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Unicity theorem for entire functions sharing one value

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Abstract. In the paper, we study the uniqueness of entire functions concerning certain nonlinear differential polynomials sharing one value and obtain two theorems which improve and supplement the related results due to X.Y. Zhang, J.F. Chen and W.C. Lin [Comput. Math. Appl., 56(2008), 1876-1883] and J.F. Chen, X.Y. Zhang, W.C. Lin and S.J. Chen [Comput. Math. Appl., 56(2008), 3000-3014].

1. Introduction, Definitions and Results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the standard notations of value distribution theory which can be found, for instance, in [6, 12]. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure.

Let $f(z)$ be a nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$ and $a \in \{\infty\} \cup \mathbb{C}$. We set $E(a; f) = \{z : f(z) - a = 0\}$, where a zero point of multiplicity m is counted m times in the set. If the zeros are counted only once, then we denote the set by $\overline{E}(a; f)$. For a positive integer k we denote by $E_{k)}(a; f)$ the set of those a-points of f with their multiplicities $\leq k$, and each a-point of f with multiplicity m is counted m times, denote by $\overline{E}_{k)}(a; f)$ the reduced form of $E_{k)}(a; f)$.

Let f and g be two nonconstant meromorphic functions. We say that f and g share the value a CM (counting multiplicities) if $f - a$ and $g - a$ have the same zeros with the same multiplicities, i.e., $E(a; f) = E(a; g)$. If we do not consider the multiplicities, then we say that f and g share the value a IM (ignoring multiplicities), i.e., $\overline{E}(a; f) = \overline{E}(a; g)$.

Corresponding to one famous question of W.K. Hayman [5], M.L. Fang and X.H. Hua [3], C.C. Yang and X.H. Hua [11] showed that similar conclusions hold for certain types of differential polynomials when they share one value. They proved the following result.

Theorem A. *Let f and g be two nonconstant entire functions, $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2000, Xu and Qiu [9] improved the above result by deriving the following result.

Theorem B. *Let f and g be two nonconstant entire functions, $n \geq 12$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 IM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

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In 2002, considering kth derivative instead of 1st derivative, M.L. Fang [4] proved the following theorems.

Theorem C. *[4] Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k+4$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem D. *[4] Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n \geq 2k+8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

In 2008, X.Y. Zhang, J.F. Chen and W.C. Lin [13] proved the following uniqueness theorem for entire functions concerning certain general nonlinear differential polynomials.

Theorem E. *[13] Let f and g be two nonconstant entire functions and let n, m and k be three positive integers with $n \geq 2k+3m+5$. Let $P(z) = a_mz^m + a_{m-1}z^{m-1} + ... + a_1z + a_0$ or $P(z) = c_0$, where $a_0 \neq 0, a_1, ... ,a_{m-1}, a_m \neq 0, c_0 \neq 0$ are complex constants. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share 1 CM, then*
(i) when $P(z) = a_mz^m + a_{m-1}z^{m-1} + ... + a_1z + a_0$, either $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = (n+m, ..., n+m-i, ..., n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, ..., m$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$R(x, y) = x^n(a_mx^m + a_{m-1}x^{m-1} + ... + a_0) - y^n(a_my^m + a_{m-1}y^{m-1} + ... + a_0);$$

(ii) when $P(z) = c_0$, either $f(z) = c_1/c_0^{\frac{1}{n}}e^{cz}$, $g(z) = c_2/c_0^{\frac{1}{n}}e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.

Now following question arises.

Question 1.1. *Whether CM sharing value can be replaced by IM sharing value in Theorems C, D and E ?*

In 2008, J.F. Chen, X.Y. Zhang, W.C. Lin and S.J. Chen [2] answered the above question for Theorems C and D. They proved the following four theorems.

Theorem F. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 5k+7$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 IM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem G. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 5k+13$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 IM, then $f \equiv g$.*

Theorem H. *Let f and g be two nonconstant entire functions and let n, m and k be three positive integers. If $E_{m)}\left(1;(f^n)^{(k)}\right) = E_{m)}\left(1;(g^n)^{(k)}\right)$, and*
(i) if $m = 1$ and $n > 4k+6$, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$; or
(ii) if $m = 2$ and $n > \frac{5k+9}{2}$, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$; or
(iii) if $m \geq 3$ and $n > 2k+4$, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.

Theorem I. *Let f and g be two nonconstant entire functions, and let n, k and m be three positive integers. If $E_{m)}\left(1;(f^n(f-1))^{(k)}\right) = E_{m)}\left(1;(g^n(g-1))^{(k)}\right)$, and*
(i)if $m = 1$ and $n > 4k+11$, then $f \equiv g$; or
(ii)if $m = 2$ and $n > \frac{5k+16}{2}$, then $f \equiv g$; or
(iii)if $m \geq 3$ and $n > 2k+7$, then $f \equiv g$.

Now it is natural to ask the following questions which are the motivation of the author.

Question 1.2. *Is it possible to replace CM sharing value by IM sharing value in Theorem E ?*

Question 1.3. *Whether one can deduce a generalised result in which all the results of J.F. Chen, X.Y. Zhang, W.C. Lin and S.J. Chen stated above will be included ?*

In the paper we will concentrate our attention on the above questions and provide an affirmative solution in this direction. We will prove two theorems second one of which will not only improve Theorem E but also improve and supplement Theorems H and I. Our first theorem will improve and supplement Theorems F and G. We now state the main results of the paper.

