Filomat 27:6 (2013), 949–954
DOI 10.2298/FIL1306949L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Quasi-metrizability of bispaces by weak bases

Kedian Li[a], Shou Lin[b]

[a] *Department of Mathematics, Zhangzhou Normal University, Zhangzhou 363000, P. R. China*
[b] *Department of Mathematics, Ningde Normal University, Ningde 352100, P. R. China*

Abstract. In this paper, some characterizations for the quasi-metrizability of bispaces are given by means of pairwise weak base g-functions, which generalizes some metrization theorems for topological spaces.

1. Introduction

Kelly [7] began first systematic discussions of bitopological spaces, and then obtained necessary and sufficient conditions that characterize the quasi-pseudo-metrizability of bispaces, see [15, 17–21]. Recently Marín [12] studied the quasi-pseudo-metrization theorem in the style of Frink's metrization theorem by weak bases, and generalization of the Fox-Künzi theorem [16] and the bitopological extension of the "double sequence" theorem of Nagata [17]. The notion of weak bases was introduced by Arhangel'skiĭ [1] to study symmetrizable spaces. Nagata [14] introduced g-functions and studied systematically the metrizability of spaces by means of g-functions. Gao [4] introduced weak base g-functions by means of weak bases to study metrizability of topological spaces. The authors of [9] presented some criteria for the quasi-pseudo-metrizability of bitopological spaces in terms of pairwise weak developments and pairwise weak base g-functions. Pairwise weak base g-functions are a powerful tool for studying the quasi-pseudo-metrizability of bitopological spaces. In this paper, we shall continue this approach. Some quasi-metrization theorems of bispaces will be given by means of pairwise weak base g-functions.

First, let us list some concepts and notations used in this paper. $\mathbb{N}$ denotes the set of all positive integers. A bispace (a bitopological space in [7]) is a triple (X, τ_i, τ_j) where X is a nonempty set, and τ_i and τ_j are two topologies on X, $i, j = 1, 2$ and $i \neq j$. For $A \subset X$, $\mathrm{cl}_{\tau_i} A$ denotes the closure of a set A in a topological space (X, τ_i), and "a sequence $\{y_n\}$ τ_i-converges to x" denotes "a sequence $\{y_n\}$ converges to x in a topological space (X, τ_i)". All spaces (X, τ_i) in this paper are assumed to be T_0. Undefined terms are given in [3].

Definition 1.1. Let (X, τ) be a topological space. A family $\mathcal{B}$ of subsets of X is a *weak base* [1] for the topology τ if for each $x \in X$ there is a subfamily $\mathcal{B}_x$ of $\mathcal{B}$ such that

(a) $x \in B$ for each $B \in \mathcal{B}_x$;
(b) if $A, B \in \mathcal{B}_x$, there is a $C \in \mathcal{B}_x$ such that $C \subset A \cap B$;
(c) a subset $U \subset X$ is open if and only if for each $x \in U$ there exists a $B \in \mathcal{B}_x$ such that $B \subset U$.

2010 *Mathematics Subject Classification*. Primary 54E35; Secondary 54E20, 54E55, 54E99
Keywords. Quasi-metrization, bispaces, weak bases, pairwise weak base g-functions
Received: 03 June 2012; Revised: 22 January 2013; Accepted: 23 January 2013
Communicated by Ljubiša D.R. Kočinac
The project is supported by the NNSF (No. 10971186, 11171162, 11201414) of China; Fujian Province Support College Research Plan Project (No. JK2011031)
Corresponding author: Shou Lin
Email addresses: likd56@126.com (Kedian Li), shoulin60@163.com (Shou Lin)

The family $\mathcal{B}_x$ is called a *local weak base* at x in X.

A topological space (X, τ) is said to have a *weak base g-function* [4], if there is a function $g : \mathbb{N} \times X \to \mathcal{P}(X)$ such that for each $x \in X$ and $n \in \mathbb{N}$

(a) $x \in g(n, x)$;

(b) $g(n+1, x) \subset g(n, x)$;

(c) $\{g(n, x) : n \in \mathbb{N}\}$ is a local weak base at x in X.

Let us recall that a function $d : X \times X \to \mathbb{R}^+$ is a *quasi-pseudo-metric* (resp. *quasi-metric*) on a set X if for all $x, y, z \in X$, it satisfies that

(a) $d(x, x) = 0$ (resp. $d(x, y) = 0$ if and only if $x = y$);

(b) $d(x, z) \leq d(x, y) + d(y, z)$.

