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On a question of Mecheri and Braha

Qingping Zeng, Huaijie Zhong

School of Mathematics and Computer Science, Fujian Normal University, Fuzhou 350007, P.R. China

Abstract. In this note we give an answer to a question posed recently by Mecheri and Braha [Oper. Matrices **6** (2012), 725–734]. More precisely, we show that if T is n-perinormal, then the nonzero points λ of its approximate point spectrum and joint approximate point spectrum are identical; but this is not the case when $\lambda = 0$.

Let $L(H)$ stand for the C^* algebra of all bounded linear operators on an infinite dimensional complex Hilbert space H. Recall that an operator $T \in L(H)$ is said to be n-perinormal if $T^{*n}T^n \geq (T^*T)^n$, where $n \geq 2$ is an integer (see [2]). For $T \in L(H)$, let $\sigma_p(T)$, $\sigma_{jp}(T)$, $\sigma_a(T)$ and $\sigma_{ja}(T)$ denote the point spectrum, joint point spectrum, approximate point spectrum and joint approximate point spectrum of T, respectively (see [2]).

In [2, Theorem 2.1], it is shown that if T is n-perinormal, $(T-\lambda)x = 0$ and $\lambda \neq 0$, then $(T-\lambda)^*x = 0$. From this result, a number of consequences are presented. For example, it is stated in [2, Theorem 3.1(1)] that the point spectrum and joint point spectrum of an n-perinormal operator are identical. But, in fact, from [2, Theorem 2.1], one could only deduce that

$$\sigma_{jp}(T)\backslash\{0\} = \sigma_p(T)\backslash\{0\}$$

when T is n-perinormal. And Example 3 below shows that in general $\sigma_p(T) \neq \sigma_{jp}(T)$ for 2-perinormal operators T.

Moreover, Mecheri and Braha posed in [2] an open question: Does $\sigma_{ja}(T) = \sigma_a(T)$ for n-perinormal operator T? In this note we give an answer to this question by proving the following theorem and giving an example of 2-perinormal operator T satisfying $0 \in \sigma_a(T)\backslash\sigma_{ja}(T)$.

Theorem 1. *Let T be n-perinormal and $0 \neq \lambda \in \mathbb{C}$. If $(T-\lambda)x_m \to 0$ for a sequence $\{x_m\}_{m=1}^{\infty}$ of unit vectors, then $(T^* - \overline{\lambda})x_m \to 0$.*

Proof. Let $(T-\lambda)x_m \to 0$ for unit vectors $\{x_m\}_{m=1}^{\infty}$ and let $l \in \mathbb{N}$. Since

$$T^l = (T-\lambda+\lambda)^l = \sum_{j=1}^{l}\binom{l}{j}\lambda^{l-j}(T-\lambda)^j + \lambda^l,$$

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Email addresses: zqpping2003@163.com (Qingping Zeng), zhonghuaijie@sina.com (Huaijie Zhong)

we have $(T^l - \lambda^l)x_m \to 0$. It then follows from

$$\Big|\|\lambda^l x_m\| - \|(T^l - \lambda^l)x_m\|\Big| \le \|T^l x_m\| = \|\lambda^l x_m + (T^l - \lambda^l)x_m\| \\ \le \|\lambda^l x_m\| + \|(T^l - \lambda^l)x_m\|$$

that $\|T^l x_m\| \to |\lambda|^l$. In particular, we have

$$\|Tx_m\| \to |\lambda| \text{ and } \|T^n x_m\| \to |\lambda|^n. \tag{1}$$

By Hölder-McCarthy inequality [1, Lemma 2.1], we have

$$\Big\||T^n|^{\frac{2}{n}} x_m\Big\| = (|T^n|^{\frac{2}{n}} x_m, |T^n|^{\frac{2}{n}} x_m)^{\frac{1}{2}} = (|T^n|^{\frac{4}{n}} x_m, x_m)^{\frac{1}{2}} \\ \le (|T^n|^2 x_m, x_m)^{\frac{2}{n}\cdot\frac{1}{2}} = (T^{*n}T^n x_m, x_m)^{\frac{1}{n}} = \|T^n x_m\|^{\frac{2}{n}},$$

which, together with (1), implies that

$$\limsup_{m\to\infty} \Big\||T^n|^{\frac{2}{n}} x_m\Big\| \le |\lambda|^2. \tag{2}$$