Theorem 1.4. *Let f and g be two nonconstant entire functions and let n, m and k be three positive integers. Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$ or $P(z) = c_0$, where $a_0 (\neq 0)$, a_1, ... ,a_{m-1}, $a_m (\neq 0)$, $c_0 (\neq 0)$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share 1 IM and $n > 5k + 6m + 7$, then the conclusions (i) and (ii) of Theorem E hold.*

Theorem 1.5. *Let f and g be two nonconstant entire functions, and let n, m, k and p be four positive integers. Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$ or $P(z) = c_0$, where $a_0 (\neq 0)$, a_1, ... ,a_{m-1}, $a_m (\neq 0)$, and $c_0 (\neq 0)$ are complex constants. Let $E_{p)}\left(1; (f^n P(f))^{(k)}\right) = E_{p)}\left(1; (g^n P(g))^{(k)}\right)$. Then the conclusions (i) and (ii) of Theorem E hold in each of the following cases:*
(i) $p = 1$, and $n > 4k + 5m + 6$; or
(ii) $p = 2$ and $n > \frac{5k+7m+9}{2}$; or
(iii) $p \geq 3$ and $n > 2k + 3m + 4$.

Remark 1.6. *Theorem 1.5 is an improvement of Theorem E.*

Remark 1.7. *If we put $P(z)$= Constant = 1, then we see that Theorems F and H are the special cases of Theorem 1.4 and 1.5 respectively. Hence Theorems 1.4 and 1.5 improves Theorems F and H respectively.*

Remark 1.8. *If we put $P(z) = z - 1$, then we see that Theorems G and I are the special cases of Theorems 1.4 and 1.5 respectively. Thus Theorems 1.4 and 1.5 improves Theorems G and I respectively.*
In this case $R(f, g) = 0$ implies

$$f^n(f - 1) = g^n(g - 1).$$

Suppose $f \not\equiv g$. Let $h = \frac{f}{g}$. Then

$$g = \frac{h^n - 1}{h^{n+1} - 1}.$$

Thus we deduce by Picard's theorem that h is a constant since g is an entire function. Hence, g is a constant, which is a contradiction. Therefore, $f \equiv g$.

Though the standard definitions and notations of the value distribution theory are available in [6], we explain some definitions and notations which are used in the paper.

Definition 1.9. *[7] For $a \in \{\infty\} \cup \mathbb{C}$ we denote by $N(r, a; f \mid= 1)$ the counting functions of simple a-points of f. For a positive integer p we denote by $N(r, a; f \mid\leq p)$ ($N(r, a; f \mid\geq p)$) the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater (less) than p, where each a-point is counted according to its multiplicity. $\overline{N}(r, a; f \mid\leq p)$ and $\overline{N}(r, a; f \mid\geq p)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities. Also $N(r, a; f \mid< p)$ and $N(r, a; f \mid> p)$ are defined analogously.*

Definition 1.10. *[8] Let p be a positive integer or infinity. We denote by $N_p(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Then*

$$N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + ... + \overline{N}(r, a; f \mid\geq p).$$

Definition 1.11. *Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. We define by $N_{11}(r, a; f)$ the counting function for common simple 1-points of f and g where multiplicity is not counted.*

Definition 1.12. *Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r, 1; f)$ the counting function of those 1-points of f and g where $p > q$, with multiplicity being not counted. $\overline{N}_L(r, 1; g)$ is defined analogously.*

2. Lemmas and Propositions

In this section we present some lemmas and propositions which will be needed in the sequel.

Proposition 2.1. *[2] Let $f(z)$ and $g(z)$ be two nonconstant entire functions and let n, k be two positive integers with $n > k$. If $[f^n]^{(k)}[g^n]^{(k)} \equiv 1$, then $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$.*

Proposition 2.2. *[13] Let $f(z)$ and $g(z)$ be two nonconstant entire functions and let n, k be two positive integers with $n > k$, and let $P(z) = a_mz^m + a_{m-1}z^{m-1} + ... + a_2z^2 + a_1z + a_0$ be a nonzero polynomial, where $a_0, a_1, a_2, \ldots, a_{m-1}, a_m$ are complex constants. If $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv 1$, then $P(z)$ is reduced to a nonzero monomial, that is, $P(z) = a_iz^i \not\equiv 0$ for some $i = 0, 1, 2, ..., m$; further, $f(z) = c_1/a_i^{\frac{1}{n+i}}e^{cz}$, $g(z) = c_2/a_i^{\frac{1}{n+i}}e^{-cz}$, where c_1, c_2 and c are three constants satisfying*

$$(-1)^k(c_1c_2)^{n+i}[(n+i)c]^{2k} = 1.$$

Proposition 2.3. *[13] Let f be a transcendental entire function, and n, m, k be three positive integers such that $n \geq k+2$, and $P(z) = a_mz^m + a_{m-1}z^{m-1} + ... + a_2z^2 + a_1z + a_0$, where $a_0, a_1, a_2, \ldots, a_{m-1}, a_m$ are complex constants. Then $[f^nP(f)]^{(k)} = 1$ has infinitely many solutions.*