If d is a quasi-pseudo-metric on X, the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ is called the *conjugate quasi-pseudo-metric* of d on X. Each quasi-pseudo-metric d on a set X induces a topology $\tau(d)$ on X, where for all $x \in X$ and all $r > 0$,

$$B_d(x, r) = \{y \in X : d(x, y) < r\}$$

is an open d-ball and the family $\{B_d(x, r) : x \in X, r > 0\}$ of open d-balls is a base for the topology $\tau(d)$. A bispace (X, τ_i, τ_j) is *quasi-pseudo-metrizable* (resp. *quasi-metrizable*) if there exists a quasi-pseudo-metric (resp. quasi-metric) d on X such that $\tau(d) = \tau_i$ and $\tau(d^{-1}) = \tau_j$ (or $\tau(d) = \tau_j$ and $\tau(d^{-1}) = \tau_i$), $i, j = 1, 2$ and $i \neq j$. It is easy to check that a space is T_0 and quasi-pseudo-metrizable if and only if it is quasi-metrizable.

A *pair cover* [12] in a bispace (X, τ_i, τ_j) is a family of pairs $(\mathcal{G}_i, \mathcal{G}_j) = \{(G_{i,\alpha}, G_{j,\alpha}) : \alpha \in I\}$ such that

(i) $\mathcal{G}_i = \{G_{i,\alpha} : \alpha \in I\}$ is a cover of X for $i = 1, 2$;

(ii) for each $x \in X$ there is an $\alpha \in I$ such that $x \in G_{1,\alpha} \cap G_{2,\alpha}$.

Let $(\mathcal{G}_i, \mathcal{G}_j)$ and $(\mathcal{G}'_i, \mathcal{G}'_j)$ be pair covers of a bispaces (X, τ_i, τ_j). We say that $(\mathcal{G}'_i, \mathcal{G}'_j)$ *refines* $(\mathcal{G}_i, \mathcal{G}_j)$, i.e., $(\mathcal{G}'_i, \mathcal{G}'_j) < (\mathcal{G}_i, \mathcal{G}_j)$ if for each pair $(G'_{i,\alpha}, G'_{j,\alpha}) \in (\mathcal{G}'_i, \mathcal{G}'_j)$ there is a pair $(G_{i,\beta}, G_{j,\beta}) \in (\mathcal{G}_i, \mathcal{G}_j)$ such that $G'_{i,\alpha} \subset G_{i,\beta}$ and $G'_{j,\alpha} \subset G_{j,\beta}$ for $i, j = 1, 2$ and $i \neq j$.

Let $(\mathcal{G}_i, \mathcal{G}_j)$ be a pair cover of a bispace (X, τ_i, τ_j). Let A be a nonempty subset of X. For $i, j = 1, 2$ and $i \neq j$, put

$$\mathrm{st}(A, \mathcal{G}_i, \mathcal{G}_j) = \cup\{G_{i,\alpha} \in \mathcal{G}_i : A \cap G_{j,\alpha} \neq \emptyset\}.$$

If $x \in X$, define

$$\mathrm{st}(x, \mathcal{G}_i, \mathcal{G}_j) = \cup\{G_{i,\alpha} \in \mathcal{G}_i : x \in G_{j,\alpha}\}$$

and

$$\mathrm{st}^2(x, \mathcal{G}_i, \mathcal{G}_j) = \mathrm{st}(\mathrm{st}(x, \mathcal{G}_i, \mathcal{G}_j), \mathcal{G}_i, \mathcal{G}_j).$$

Definition 1.2. ([9]) A *pairwise weak development* in a bispace (X, τ_i, τ_j) is a sequence $\{(\mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in \mathbb{N}\}$ of pair covers of X such that for each $x \in X$ $\{\mathrm{st}(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in \mathbb{N}\}$ is a weak base of τ_i-neighborhoods of x in X.

A bispace (X, τ_i, τ_j) is *pairwise weak developable* if it has a pairwise weak development $\{(\mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in \mathbb{N}\}$ such that $(\mathcal{G}_{i,n+1}, \mathcal{G}_{j,n+1}) < (\mathcal{G}_{i,n}, \mathcal{G}_{j,n})$ for each $n \in \mathbb{N}$.