Since T is n-perinormal, $|T^n|^{\frac{2}{n}} - |T|^2$ is positive. It then follows from

$$\Big\|(|T^n|^{\frac{2}{n}} - |T|^2)^{\frac{1}{2}} x_m\Big\|^2 = (|T^n|^{\frac{2}{n}} x_m, x_m) - (|T|^2 x_m, x_m) \le \Big\||T^n|^{\frac{2}{n}} x_m\Big\| - \|Tx_m\|^2$$

that $(|T^n|^{\frac{2}{n}} - |T|^2)^{\frac{1}{2}} x_m \to 0$ and so $(|T^n|^{\frac{2}{n}} - |T|^2)x_m \to 0$. By (2) and the fact that

$$\|T^*\lambda x_m\| - \|T^*(T-\lambda)x_m\| \le \|T^*Tx_m\| \le \Big\|(|T^n|^{\frac{2}{n}} - |T|^2)x_m\Big\| + \Big\||T^n|^{\frac{2}{n}} x_m\Big\|,$$

we get

$$\limsup_{m\to\infty} \|T^* x_m\| \le |\lambda|.$$

Since

$$\begin{aligned} \|T^*x_m - \overline{\lambda}x_m\|^2 &= (T^*x_m - \overline{\lambda}x_m, T^*x_m - \overline{\lambda}x_m) \\ &= (T^*x_m, T^*x_m) - \overline{\lambda}(x_m, T^*x_m) - \lambda(T^*x_m, x_m) + |\lambda|^2 \\ &= \|T^*x_m\|^2 - \overline{\lambda}(Tx_m, x_m) - \lambda(x_m, Tx_m) + |\lambda|^2 \\ &= \|T^*x_m\|^2 - \overline{\lambda}((T-\lambda)x_m, x_m) - \lambda(x_m, (T-\lambda)x_m) - |\lambda|^2, \end{aligned}$$

we have

$$\limsup_{m\to\infty} \|T^*x_m - \overline{\lambda}x_m\|^2 \le |\lambda|^2 - |\lambda|^2 = 0.$$

This establishes that $(T^* - \overline{\lambda})x_m \to 0$. □

Corollary 2. *If T is n-perinormal, then*

$$\sigma_{ja}(T)\backslash\{0\} = \sigma_a(T)\backslash\{0\}.$$

The next example shows that there exists a 2-perinormal operator T satisfying

$$0 \in \sigma_p(T)\backslash\sigma_{jp}(T) \text{ and } 0 \in \sigma_a(T)\backslash\sigma_{ja}(T).$$

Example 3. Let U be the unilateral right shift operator on $l_2(\mathbb{N})$ with the canonical orthogonal basis $\{e_n\}_{n=1}^{\infty}$ and

$$T = \begin{pmatrix} 2+U & e_1 \otimes e_1 \\ 0 & 0 \end{pmatrix} \text{ on } H = l_2(\mathbb{N}) \oplus \mathbb{C}e_1.$$

Put $S = (2+U^*)(2+U)$. Then

$$T^* = \begin{pmatrix} 2+U^* & 0 \\ e_1 \otimes e_1 & 0 \end{pmatrix},$$

$$T^*T = \begin{pmatrix} S & 2e_1 \otimes e_1 \\ 2e_1 \otimes e_1 & e_1 \otimes e_1 \end{pmatrix}$$

and

$$(T^*T)^2 = \begin{pmatrix} S^2 + 4e_1 \otimes e_1 & S \cdot 2e_1 \otimes e_1 + 2e_1 \otimes e_1 \\ 2e_1 \otimes e_1 \cdot S + 2e_1 \otimes e_1 & 5e_1 \otimes e_1 \end{pmatrix}.$$