Lemma 2.4. *[10] Let f be a nonconstant meromorphic function and let $a_n(z)(\not\equiv 0), a_{n-1}(z), \ldots, a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, ..., n$. Then*

$$T(r, a_nf^n + a_{n-1}f^{n-1} + ... + a_1f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.5. *[6, 12] Let f be a transcendental entire function, and let k be a positive integer. Then for any non-zero finite complex number c*

$$\begin{aligned} T(r,f) &\leq N(r,0;f) + N(r,c;f^{(k)}) - N(r,0;f^{(k+1)}) + S(r,f) \\ &\leq N_{k+1}(r,0;f) + \overline{N}(r,c;f^{(k)}) - N_0(r,0;f^{(k+1)}) + S(r,f), \end{aligned}$$

where $N_0(r,0;f^{(k+1)})$ denotes the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)}-c) \neq 0$.

Lemma 2.6. *[1] For two positive integers p and k*

$$N_p(r,0;f^{(k)}) \leq N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) - \sum_{m=p+1}^{\infty} \overline{N}\left(r,0;\frac{f^{(k)}}{f} \mid\geq m\right) + S(r,f).$$

Lemma 2.7. *[6, 12] Let f be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two distinct meromorphic functions such that $T(r, a_i(z)) = S(r, f)$, i=1,2. Then*

$$T(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f).$$

Lemma 2.8. *[2] Let f and g be two nonconstant entire functions, and let p and k be two positive integers. If $f^{(k)}$ and $g^{(k)}$ share 1 IM, then one of the following two cases holds:*
(i)

$$\begin{aligned} T(r,f) + T(r,g) \leq\ & 2\left[N_{k+1}(r,0;f) + N_{k+1}(r,0;g) + \overline{N}(r,0;f) + \overline{N}(r,0;g)\right] \\ & +3\left[\overline{N}_L\left(r,1;f^{(k)}\right) + \overline{N}_L\left(r,1;g^{(k)}\right)\right] + S(r,f) + S(r,g); \end{aligned}$$

(ii) $\frac{1}{f^{(k)}-1} = \frac{bg^{(k)}+a-b}{g^{(k)}-1}$,
where $a \neq 0$, b are two constants.

Lemma 2.9. *Let f and g be two nonconstant entire functions, and let p and k be two positive integers. If* $E_{p)}\left(1, f^{(k)}\right) = E_{p)}\left(1, g^{(k)}\right)$*, then one of the following two cases holds:*
(i)

$$\begin{aligned} T(r,f) + T(r,g) \le\ & N_{k+1}(r,0;f) + N_{k+1}(r,0;g) + \overline{N}(r,0;f) \\ & + \overline{N}(r,0;g) + \overline{N}\left(r,1;f^{(k)}\right) + \overline{N}\left(r,1;g^{(k)}\right) \\ & - N_{11}\left(r,1;f^{(k)}\right) + \overline{N}\left(r,1;f^{(k)} \mid \ge p+1\right) \\ & + \overline{N}\left(r,1;g^{(k)} \mid \ge p+1\right) + S(r,f) + S(r,g); \end{aligned}$$

(ii) $\frac{1}{f^{(k)}-1} = \frac{bg^{(k)}+a-b}{g^{(k)}-1}$,
where $a \ne 0$, b *are two constants.*

Proof. Let

$$H(z) = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}. \tag{1}$$

where $F \equiv f^{(k)}$ and $G \equiv g^{(k)}$. Since $E_{p)}\left(1, f^{(k)}\right) = E_{p)}\left(1, g^{(k)}\right)$, from (1) we see that if z_0 is a common simple 1-point of F and G, then it is a zero of H. Now we consider two cases: $H(z) \not\equiv 0$ and $H(z) \equiv 0$. Let $H(z) \not\equiv 0$. Then we have

$$\begin{aligned} N_{11}(r,1;F) = N_{11}(r,1;G) \le\ & \overline{N}(r,0;H) \le T(r,H) + O(1) \\ \le\ & N(r,\infty;H) + S(r,f) + S(r,g). \end{aligned} \tag{2}$$

It is easy to see that $H(z)$ have poles only at zeros of F' and G' and 1-points of F whose multiplicities are not equal to the multiplicities of the corresponding 1-points of G. So from (1) we have

$$\begin{aligned} N(r,\infty;H) \le\ & \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,1;F \mid \ge p+1) \\ & + \overline{N}(r,1;G \mid \ge p+1) + N_0(r,0;F') + N_0(r,0;G'), \end{aligned} \tag{3}$$

where $N_0(r,0;F')$ and $N_0(r,0;G')$ has the same meaning as in Lemma 2.5. By Lemma 2.5 we have

$$T(r,f) \le N_{k+1}(r,0;f) + \overline{N}(r,1;F) - N_0(r,0;F') + S(r,f), \tag{4}$$

and

$$T(r,g) \le N_{k+1}(r,0;g) + \overline{N}(r,1;G) - N_0(r,0;G') + S(r,g). \tag{5}$$