A bispace (X, τ_i, τ_j) is said to have a *pairwise weak base g-function* if there are functions $g_i, g_j : \mathbb{N} \times X \to \mathcal{P}(X)$ $(i = 1, 2)$ such that for $i, j = 1, 2$ and $i \neq j$

(a) $x \in g_i(n, x) \cap g_j(n, x)$ for all $x \in X$ and $n \in \mathbb{N}$;

(b) $g_i(n+1, x) \subset g_i(n, x)$ and $g_j(n+1, x) \subset g_j(n, x)$ for all $n \in \mathbb{N}$;

(c) $\{g_i(n, x) : n \in \mathbb{N}, x \in X\}$ is a weak base for the space (X, τ_i), and $\{g_j(n, x) : n \in \mathbb{N}, x \in X\}$ is a weak base for the space (X, τ_j).

Let (g_i, g_j) be a pairwise weak base g-function for a bispace (X, τ_i, τ_j) and $k \in \mathbb{N}$. Define

$$g_i^1(n, x) = g_i(n, x) \text{ and } g_i^{k+1}(n, x) = \cup\{g_i^k(n, y) : y \in g_i(n, x)\}.$$

It is easy to verify that $g_i^{k+1}(n, x) = \bigcup\{g_i(n, y) : y \in g_i^k(n, x)\}$ by inductions on $k \in \mathbb{N}$.

2. Main results

Lemma 2.1. ([9]) *A T_1-bispace (X, τ_i, τ_j) is quasi-metrizable if and only if it has a pairwise weak development $\{(\mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in \mathbb{N}\}$ such that $\{st^2(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in \mathbb{N}, x \in X\}$ is a weak base for a space (X, τ_i), $i, j = 1, 2$ and $i \neq j$.*

Theorem 2.2. *For a T_1-bispace (X, τ_i, τ_j) the following are equivalent:*

(1) *(X, τ_i, τ_j) is quasi-metrizable;*

(2) *There is a pairwise weak base g-function (g_i, g_j) for (X, τ_i, τ_j) such that if a sequence $\{y_n\}$ τ_i-converges to x and $g_j(n, x_n) \cap g_i(n, y_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ τ_i-converges to x;*

(3) *There is a pairwise weak base g-function (g_i, g_j) for (X, τ_i, τ_j) such that*

(3.1) If a sequence $\{y_n\}$ τ_i-converges to x and $x_n \in g_i(n, y_n)$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ τ_i-converges to x.

(3.2) If a sequence $\{y_n\}$ τ_i-converges to x and $y_n \in g_j(n, x_n)$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ τ_i-converges to x.

(4) *There is a pairwise weak base g-function (g_i, g_j) for (X, τ_i, τ_j) such that if $x \in g_j(n, z_n)$, $g_i(n, z_n) \cap g_j(n, y_n) \neq \emptyset$ and $x_n \in g_i(n, y_n)$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ τ_i-converges to x.*

Proof. (1) ⇒ (2) Suppose that (X, τ_i, τ_j) is quasi-metrizable. For each $r > 0$, $i, j = 1, 2$ and $i \neq j$, put

$$B_i(x, r) = \{y \in X : d(x, y) < r\},\ B_j(x, r) = \{y \in X : d(y, x) < r\}$$

and for each $x \in X$ and $n \in \mathbb{N}$, let

$$g_i(n, x) = B_i(x, \frac{1}{2^n}),\ g_j(n, x) = B_j(x, \frac{1}{2^n}).$$

Then (g_i, g_j) is a pairwise weak base g-function for (X, τ_i, τ_j) satisfying the condition (2). In fact, if a sequence $\{y_n\}$ τ_i-converges to x and $g_j(n, x_n) \cap g_i(n, y_n) \neq \emptyset$ for all $n \in \mathbb{N}$, let U be a τ_i-neighborhood of x in X, then there exists an $k \in \mathbb{N}$ such that $g_i(k, x) = B_i(x, \frac{1}{2^k}) \subset U$. Since each $B_i(x, r)$ is open in (X, τ_i) and the sequence $\{y_n\}$ τ_i-converges to x, then $\{y_n : n > m\} \subset g_i(3k, x)$ for some $m \in \mathbb{N}$. Let $n_0 = \max\{3k, 3m\}$. We can choose $t_n \in g_j(n, x_n) \cap g_i(n, y_n)$ for each $n > n_0$ by $g_j(n, x_n) \cap g_i(n, y_n) \neq \emptyset$ for all $n \in \mathbb{N}$. Thus

$$d(x, x_n) \leq d(x, y_n) + d(y_n, t_n) + d(t_n, x_n) \leq \frac{1}{2^{3k}} + \frac{1}{2^n} + \frac{1}{2^n} < \frac{1}{2^k}.$$

That is $x_n \in U$ for each $n > n_0$, therefore the sequence $\{x_n\}$ τ_i-converges to x.

