, n_2\}$. Then for $n \geq n_o$, we will get a $s \in \mathbb{N}$ such that

$$\mu\left(x_s - L_1, \frac{t}{2}\right) > \frac{1}{\lambda_n} \sum_{k \in J_n} \mu\left(x_k - L_1, \frac{t}{2}\right) > 1 - \beta$$

and

$$\mu\left(x_s - L_2, \frac{t}{2}\right) > \frac{1}{\lambda_n} \sum_{k \in J_n} \mu\left(x_k - L_2, \frac{t}{2}\right) > 1 - \beta.$$

Then, we have

$$\begin{aligned} \mu(L_1 - L_2, t) &\geq \mu\left(x_s - L_1, \frac{t}{2}\right) * \mu\left(x_s - L_2, \frac{t}{2}\right) \\ &> (1 - \beta) * (1 - \beta) > 1 - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\mu(L_1 - L_2, t) = 1$ for all $t > 0$, by using a similar technique, it can be proved that $\nu(L_1 - L_2, t) = 0$ for all $t > 0$, which implies that $L_1 = L_2$. \square

Theorem 2.7. Let $(X, \mu, \nu, *, o)$ be an IFNLS and let $x = (x_k)$ in X . If $(\mu, \nu)^\lambda - \lim x = L$, then $I_\lambda^{(\mu, \nu)} - \lim x = L$.

Proof. Let $(\mu, \nu)^\lambda - \lim x = L$, then for every $t > 0$ and given $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L, t) > 1 - \varepsilon \quad \text{and} \quad \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L, t) < \varepsilon$$

for all $n \geq n_0$. Therefore the set

$$B = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L, t) \leq 1 - \varepsilon \quad \text{or} \quad \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L, t) \geq \varepsilon \right\}$$

$$\subseteq \{1, 2, \dots, n_0 - 1\}.$$

But, with I being admissible, we have $B \in I$. Hence $I_\lambda^{(\mu, \nu)} - \lim x = L$. \square

Theorem 2.8. Sequential method I_λ is a regular method.

Proof. The proof follows from the fact that I is admissible and Theorem 2.7. \square

Theorem 2.9. Let $(X, \mu, \nu, *, o)$ be an IFNLS and let $x = (x_k)$ in X . If $(\mu, \nu)^\lambda - \lim x = L$, then there exists a subsequence (x_{m_k}) of $x = (x_k)$ such that $(\mu, \nu) - \lim x_{m_k} = L$.

Proof. Let $(\mu, \nu)^\lambda - \lim x = L$. Then, for every $t > 0$ and given $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L, t) > 1 - \varepsilon \quad \text{and} \quad \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L, t) < \varepsilon$$

for all $n \geq n_0$. Clearly, for each $n \geq n_0$, we can select an $m_k \in J_n$ such that

$$\mu(x_{m_k} - L, t) > \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L, t) > 1 - \varepsilon$$

and

$$\nu(x_{m_k} - L, t) < \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L, t) < \varepsilon.$$

It follows that $(\mu, \nu) - \lim x_{m_k} = L$. \square

Definition 2.10. Let $(X, \mu, \nu, *, o)$ be an IFNLS and let $x = (x_k)$ be a sequence in X . Then:

1. An element $L \in X$ is said to be an I_λ -limit point of $x = (x_k)$ if there is a set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \subset \mathbb{N}$ such that the set $M^t = \{n \in \mathbb{N} : m_k \in J_n\} \notin I$ and $(\mu, \nu)^\lambda - \lim x_{m_k} = L$.
2. An element $L \in X$ is said to be an I_λ -cluster point of $x = (x_k)$ if for every $t > 0$ and $\varepsilon \in (0, 1)$, we have

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L, t) > 1 - \varepsilon \quad \text{and} \quad \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L, t) < \varepsilon \right\} \notin I.$$

Let $\Lambda_{(\mu, \nu)^\lambda}(x)$ denote the set of all I_λ -limit points and $\Gamma_{(\mu, \nu)^\lambda}(x)$ denote the set of all I_λ -cluster points in X , respectively.

Theorem 2.11. Let $(X, \mu, \nu, *, o)$ be an IFNLS. For each sequence $x = (x_k)$ in X , we have $\Lambda_{(\mu, \nu)^\lambda}(x) \subset \Gamma_{(\mu, \nu)^\lambda}(x)$.