Using (2), (3), (4) and (5) we obtain

$$\begin{aligned} T(r,f) + T(r,g) \le\ & N_{k+1}(r,0;f) + N_{k+1}(r,0;g) + \overline{N}(r,1;F) \\ & + \overline{N}(r,1;G) - N_0(r,0;F') - N_0(r,0;G') \\ & + S(r,f) + S(r,g) \\ \le\ & N_{k+1}(r,0;f) + N_{k+1}(r,0;g) + \overline{N}(r,1;F) \\ & + \overline{N}(r,1;G) + \overline{N}(r,0;f) + \overline{N}(r,0;g) \\ & - N_{11}(r,1;F) + \overline{N}(r,1;F \mid \ge p+1) \\ & + \overline{N}(r,1;G \mid \ge p+1) + S(r,f) + S(r,g), \end{aligned}$$

which is (i).
If $H(z) \equiv 0$, then from (1) we obtain

$$\frac{F''}{F'} - \frac{2F'}{F-1} \equiv \frac{G''}{G'} - \frac{2G'}{G-1}.$$

Integrating both sides twice we get

$$\frac{1}{F-1} \equiv \frac{bG+a-b}{G-1},$$

where $a(\neq 0)$ and b are constants, which is (ii). □

Lemma 2.10. *Let f be a nonconstant meromorphic function, and let n, m and k be positive integers with $n > k$. If $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_2 z^2 + a_1 z + a_0$, where $a_0, a_1, a_2, \dots, a_{m-1}, a_m$ are complex constants, then*

$$\begin{aligned} N\left(r,1;[f^nP(f)]^{(k)}\right) - \overline{N}\left(r,1;[f^nP(f)]^{(k)}\right) &\leq (k+1)[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)] \\ &\quad +mT(r,f)+S(r,f). \end{aligned}$$

Proof. We put $F = [f^nP(f)]^{(k)}$. Then using Lemma 2.6 we have

$$\begin{aligned} N(r,1;F) - \overline{N}(r,1;F) &\leq N\left(r,\infty;\frac{F}{F'}\right) \\ &\leq N\left(r,\infty;\frac{F'}{F}\right) + S(r,f) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq N_{k+1}(r,0;[f^nP(f)]) + (k+1)\overline{N}(r,\infty;f) + S(r,f) \\ &\leq N_{k+1}(r,0;f^n) + N_{k+1}(r,0;P(f)) + (k+1)\overline{N}(r,\infty;f) + S(r,f) \\ &\leq (k+1)\left[\overline{N}(r,0;f) + \overline{N}(r,\infty;f)\right] + mT(r,f) + S(r,f). \end{aligned}$$

□

3. Proofs of the Theorems

Proof. [Proof of Theorem 1.5] Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$, where $a_0 \neq 0, a_1, ... ,a_{m-1}, a_m \neq 0$ are complex constants. We consider $F(z) = f^nP(f)$ and $G(z) = g^nP(g)$.
(i) Let $p = 1$. Then by Lemma 2.10 we have

$$\begin{aligned} \overline{N}\left(r,1;F^{(k)} \mid\geq 2\right) &\leq N\left(r,1;F^{(k)}\right) - \overline{N}\left(r,1;F^{(k)}\right) \\ &\leq (k+1)\overline{N}(r,0;f) + mT(r,f) + S(r,f). \end{aligned}$$

Similarly, we have

$$\overline{N}\left(r,1;G^{(k)} \mid\geq 2\right) \leq (k+1)\overline{N}(r,0;g) + mT(r,g) + S(r,g).$$

Suppose (i) of Lemma 2.9 holds for F and G. Then

$$\begin{aligned} T(r,F) + T(r,G) &\leq N_{k+1}(r,0;F) + N_{k+1}(r,0;G) + \overline{N}(r,0;F) \\ &\quad +\overline{N}(r,0;G) + \overline{N}\left(r,1;F^{(k)}\right) + \overline{N}\left(r,1;G^{(k)}\right) \\ &\quad -N_{11}\left(r,1;F^{(k)}\right) + \overline{N}\left(r,1;F^{(k)} \mid\geq 2\right) \\ &\quad +\overline{N}\left(r,1;G^{(k)} \mid\geq 2\right) + S(r,F) + S(r,G). \end{aligned} \tag{6}$$

Now it is clear that

$$\begin{aligned} N_{k+1}(r,0;F) &\leq N_{k+1}(r,0;f^n) + N_{k+1}(r,0;P(f)) \\ &\leq (k+1)\overline{N}(r,0;f) + mT(r,f), \end{aligned}$$

$$N_{k+1}(r,0;G) \leq (k+1)\overline{N}(r,0;g) + mT(r,g),$$

$$\begin{aligned}
\overline{N}(r,0;F) &\leq \overline{N}(r,0;f)+mT(r,f),\\
\overline{N}(r,0;G) &\leq \overline{N}(r,0;g)+mT(r,g),\\
\overline{N}\left(r,1;F^{(k)}\right) &\leq \frac{1}{2}N\left(r,1;F^{(k)}\mid\leq 1\right)+\frac{1}{2}N\left(r,1;F^{(k)}\right),\\
\overline{N}\left(r,1;G^{(k)}\right) &\leq \frac{1}{2}N\left(r,1;G^{(k)}\mid\leq 1\right)+\frac{1}{2}N\left(r,1;G^{(k)}\right),
\end{aligned}$$

$$N_{11}\left(r,1;F^{(k)}\right)=N\left(r,1;F^{(k)}\mid\leq 1\right)=N\left(r,1;G^{(k)}\mid\leq 1\right),$$

$$\begin{aligned}
N\left(r,1;F^{(k)}\right) &\leq T\left(r,F^{(k)}\right)+O(1)\\
&\leq T(r,F)+S(r,f)\\
&\leq (n+m)T(r,f)+S(r,f),\\
N\left(r,1;G^{(k)}\right) &\leq T\left(r,G^{(k)}\right)+O(1)\\
&\leq (n+m)T(r,g)+S(r,g).
\end{aligned}$$