(2) ⇒ (3) Let (g_i, g_j) be a pairwise weak base g-function satisfying the condition (2). Suppose that a sequence $\{y_n\}$ τ_i-converges to x and $x_n \in g_i(n, y_n)$ for all $n \in \mathbb{N}$. Then $x_n \in g_j(n, x_n) \cap g_i(n, y_n)$, thus the sequence $\{x_n\}$ τ_i-converges to x, and (3.1) holds. By a similar proof, (3.2) holds.

(3) ⇒ (4) Let (g_i, g_j) be a pairwise weak base g-function satisfying the condition (3). Suppose that $x \in g_j(n, z_n)$, $g_i(n, z_n) \cap g_j(n, y_n) \neq \emptyset$ and $x_n \in g_i(n, y_n)$ for all $n \in \mathbb{N}$. Since $x \in g_j(n, z_n)$, then the sequence $\{z_n\}$ τ_i-converges to x by (3.2). Take $t_n \in g_i(n, z_n) \cap g_j(n, y_n)$ for all $n \in \mathbb{N}$, then the sequence $\{t_n\}$ τ_i-converges to x by (3.1), and the sequence $\{y_n\}$ τ_i-converges to x by (3.2). Since $x_n \in g_i(n, y_n)$ and the sequence $\{y_n\}$ τ_i-converges to x, the sequence $\{x_n\}$ τ_i-converges to x by (3.1).

(4) ⇒ (1) Let (g_i, g_j) be a pairwise weak base g-function satisfying the condition (4). For $i = 1, 2$ and $n \in \mathbb{N}$, let

$$\mathcal{G}_{i,n} = \{g_i(n, x) : x \in X\}.$$

Then $(\mathcal{G}_{i,n+1}, \mathcal{G}_{j,n+1}) < (\mathcal{G}_{i,n}, \mathcal{G}_{j,n})$ for each $n \in \mathbb{N}$. By Lemma 2.1, we only need to show that $\{\mathrm{st}^2(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : x \in X, n \in \mathbb{N}\}$ is a weak base for (X, τ_i), $i, j = 1, 2$ and $i \neq j$.

Let $U \subset X$ in which for any $x \in U$ there is some $n \in \mathbb{N}$ such that $\mathrm{st}^2(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) \subset U$. Then $g_i(n, x) \subset U$. Since $\{g_i(n, x) : x \in X, n \in \mathbb{N}\}$ is a weak base for (X, τ_i), thus U is τ_i-open. On the other hand, suppose U is τ_i-open and $x \in U$. We want to verify $\mathrm{st}^2(x, \mathcal{G}_{i,m}, \mathcal{G}_{j,m}) \subset U$ for some $m \in \mathbb{N}$. If not, take $x_n \in \mathrm{st}^2(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) - U$ for each $n \in \mathbb{N}$. Also, we can get $y_n \in X$ such that $x_n \in g_i(n, y_n)$ with $g_j(n, y_n) \cap \mathrm{st}(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) \neq \emptyset$, and thus there exist $z_n, s_n \in X$ with $s_n \in g_i(n, z_n) \cap g_j(n, y_n)$ and $x \in g_j(n, z_n)$. Then the sequence $\{x_n\}$ τ_i-converges to x by (4). This is a contradiction.

Hence, (X, τ_i, τ_j) is quasi-metrizable by Lemma 2.1. □

Lemma 2.3. ([11]) *Let $\mathcal{B}_i = \bigcup\{\mathcal{B}(i,x) : x \in X\}$ be a weak base for a T_2-space (X,τ_i). For each $x \in X$ and $B \in \mathcal{B}(i,x)$, if a sequence $\{x_n\}$ τ_i-converges to x, then $\{x_n : n > m\} \subset B$ for some $m \in \mathbb{N}$.*

Theorem 2.4. *For a T_2-bispace (X,τ_i,τ_j) the following are equivalent:*

(1) *(X,τ_i,τ_j) is quasi-metrizable;*

(2) *There is a pairwise weak base g-function (g_i,g_j) for (X,τ_i,τ_j) such that if $y_n \in g_i(n,x)$ and $g_j(n,x_n)\cap g_i(n,y_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ τ_i-converges to x;*