Proof. Let $L \in \Lambda_{(\mu, \nu)^\lambda}(x)$, then there exists a set $M \subset \mathbb{N}$ such that $M' \notin I$, where M and M' are as in the Definition 2.10, satisfies $(\mu, \nu)^\lambda - \lim x_{m_k} = L$. Thus, for every $t > 0$ and $\varepsilon \in (0, 1)$, there exists $n_o \in \mathbb{N}$ such that

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_{m_k} - L, t) > 1 - \varepsilon \text{ and } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_{m_k} - L, t) < \varepsilon$$

for all $n \geq n_o$. Therefore,

$$B = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L, t) > 1 - \varepsilon \text{ and } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L, t) < \varepsilon \right\}$$

$$\supseteq M' \setminus \{m_1, m_2, \dots, m_{k_o}\}.$$

Now, with I being admissible, we must have $M' \setminus \{m_1, m_2, \dots, m_{k_o}\} \notin I$ and as such $B \notin I$. Hence $L \in \Gamma_{(\mu, \nu)^\lambda}(x)$. \square

Theorem 2.12. Let $(X, \mu, \nu, *, o)$ be an IFNLS. For each sequence $x = (x_k)$ in X , the set $\Gamma_{(\mu, \nu)^\lambda}(x)$ is closed set in X with respect to the usual topology induced by the intuitionistic fuzzy norm $(\mu, \nu)^\lambda$.

Proof. Let $y \in \overline{\Gamma_{(\mu, \nu)^\lambda}(x)}$. Take $t > 0$ and $\varepsilon \in (0, 1)$. Then there exists $L_o \in \Gamma_{(\mu, \nu)^\lambda}(x) \cap B(y, \varepsilon, t)$. Choose $\delta > 0$ such that $B(L_o, \delta, t) \subset B(y, \varepsilon, t)$. We have

$$G = \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - y, t) > 1 - \varepsilon \text{ and } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - y, t) < \varepsilon\}$$

$$\supseteq \{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L_o, t) > 1 - \delta \text{ and } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L_o, t) < \delta\} = H.$$

Thus $H \notin I$ and so $G \notin I$. Hence $y \in \Gamma_{(\mu, \nu)^\lambda}(x)$. \square

Theorem 2.13. Let $(X, \mu, \nu, *, o)$ be an IFNLS and let $x = (x_k)$ in X . Then the following statements are equivalent:

- (1) L is a I_λ -limit point of x ,
- (2) There exist two sequences y and z in X such that $x = y + z$ and $(\mu, \nu)^\lambda - \lim y = L$ and $\{n \in \mathbb{N} : k \in J_n, z_k \neq \theta\} \in I$, where θ is the zero element of X .

Proof. Suppose that (1) holds. Then there exist sets M and M' as in Definition 2.10 such that $M' \notin I$ and $(\mu, \nu)^\lambda - \lim x_{m_k} = L$. Define the sequences y and z as follows:

$$y_k = \begin{cases} x_k, & \text{if } k \in J_n; n \in M' \\ L, & \text{otherwise.} \end{cases}$$

and

$$z_k = \begin{cases} \theta, & \text{if } k \in J_n; n \in M' \\ x_k - L, & \text{otherwise.} \end{cases}$$

It suffices to consider the case $k \in J_n$ such that $n \in \mathbb{N} - M'$. Then for each $\varepsilon \in (0, 1)$ and $t > 0$, we have $\mu(y_k - L, t) = 1 > 1 - \varepsilon$ and $\nu(y_k - L, t) = 0 < \varepsilon$. Thus, in this case,

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_k - L, t) = 1 > 1 - \varepsilon \text{ and } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_k - L, t) = 0 < \varepsilon.$$

Hence $(\mu, \nu)^\lambda - \lim y = L$. Now $\{n \in \mathbb{N} : k \in J_n, z_k \neq \theta\} \subset \mathbb{N} - M'$ and so $\{n \in \mathbb{N} : k \in J_n, z_k \neq \theta\} \in I$.