So from (6) we obtain

$$\begin{aligned}
(n+m)[T(r,f)+T(r,g)] \leq\ & 2(k+1)\overline{N}(r,0;f)+2mT(r,f)+2(k+1)\overline{N}(r,0;g)\\
&+2mT(r,g)+\overline{N}(r,0;f)+mT(r,f)+\overline{N}(r,0;g)\\
&+mT(r,g)+\frac{n+m}{2}T(r,f)+\frac{n+m}{2}T(r,g)\\
&+S(r,f)+S(r,g),
\end{aligned}$$

i.e.,

$$\left[\frac{n}{2}-(2k+\frac{5}{2}m+3)\right]\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

which is a contradiction because $n>4k+5m+6$. Hence by Lemma 2.9 we have

$$\frac{1}{F^{(k)}-1}=\frac{bG^{(k)}+a-b}{G^{(k)}-1}, \tag{7}$$

where $a\neq 0$, b are two constants. We now discuss the following three cases seperately.

Case 3.1. *Let $b\neq 0$ and $a=b$. Then from (7) we get*

$$\frac{1}{F^{(k)}-1}=\frac{bG^{(k)}}{G^{(k)}-1}. \tag{8}$$

If $b=-1$, then from (8) we obtain

$$F^{(k)}G^{(k)}=1,$$

i.e.,

$$\left[f^n(a_mf^m+...+a_0)\right]^{(k)}\left[g^n(a_mg^m+...+a_0)\right]^{(k)}=1,$$

which by the assumptions and Proposition 2.2 is a contradiction.
If $b\neq -1$, then it follows from (8) and the fact that f and g are entire that

$$F^{(k)}-\left(1+\frac{1}{b}\right)=-\frac{1}{bG^{(k)}}\neq 0.$$

So using Lemma 2.5 we get

$$\begin{aligned}(n+m)T(r,f) &= T(r,F)+O(1)\\ &\leq N_{k+1}(r,0;F)+S(r,f)\\ &\leq (k+1)\overline{N}(r,0;f)+mT(r,f)+S(r,f)\\ &\leq (k+1+m)T(r,f)+S(r,f),\end{aligned}$$

i.e.,

$$[n-(k+1)]T(r,f)\leq S(r,f),$$

which is a contradiction as $n>4k+5m+6$.

Case 3.2. *Let $b\neq 0$ and $a\neq b$. Then from (7) we get*

$$G^{(k)}+\frac{a-b}{b}=-\frac{a}{b^2(F^{(k)}-(1+\frac{1}{b}))}\neq 0.$$

So by Lemma 2.5 we have

$$\begin{aligned}(n+m)T(r,g) &= T(r,G)+O(1)\\ &\leq N_{k+1}(r,0;G)+S(r,G).\end{aligned}$$

Proceeding as Case 3.1 we obtain

$$[n-(k+1)]T(r,g)\leq S(r,g),$$

which is again a contradiction as $n>4k+5m+6$.

Case 3.3. *Let $b=0$ and $a\neq 0$. Then from (7) we obtain*

$$F^{(k)}=\frac{1}{a}G^{(k)}+1-\frac{1}{a}, \tag{9}$$

i.e.,

$$F=\frac{1}{a}G+\phi(z), \tag{10}$$

where $\phi(z)$ is a polynomial of degree at most k. By (10) and Lemma 2.4 we can say that

$$T(r,f)=T(r,g)+S(r,f). \tag{11}$$

By the assumptions and Proposition 2.3, it is clear that both f and g are transcendental entire functions or both are polynomials. First we suppose that both f and g are transcendental entire functions. If $\phi(z)\not\equiv 0$, then by Lemma 2.7, (10) and (11) we obtain

$$\begin{aligned}(n+m)T(r,f) &= T(r,F)+O(1)\\ &\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,f)\\ &\leq 2(m+1)T(r,f)+S(r,f),\end{aligned}$$

which is a contradiction because $n>4k+5m+6$. Hence in this case $\phi(z)\equiv 0$.
Now we assume that both f and g are polynomials. We suppose that f and g have γ and δ pairwise distinct zeros respectively. Then f and g are of the form

$$f(z)=c(z-a_1)^{l_1}(z-a_2)^{l_2}...(z-a_\gamma)^{l_\gamma},$$

$$g(z)=d(z-b_1)^{m_1}(z-b_2)^{m_2}...(z-b_\delta)^{m_\delta},$$

so that

$$f^n(z)=c^n(z-a_1)^{nl_1}(z-a_2)^{nl_2}...(z-a_\gamma)^{nl_\gamma}, \tag{12}$$

$$g^n(z) = d^n(z-b_1)^{nm_1}(z-b_2)^{nm_2}...(z-b_\delta)^{nm_\delta}, \tag{13}$$

where c and d are nonzero constants, $nl_i > 4k+5m+6$, $nm_j > 4k+5m+6$, $i = 1,2,...,\gamma$, and $j = 1,2,...,\delta$. Differentiating (9) we obtain

$$F^{(k+1)} = \frac{1}{a}G^{(k+1)},$$

i.e.,

$$(a_m f^{n+m})^{(k+1)} + ... + (a_0 f^n)^{(k+1)} = \frac{1}{a}\left[(a_m g^{n+m})^{(k+1)} + ... + (a_0 g^n)^{(k+1)}\right]. \tag{14}$$