(3) *There is a pairwise weak base g-function (g_i,g_j) for (X,τ_i,τ_j) such that if $g_i(n,x)\cap g_j(n,y_n) \neq \emptyset$ and $g_j(n,x_n)\cap g_i(n,y_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ τ_i-converges to x;*

(4) *There is a pairwise weak base g-function (g_i,g_j) for (X,τ_i,τ_j) such that if $g_i(n,x)\cap g_j(n,y_n) \neq \emptyset$ and $x_n \in g_i(n,y_n)$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ τ_i-converges to x.*

Proof. (1) ⇒ (2) Since (X,τ_i,τ_j) is quasi-metrizable, there is a pairwise weak base g-function (g_i,g_j) for (X,τ_i,τ_j) satisfying the condition (2) in Theorem 2.2. Suppose that $y_n \in g_i(n,x)$ and $g_j(n,x_n)\cap g_i(n,y_n) \neq \emptyset$ for all $n \in \mathbb{N}$. Since $\{g_i(n,x) : n \in \mathbb{N}\}$ is a local weak base at x for the space (X,τ_i), the sequence $\{y_n\}$ τ_i-converges to x by $y_n \in g_i(n,x)$. Then the sequence $\{x_n\}$ τ_i-converges to x by $g_j(n,x_n)\cap g_i(n,y_n) \neq \emptyset$ and (2) in Theorem 2.2.

(2) ⇒ (3) Let (g_i,g_j) be a pairwise weak base g-function satisfying the condition (2). Suppose that $g_i(n,x)\cap g_j(n,y_n) \neq \emptyset$ and $g_j(n,x_n)\cap g_i(n,y_n) \neq \emptyset$ for all $n \in \mathbb{N}$. If the sequence $\{x_n\}$ does not τ_i-converge to x, then there are a neighborhood U of x in X and a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \notin U$ for all $l \in \mathbb{N}$. Take $z_l \in g_i(n_l,x)\cap g_j(n_l,y_{n_l})$ for all $l \in \mathbb{N}$. Since $z_l \in g_i(n_l,x) \subset g_i(l,x)$ and $z_l \in g_j(n_l,y_{n_l})\cap g_i(l,z_l) \subset g_j(l,y_{n_l})\cap g_i(l,z_l)$, the sequence $\{y_{n_l}\}$ τ_i-converges to x by (2). By Lemma 2.3, there is a subsequence $\{y_{n_{l_k}}\}$ of $\{y_{n_l}\}$ such that $y_{n_{l_k}} \in g_i(k,x)$ for all $k \in \mathbb{N}$. Since $g_j(k,x_{n_{l_k}})\cap g_i(k,y_{n_{l_k}}) \supset g_j(n_{l_k},x_{n_{l_k}})\cap g_i(n_{l_k},y_{n_{l_k}}) \neq \emptyset$, the subsequence $\{x_{n_{l_k}}\}$ τ_i-converges to x by (2). That is a contradiction with $x_{n_{l_k}} \notin U$ for all $k \in \mathbb{N}$. Thus the sequence $\{x_n\}$ τ_i-converges to x.

(3) ⇒ (4) Obviously.

(4) ⇒ (1) Let (g_i,g_j) be a pairwise weak base g-function satisfying the condition (4). It is enough to show the (g_i,g_j) satisfies the condition (4) in Theorem 2.2. Suppose that $x \in g_j(n,z_n)$, $g_i(n,z_n)\cap g_j(n,y_n) \neq \emptyset$ and $x_n \in g_i(n,y_n)$ for all $n \in \mathbb{N}$. Take $t_n \in g_i(n,z_n)\cap g_j(n,y_n)$ for each $n \in \mathbb{N}$. Then the sequence $\{t_n\}$ τ_i-converges to x by $g_i(n,x)\cap g_j(n,z_n) \neq \emptyset$ and (4). By Lemma 2.3, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ with $t_{n_k} \in g_i(k,x)$ for all $k \in \mathbb{N}$. Then $t_{n_k} \in g_i(k,x)\cap g_j(n_k,y_{n_k}) \subset g_i(k,x)\cap g_j(k,y_{n_k})$ and $x_{n_k} \in g_i(n_k,y_{n_k}) \subset g_i(k,y_{n_k})$ for all $k \in \mathbb{N}$. Again, by (4), the sequence $\{x_{n_k}\}$ τ_i-converges to x. By a similar method in (2) ⇒ (3) above, the sequence $\{x_n\}$ τ_i-converges to x. □

Let $k \in \mathbb{N}$. Consider the following conditions about a pairwise weak base g-function (g_i,g_j) for a bispace (X,τ_i,τ_j).