Now, suppose that (2) holds. Let $M' = \{n \in \mathbb{N} : k \in J_n, z_k = \theta\}$. Then, clearly $M' \in F(I)$ and so it is an infinite set. Construct the set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \subset \mathbb{N}$ such that $m_k \in J_n$ and $z_{m_k} = \theta$. Since $x_{m_k} = y_{m_k}$ and $(\mu, \nu)^\lambda - \lim y = L$ we obtain $(\mu, \nu)^\lambda - \lim x_{m_k} = L$. This completes the proof. \square

Theorem 2.14. Let $(X, \mu, \nu, *, o)$ be an IFNLS and $x = (x_k)$ be a sequence in X . Let I be a non-trivial ideal in \mathbb{N} . If there is a $I_\lambda^{(\mu, \nu)}$ -convergent sequence $y = (y_k)$ in X such that $\{k \in \mathbb{N} : y_k \neq x_k\} \in I$, then x is also $I_\lambda^{(\mu, \nu)}$ -convergent.

Proof. Suppose that $\{k \in \mathbb{N} : y_k \neq x_k\} \in I$ and $I_\lambda^{(\mu, \nu)} - y = \ell$. Then for every $\varepsilon \in (0, 1)$ and $t > 0$, the set

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_k - L, t) \leq 1 - \varepsilon \text{ or } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_k - L, t) \geq \varepsilon \right\} \in I.$$

For every $0 < \varepsilon < 1$ and $t > 0$, we have

$$\begin{aligned} & \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L, t) \geq \varepsilon \right\} \\ & \subseteq \{k \in \mathbb{N} : y_k \neq x_k\} \cup \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(y_k - L, t) \leq 1 - \varepsilon \text{ or } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(y_k - L, t) \geq \varepsilon \right\}. \end{aligned} \tag{1}$$

As both the sets of right-hand side of (1) are in I , we have that

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - L, t) \geq \varepsilon \right\} \in I.$$

This completes the proof of the theorem. \square

Definition 2.15. Let $(X, \mu, \nu, *, o)$ be an IFNLS. A sequence $x = (x_k)$ in X is said to be a *Cauchy sequence* with respect to the intuitionistic fuzzy norm $(\mu, \nu)^\lambda$ if, for every $t > 0$ and $\varepsilon \in (0, 1)$, there exist $n_o, m \in \mathbb{N}$ satisfying

$$\frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - x_m, t) > 1 - \varepsilon \text{ and } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - x_m, t) < \varepsilon$$

for all $n \geq n_o$

Definition 2.16. Let $(X, \mu, \nu, *, o)$ be an IFNLS. A sequence $x = (x_k)$ in X is said to be an I_λ -*Cauchy sequence* with respect to the intuitionistic fuzzy norm $(\mu, \nu)^\lambda$ if, for every $t > 0$ and $\varepsilon \in (0, 1)$, there exists $m \in \mathbb{N}$ satisfying

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in J_n} \mu(x_k - x_m, t) > 1 - \varepsilon \text{ and } \frac{1}{\lambda_n} \sum_{k \in J_n} \nu(x_k - x_m, t) < \varepsilon \right\} \in F(I)$$

Definition 2.17. Let $(X, \mu, \nu, *, o)$ be an IFNLS. A sequence $x = (x_k)$ in X is said to be an I_λ^* -*Cauchy sequence* with respect to the intuitionistic fuzzy norm $(\mu, \nu)^\lambda$ if, there exists a set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \subset \mathbb{N}$ such that the set $M' = \{n \in \mathbb{N} : m_k \in J_n\} \in F(I)$ and the subsequence (x_{m_k}) of $x = (x_k)$ is a Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)^\lambda$.

The proof of the following result is straightforward from the definitions.

Theorem 2.18. Let $(X, \mu, \nu, *, o)$ be an IFNLS. If a sequence $x = (x_k)$ in X is Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)^\lambda$, then it is I_λ -Cauchy sequence with respect to the same norm.

We formulate also the following two results without proofs, because they can be easily established.

Theorem 2.19. Let $(X, \mu, \nu, *, o)$ be an IFNLS. If a sequence $x = (x_k)$ in X is Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)^\lambda$, then there is a subsequence of $x = (x_k)$ which is ordinary Cauchy sequence with respect to the same norm.

Theorem 2.20. Let $(X, \mu, \nu, *, o)$ be an IFNLS. If a sequence $x = (x_k)$ in X is I_λ^* -Cauchy sequence with respect to the intuitionistic fuzzy norm $(\mu, \nu)^\lambda$, then it is I_λ -Cauchy sequence as well.

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