Using (12) and (13), (14) can be written as

$$\begin{aligned}(z-a_1)^{nl_1-(k+1)}(z-a_2)^{nl_2-(k+1)}...(z-a_\gamma)^{nl_\gamma-(k+1)}p_1(z) = (z-b_1)^{nm_1-(k+1)}\\ (z-b_2)^{nm_2-(k+1)}...(z-b_\delta)^{nm_\delta-(k+1)}p_2(z),\end{aligned} \tag{15}$$

where $p_1(z)$ and $p_2(z)$ are polynomials such that $degp_1 = m\sum_{i=1}^{\gamma} l_i + (\gamma-1)(k+1)$ and $degp_2 = m\sum_{j=1}^{\delta} m_j + (\delta-1)(k+1)$, respectively. Now

$$\begin{aligned}\sum_{i=1}^{\gamma}[nl_i-(k+1)] - m\sum_{i=1}^{\gamma} l_i &= \sum_{i=1}^{\gamma}[(n-m)l_i-(k+1)]\\ &> \gamma(3k+4m+5)\\ &> (\gamma-1)(k+1),\end{aligned}$$

i.e.,

$$\sum_{i=1}^{\gamma}[nl_i-(k+1)] > m\sum_{i=1}^{\gamma} l_i + (\gamma-1)(k+1).$$

Similarly,

$$\sum_{j=1}^{\delta}[nm_j-(k+1)] > m\sum_{j=1}^{\delta} m_j + (\delta-1)(k+1).$$

Thus from (15) we deduce that there is α such that

$$f^n(\alpha)\left[a_m f^m(\alpha) + ... + a_1 f(\alpha) + a_0\right] = g^n(\alpha)\left[a_m g^m(\alpha) + ... + a_1 g(\alpha) + a_0\right] = 0,$$

where α has multiplicity greater than $4k+5m+6$. This together with (10) implies $\phi(z) = 0$. Thus from (9) and (10) we obtain $a = 1$ and so

$$f^n\left[a_m f^m + ... + a_1 f + a_0\right] \equiv g^n\left[a_m g^m + ... + a_1 g + a_0\right]. \tag{16}$$

Let $h = \frac{f}{g}$. If h is a constant, by putting $f = gh$ in (16) we get

$$a_m g^{n+m}(h^{n+m}-1) + a_{m-1}g^{n+m-1}(h^{n+m-1}-1) + ... + a_0 g^n(h^n-1) = 0,$$

which implies $h^d = 1$, where $d = (n+m,...,n+m-i,...,n)$, $a_{m-i} \neq 0$ for some $i = 0,1,...,m$. Thus $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, $d = (n+m,...,n+m-i,...,n)$, $a_{m-i} \neq 0$ for some $i = 0,1,...,m$.

If h is not a constant, then from (16) we can say that f and g satisfy the algebraic equation $R(f,g) = 0$, where

$$R(x,y) = x^n(a_m x^m + a_{m-1}x^{m-1} + ... + a_0) - y^n(a_m y^m + a_{m-1}y^{m-1} + ... + a_0).$$

This completes the proof for $p = 1$.

(ii) Let $p = 2$. By Lemma 2.10 we have

$$\begin{aligned}\overline{N}\left(r,1;F^{(k)} \mid\geq 3\right) &\leq \frac{1}{2}\left[N\left(r,1;F^{(k)}\right) - \overline{N}\left(r,1;F^{(k)}\right)\right]\\ &\leq \frac{(k+1)}{2}\overline{N}(r,0;f) + \frac{m}{2}T(r,f) + S(r,f).\end{aligned}$$

Similarly, we have

$$\overline{N}\left(r,1;G^{(k)} \mid\geq 3\right) \leq \frac{(k+1)}{2}\overline{N}(r,0;g) + \frac{m}{2}T(r,g) + S(r,g).$$

Suppose (i) of Lemma 2.9 holds for F and G. Then

$$\begin{aligned}T(r,F) + T(r,G) &\leq N_{k+1}(r,0;F) + N_{k+1}(r,0;G) + \overline{N}(r,0;F)\\ &+\overline{N}(r,0;G) + \overline{N}\left(r,1;F^{(k)}\right) + \overline{N}\left(r,1;G^{(k)}\right)\\ &-N_{11}\left(r,1;F^{(k)}\right) + \overline{N}\left(r,1;F^{(k)} \mid\geq 3\right)\\ &+\overline{N}\left(r,1;G^{(k)} \mid\geq 3\right) + S(r,F) + S(r,G).\end{aligned} \tag{17}$$