$(p\sigma')$ If $x \in g_j^2(n,x_n)$ for all $n \in \mathbb{N}$, then $\{x_n\}$ τ_i-converges to x.

(pN') For any $A \subset X$ and each $n \in \mathbb{N}$, $\mathrm{cl}_{\tau_i}A \subset \cup\{g_j(n,x) : x \in A\}$.

(pS') If $\{y_n\}$ τ_i-converges to x and $y_n \in g_j(n,x_n)$ for all $n \in \mathbb{N}$, then $\{x_n\}$ τ_i-converges to x.

Theorem 2.5. *A T_2-bispace (X,τ_i,τ_j) is quasi-metrizable if and only if it has a pairwise weak base g-function (g_i,g_j) satisfying $(p\sigma')$ and (pN').*

Proof. Necessity. Let (X,τ_i,τ_j) be a quasi-pseudo-metrizable bispace. Let g_i, g_j be the functions defined by the proof of (1) ⇒ (2) in Theorem 2.2.

First, $(p\sigma')$ holds. Let $x \in g_j^2(n,x_n)$ for all $n \in \mathbb{N}$, then $x \in g_j(n,t_n)$ and $t_n \in g_j(n,x_n)$. $\{t_n\}$ τ_i-converges to x by the condition (2) of Theorem 2.2. Again by the condition (2) of Theorem 2.2, then $\{x_n\}$ τ_i-converges to x.

Secondly, (pN') holds. Assume that there are a subset $A \subset X$ and an $m \in \mathbb{N}$ such that $\mathrm{cl}_{\tau_i}A \not\subset \cup\{g_j(m,y) : y \in A\}$, then there exists a point $x \in \mathrm{cl}_{\tau_i}A - \cup\{g_j(m,y) : y \in A\}$. Since (X,τ_i) is first-countable, there is a sequence $\{y_n\} \subset A$ such that $\{y_n\}$ τ_i-converges to x. For $k \in \mathbb{N}$ and $k > m$, since $g_i(k,x)$ is open in (X,τ_i), then $\{y_n : n > n_0\} \subset g_i(k,x)$ for some $n_0 \in \mathbb{N}$.

Because $x \notin \cup\{g_j(m, y) : y \in A\}$, then $x \notin g_j(m, y_n)$ for any $n \in \mathbb{N}$. Let $k > m$ and $n > \max\{m, n_0\}$, then $y_n \in g_i(k, x)$ and $x \notin g_j(m, y_n)$. We have $d(x, y_n) < \frac{1}{2^k} < \frac{1}{2^m}$ and $d(x, y_n) \geqslant \frac{1}{2^m}$, this is a contradiction. Therefore, the condition (pN') holds.

Sufficiency. Let (g_i, g_j) be a pairwise weak base g-function for a bispace (X, τ_i, τ_j) satisfying the conditions $(p\sigma')$ and (pN'). For each $x \in X$ and $n \in \mathbb{N}$, put

$$h_i(n, x) = g_i(n, x) - \mathrm{cl}_{\tau_i}\{y \in X : x \notin g_j(n, y)\}.$$

By (pN'), $x \notin \mathrm{cl}_{\tau_i}\{y \in X : x \notin g_j(n, y)\}$, i.e.,

$$x \in g_i(n, x) - \mathrm{cl}_{\tau_i}\{y \in X : x \notin g_j(n, y)\} = h_i(n, x).$$

Hence (h_i, h_j) is a pairwise weak base g-function for (X, τ_i, τ_j) with the following property:

$$\text{If } y \in h_i(n, x), \text{ then } y \in g_i(n, x) \text{ and } x \in g_j(n, y).$$

Now, suppose that $z_n \in h_i(n, x) \cap h_j(n, y_n)$ and $x_n \in h_i(n, y_n)$ for all $n \in \mathbb{N}$. Then $z_n \in g_i(n, x)$, $x \in g_j(n, z_n)$, $z_n \in g_j(n, y_n)$ and $y_n \in g_i(n, z_n)$. It is obvious that $x \in g_j^2(n, y_n)$. It follows from $(p\sigma')$ that the sequence $\{y_n\}$ τ_i-converges to x. There is a subsequence $\{y_{n_m}\}$ of $\{y_n\}$ such that $y_{n_m} \in h_i(m, x)$, then $y_{n_m} \in g_i(m, x)$ and $x \in g_j(m, y_{n_m})$ for all $m \in \mathbb{N}$. Since $x_{n_m} \in h_i(m, y_{n_m})$, we have that $x_{n_m} \in g_i(m, y_{n_m})$ and $y_{n_m} \in g_j(m, x_{n_m})$. Thus $x \in g_j^2(m, x_{n_m})$ for all $m \in \mathbb{N}$. Again, by $(p\sigma')$, the sequence $\{x_{n_m}\}$ τ_i-converges to x, and thus the sequence $\{x_n\}$ τ_i-converges to x. The quasi-metrizability of the bispace (X, τ_i, τ_j) now follows from (1) $\Leftrightarrow$ (4) of Theorem 2.4. □

Corollary 2.6. *A T_2-bispace (X, τ_i, τ_j) is quasi-metrizable if and only if it has a pairwise weak base g-function (g_i, g_j) satisfying (pS') and (pN').*

Proof. Necessity is from the (2) of Theorem 2.2 and the necessity of Theorem 2.5.

Sufficiency. By Theorem 2.5, we only need to show that $(pS') \Rightarrow (p\sigma')$.

Let (g_i, g_j) be a pairwise weak base g-function for (X, τ_i, τ_j) satisfying (pS'). Let $x \in g_j^2(n, x_n)$ for each $n \in \mathbb{N}$. There is $t_n \in g_j(n, x_n)$ such that $x \in g_j(n, t_n)$ for each $n \in \mathbb{N}$. It follows from (pS') that the sequence $\{t_n\}$ τ_i-converges to x, and the sequence $\{x_n\}$ τ_i-converges to x. Hence, $(pS') \Rightarrow (p\sigma')$. □

The following result was obtained in [9].

Theorem 2.7. ([9]) *A T_1-bispace (X, τ_i, τ_j) is quasi-metrizable if and only if it has a pairwise weak base g-function (g_i, g_j) satisfying that*

(1) *There exists an $m \in \mathbb{N}$ such that $x \notin cl_{\tau_i} \cup \{g_j(m, y) : y \in X - U\}$ for each $x \in X$ and a τ_i-neighborhood U of x.*

(2) *For any $Y \subset X$ and each $n \in \mathbb{N}$, $cl_{\tau_i} Y \subset \cup\{cl_{\tau_i} g_j^2(n, y) : y \in Y\}$.*

By the similar method in the proof of Theorem 2.2 in [9], we can prove the following theorem.

Theorem 2.8. *Let $k > 2$. A T_1-bispace (X, τ_i, τ_j) is quasi-metrizable if and only if it has a pairwise weak base g-function (g_i, g_j) satisfying that*

(1) *There exists an $m \in \mathbb{N}$ such that $x \notin cl_{\tau_i}(\cup\{g_j(m, y) : y \in X - U\})$ for each $x \in X$ and τ_i-neighborhood U of x.*

(2) *For any $Y \subset X$ and $n \in \mathbb{N}$, $cl_{\tau_i} Y \subset \cup\{cl_{\tau_i} g_j^k(n, y) : y \in Y\}$.*

Remark 2.9. It is well known that a bispace is pairwise stratifiable if and only if it has a pairwise g-function satisfying (1) of Theorem 2.8 [8]. We may say that (2) of Theorem 2.8 give a difference between quasi-metrizable and pairwise stratifiable spaces.

Assume that $\tau_1 = \tau_2 = \tau$, a bispace (X, τ_1, τ_2) is a topological space (X, τ) and the quasi-metrizability of bispaces is equivalent to the metrizability of topological spaces. Thus we have the following corollaries.

Corollary 2.10. ([5, Theorem 6]) *Let $k > 2$. A T_1-space (X, τ) is metrizable if and only if it has a weak base g-function g for X satisfying that*

(1) For each $x \in X$ and a neighborhood U of x, there exists an $m \in \mathbb{N}$ such that

$$x \notin \overline{\cup\{g(m, y) : y \in X - U\}}.$$

(2) For any $Y \subset X$ and each $n \in \mathbb{N}$,

$$\overline{Y} \subset \cup\{\overline{g^k(n, y)} : y \in Y\}.$$

Corollary 2.11. ([23, Theorem 2.3]) *The following are equivalent for a T_2-space (X, τ):*

(1) *X is metrizable;*

(2) *There is a weak base g-function g for X such that if a sequence $\{y_n\}$ converges to x and $g(n, x_n) \cap g(n, y_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges to x;*