Now it is clear that

$$N_{k+1}(r,0;F) \leq (k+1)\overline{N}(r,0;f) + mT(r,f),$$

$$N_{k+1}(r,0;G) \leq (k+1)\overline{N}(r,0;g) + mT(r,g),$$

$$\overline{N}(r,0;F) \leq \overline{N}(r,0;f) + mT(r,f),$$

$$\overline{N}(r,0;G) \leq \overline{N}(r,0;g) + mT(r,g),$$

$$\overline{N}\left(r,1;F^{(k)}\right) + \frac{1}{2}\overline{N}\left(r,1;F^{(k)} \mid\geq 3\right) \leq \frac{1}{2}N\left(r,1;F^{(k)} \mid\leq 1\right) + \frac{1}{2}N\left(r,1;F^{(k)}\right),$$

$$\overline{N}\left(r,1;G^{(k)}\right) + \frac{1}{2}\overline{N}\left(r,1;G^{(k)} \mid\geq 3\right) \leq \frac{1}{2}N\left(r,1;G^{(k)} \mid\leq 1\right) + \frac{1}{2}N\left(r,1;G^{(k)}\right),$$

$$N_{11}\left(r,1;F^{(k)}\right) = N\left(r,1;F^{(k)} \mid\leq 1\right) = N\left(r,1;G^{(k)} \mid\leq 1\right),$$

$$\begin{aligned}N\left(r,1;F^{(k)}\right) &\leq T\left(r,F^{(k)}\right) + O(1)\\ &\leq (n+m)T(r,f) + S(r,f),\end{aligned}$$

$$\begin{aligned}N\left(r,1;G^{(k)}\right) &\leq T\left(r,G^{(k)}\right) + O(1)\\ &\leq (n+m)T(r,g) + S(r,g).\end{aligned}$$

So from (17) we obtain

$$\begin{aligned}(n+m)[T(r,f) + T(r,g)] &\leq (k+1)\overline{N}(r,0;f) + mT(r,f) + (k+1)\overline{N}(r,0;g)\\ &+mT(r,g) + \overline{N}(r,0;f) + mT(r,f) + \overline{N}(r,0;g)\\ &+mT(r,g) + \frac{n+m}{2}T(r,f) + \frac{n+m}{2}T(r,g)\\ &+\frac{k+1}{4}\overline{N}(r,0;f) + \frac{m}{4}T(r,f) + \frac{k+1}{4}\overline{N}(r,0;g) + \frac{m}{4}T(r,g)\\ &+S(r,f) + S(r,g),\end{aligned}$$

i.e.,

$$\left(\frac{n}{2}-\frac{5k+7m+9}{4}\right)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

contradicts with the fact that $n>\frac{5k+7m+9}{2}$. Hence by Lemma 2.9 we have

$$\frac{1}{F^{(k)}-1}=\frac{bG^{(k)}+a-b}{G^{(k)}-1},$$

where $a\neq 0$, b are two constants. Now, by using the argument as in (i) before, we can prove the case when $p=2$.
(iii) Let $p\geq 3$. Suppose (i) of Lemma 2.9 holds for F and G. Then

$$\begin{aligned}T(r,F)+T(r,G) \leq\ & N_{k+1}(r,0;F)+N_{k+1}(r,0;G)+\overline{N}(r,0;F)\\ &+\overline{N}(r,0;G)+\overline{N}\left(r,1;F^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right)\\ &-N_{11}\left(r,1;F^{(k)}\right)+\overline{N}\left(r,1;F^{(k)}\mid\geq p+1\right)\\ &+\overline{N}\left(r,1;G^{(k)}\mid\geq p+1\right)+S(r,F)+S(r,G).\end{aligned}\tag{18}$$

Now it is clear that

$$N_{k+1}(r,0;F)\leq (k+1)\overline{N}(r,0;f)+mT(r,f),$$

$$N_{k+1}(r,0;G)\leq (k+1)\overline{N}(r,0;g)+mT(r,g),$$

$$\overline{N}(r,0;F)\leq \overline{N}(r,0;f)+mT(r,f),$$

$$\overline{N}(r,0;G)\leq \overline{N}(r,0;g)+mT(r,g),$$

$$\overline{N}\left(r,1;F^{(k)}\right)+\overline{N}\left(r,1;F^{(k)}\mid\geq p+1\right)\leq \frac{1}{2}N\left(r,1;F^{(k)}\mid\leq 1\right)+\frac{1}{2}N\left(r,1;F^{(k)}\right),$$

$$\overline{N}\left(r,1;G^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\mid\geq p+1\right)\leq \frac{1}{2}N\left(r,1;G^{(k)}\mid\leq 1\right)+\frac{1}{2}N\left(r,1;G^{(k)}\right),$$

$$N_{11}\left(r,1;F^{(k)}\right)=N\left(r,1;F^{(k)}\mid\leq 1\right)=N\left(r,1;G^{(k)}\mid\leq 1\right),$$

$$N\left(r,1;F^{(k)}\right)\leq (n+m)T(r,f)+S(r,f),$$

$$N\left(r,1;G^{(k)}\right)\leq (n+m)T(r,g)+S(r,g).$$

So by (18) we obtain

$$\begin{aligned}(n+m)[T(r,f)+T(r,g)] \leq\ & (k+1)\overline{N}(r,0;f)+mT(r,f)+(k+1)\overline{N}(r,0;g)\\ &+mT(r,g)+\overline{N}(r,0;f)+mT(r,f)+\overline{N}(r,0;g)\\ &+mT(r,g)+\frac{n+m}{2}T(r,f)+\frac{n+m}{2}T(r,g)\\ &+S(r,f)+S(r,g),\end{aligned}$$

i.e.,

$$\left(\frac{n}{2}-\frac{2k+3m+4}{2}\right)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

a contradiction with $n > 2k + 3m + 4$. Hence by Lemma 2.9 and by using the argument as in (i) before, we can prove the case for $p \geq 3$.

We omit the proof the case when $P(z) = c_0$, where $c_0(\neq 0)$ is a complex constant, since using Proposition 2.1 and proceeding in the same way the proof can be carried out in the line of proof as before in each cases ($p = 1, p = 2, p \geq 3$). This completes the proof of Theorem 1.5. □

Proof. [Proof of Theorem 1.4] Let $P(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0$, where $a_0(\neq 0), a_1, ...\ ,a_{m-1}, a_m(\neq 0)$ are complex constants; $F(z) = f^n P(f)$ and $G(z) = g^n P(g)$. Clearly $F^{(k)}$ and $G^{(k)}$ share the value 1 IM. We suppose that F and G satisfy (i) of Lemma 2.8. Then

$$\begin{aligned} T(r,F) + T(r,G) \quad \leq \quad & 2\left[N_{k+1}(r,0;F) + N_{k+1}(r,0;G) + \overline{N}(r,0;F) + \overline{N}(r,0;G)\right] \\ & +3\left[\overline{N}_L\left(r,1;F^{(k)}\right) + \overline{N}_L\left(r,1;G^{(k)}\right)\right] \\ & +S(r,F) + S(r,G). \end{aligned} \tag{19}$$

By Lemma 2.10, we have

$$\begin{aligned} \overline{N}_L\left(r,1;F^{(k)}\right) \quad & \leq \quad N\left(r,1;F^{(k)}\right) - \overline{N}\left(r,1;F^{(k)}\right) \\ & \leq \quad (k+1)\overline{N}(r,0;f) + mT(r,f) + S(r,f). \end{aligned}$$

Similarly, we obtain

$$\overline{N}_L\left(r,1;G^{(k)}\right) \quad \leq \quad (k+1)\overline{N}(r,0;g) + mT(r,g) + S(r,g).$$

$$N_{k+1}(r,0;F) \leq (k+1)\overline{N}(r,0;f) + mT(r,f),$$

$$N_{k+1}(r,0;G) \leq (k+1)\overline{N}(r,0;g) + mT(r,g),$$

$$\overline{N}(r,0;F) \leq \overline{N}(r,0;f) + mT(r,f),$$

$$\overline{N}(r,0;G) \leq \overline{N}(r,0;g) + mT(r,g).$$

Hence from (19) we get

$$[n - (5k + 6m + 7)]\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

a contradiction with $n > 5k + 6m + 7$. Thus by Lemma 2.8 and by using the argument as in (i) of Theorem 1.5, we can easily obtain the result of Theorem 1.4.

We omit the proof when $P(z) = c_0$, where $c_0 \neq 0$ is a complex constant, since the proof is similar one as the case in Theorem 1.5. This completes the proof of Theorem 1.4. □

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References

[1] A. Banerjee, Uniqueness of meromorphic functions sharing a small function with their derivatives, Math. Vesnik., 60 (2008), 121-135.

[2] J.F. Chen, X.Y. Zhang, W.C. Lin and S.J. Chen, Uniqueness of entire functions that share one value, Comput. Math. Appl., 56 (2008), 3000-3014.

[3] M.L. Fang and X.H. Hua, Entire functions that share one value, J. Nanjing Univ. Math. Biquarterly, 13 (1996), 44-48.

[4] M.L. Fang, Uniqueness and value sharing of entire functions, Comput. Math. Appl., 44 (2002), 823-831.

[5] W.K. Hayman, Picard values of meromorphic functions and their derivatives, Ann. of Math., 70 (1959), 9-42.

[6] W.K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford (1964).

[7] I. Lahiri, Value distribution of certain differential polynomials, Int. J. Math. Math. Sc., 28 (2001), 83-91.

[8] I. Lahiri, Weighted sharing of three values, Z. Anal. Anwendungen, 23 (2004), 237 - 252.

[9] Y. Xu and H.L. Qiu, Entire functions sharing one value IM, Indian J. Pure Appl. Math., 31 (2000), 849-855.

[10] C.C. Yang, On deficiencies of differential polynomials II, Math. Z., 125 (1972), 107-112.

[11] C.C. Yang and X.H. Hua, Uniqueness and value sharing of meromorphic functions, Ann. Acad. Sci. Fenn. Math., 22 (1997), 395-406.

[12] L. Yang, Value Distribution Theory, Springer- Verlag, Berlin, 1993.

[13] X.Y. Zhang, J.F. Chen and W.C. Lin, Entire or meromorphic functions sharing one value, Comput. Math. Appl., 56 (2008), 1876-1883.