(3) *There is a weak base g-function g for X such that if $y_n \in g(n, x)$ and $g(n, x_n) \cap g(n, y_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges to x;*

(4) *There is a weak base g-function g for X such that if $g(n, x) \cap g(n, y_n) \neq \emptyset$ and $g(n, x_n) \cap g(n, y_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges to x;*

(5) *There is a weak base g-function g for X such that if $g(n, x) \cap g(n, y_n) \neq \emptyset$ and $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges to x;*

(6) *There is a weak base g-function g for X such that if $x \in g(n, z_n)$, $g(n, z_n) \cap g(n, y_n) \neq \emptyset$ and $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges to x.*

Corollary 2.12. ([22, Conditions (1) and (5) in Theorem 2.2]) *A T_1-space X is metrizable if and only if there is weak base g-function (i.e., a CWC-mapping) g for X satisfying that:*

(*I*) *For sequences $\{x_n\}$, $\{y_n\}$ if the sequence $\{y_n\}$ converges to x and $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges to x.*

(*II*) *For sequences $\{x_n\}$, $\{y_n\}$ if the sequence $\{y_n\}$ converges to x and $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges to x.*

Acknowledgments. We wish to thank the referees for the detailed list of corrections, suggestions to the paper, and all their efforts in order to improve the paper.

References

[1] A.V. Arhangel'skiĭ, Mappings and spaces, Russian Math. Surveys 21 (1966) 115–162.
[2] B. Chen, S.L. Jiang, Metrizability and weak base g-functions, New Zealand J. Math. 37 (2008) 15–20.
[3] R. Engelking, General Topology (Revised and completed edition), Heldermann Verlag, Berlin, 1989.
[4] Zh. Gao, Metrizability of spaces and weak base g-functions, Topology Appl.146-147 (2005) 279–288.
[5] Zh. Gao, Metrizability of spaces and J. Nagata's problem, Acta Math. Scientia, 26B (2006) 25–30.
[6] P. Gartside, Generalized metric spaces, I, in: K.P. Hart, J. Nagata and J.V. Vaughan (eds.), Encyclopedia of General Topology, Elsevier Science Publishers, 2004, 273–275.
[7] J.C. Kelly, Bitoplogical spaces, Proc. London Math. Soc. 13 (1963) 71–89.
[8] K. Li, Characterizations of pairwise stratifiable spaces, Acta Math. Scientia, 30A (2010) 649–655.
[9] K. Li, F. Lin, Characterizations of quasi-pseudo-metrizable spaces, Topology Proc. 40 (2012) 121–129.
[10] K. Li, S. Lin, Notes on symmetric g-functions, Acta Math. Hungar. 116 (2007) 73–77.
[11] S. Lin, Generalized Metric Spaces and Mappings (second edition), China Science Publishers, Beijing, 2007.
[12] J. Marín, Weak bases and quasi-pseudo-metrization of bispaces, Topology Appl. 156 (2009) 3070–3076.
[13] H.W. Martin, Weak bases and metrization, Trans. Amer. Math. Soc. 222 (1976) 338–344.
[14] J. Nagata, Characterizations of metrizable and Lasnev spaces in terms of g-functions, Questions Answers Gen. Topology. 4 (1986) 129–139.
[15] C.W. Patty, Bitoplogical spaces, Duke Math. J. 34 (1967) 387–392.
[16] T.G. Raghavan, On quasi-metrizability, Indian J. Pure Appl. Math. 15 (1984) 1084–1089.
[17] T.G. Raghavan, I.L. Reilly, Characterizations of quasi-metrizable bitopological spaces, J. Aust. Math. Soc. 44 (1984) 271–274.
[18] S. Romaguera, Two characterizations of quasi-pseudo-metrizable bitopological spaces, J. Aust. Math. Soc. 35 (1983) 327–333.
[19] S. Romaguera, On bitopological quasi-pseudo-metrization, J. Aust. Math. Soc., **36** (1984), 126-129.
[20] S. Romaguera, J. Marín, On the bitopological extension of the Bing metrization theorem, J. Aust. Math. Soc. 44 (1988) 233–241.
[21] S. Salbany, Quasi-metrization of bitopological spaces, Arch. Math. (Basel) 23 (1972) 299–306.
[22] P.F. Yan, S. Lin, CWC-mappings and metrization theorems, Advances Math. (China) 36 (2007) 153–158.
[23] E.G. Yang, W.X. Shi, Weak base g-functions and metrizability of topological spaces, Topology Appl. 158 (2011) 238–243.