Moreover,

$$T^2 = \begin{pmatrix} (2+U)^2 & (2+U) \cdot e_1 \otimes e_1 \\ 0 & 0 \end{pmatrix},$$

$$T^{*2} = \begin{pmatrix} (2+U^*)^2 & 0 \\ e_1 \otimes e_1 \cdot (2+U^*) & 0 \end{pmatrix}$$

and

$$T^{*2}T^2 = \begin{pmatrix} (2+U^*)S(2+U) & (2+U^*)S \cdot e_1 \otimes e_1 \\ e_1 \otimes e_1 \cdot S(2+U) & e_1 \otimes e_1 \cdot S \cdot e_1 \otimes e_1 \end{pmatrix}.$$

Since $S = (2+U^*)(2+U)$, a routine calculation shows that

$$(2+U^*)S(2+U) = S^2 + 4e_1 \otimes e_1,$$

$$(2+U^*)S \cdot e_1 \otimes e_1 = S \cdot 2e_1 \otimes e_1 + 2e_1 \otimes e_1$$

and

$$e_1 \otimes e_1 \cdot S \cdot e_1 \otimes e_1 = 5e_1 \otimes e_1.$$

Thus $T^{*2}T^2 = (T^*T)^2$ and hence T is 2-perinormal.

Next, we show that

$$0 \in \sigma_p(T) \backslash \sigma_{jp}(T) \text{ and } 0 \in \sigma_a(T) \backslash \sigma_{ja}(T).$$

Clearly, $\ker(T) = \{-(2+U)^{-1}ae_1 \oplus ae_1 : a \in \mathbb{C}\}$ and $\ker(T^*) = \{0\} \oplus \mathbb{C}e_1$, hence

$$\ker(T) \cap \ker(T^*) = \{0\} \oplus \{0\}.$$

Consequently, $0 \in \sigma_p(T) \backslash \sigma_{jp}(T)$. Evidently, $0 \in \sigma_a(T)$. We claim that $0 \notin \sigma_{ja}(T)$. Otherwise, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of unit vectors satisfying $Tx_n \to 0$ and $T^*x_n \to 0$. For $n \in \mathbb{N}$, let $x_n = (b_{1,n}, b_{2,n}, \cdots) \oplus a_n e_1 \in l_2(\mathbb{N}) \oplus \mathbb{C}e_1$. Then

$$a_n^2 + \sum_{k=1}^{\infty} b_{k,n}^2 = 1, \tag{3}$$

$$(2b_{1,n} + a_n)^2 + \sum_{k=1}^{\infty} (2b_{k+1,n} + b_{k,n})^2 \to 0, \tag{4}$$

and

$$\sum_{k=1}^{\infty} (2b_{k,n} + b_{k+1,n})^2 + b_{1,n}^2 \to 0. \tag{5}$$

By (5), (4) and (3), we have

$$b_{1,n}^2 \to 0, \quad (2b_{1,n} + a_n)^2 \to 0, \quad a_n^2 \to 0$$

and

$$\sum_{k=1}^{\infty} b_{k,n}^2 \to 1, \quad \sum_{k=2}^{\infty} b_{k,n}^2 \to 1.$$

Then by (4), we have

$$\sum_{k=1}^{\infty} (2b_{k+1,n} + b_{k,n})^2 = \sum_{k=1}^{\infty} (4b_{k+1,n}^2 + 4b_{k+1,n}b_{k,n} + b_{k,n}^2) \to 0.$$

Thus

$$\sum_{k=1}^{\infty} 4b_{k+1,n}b_{k,n} \to -5,$$

which contradicts to the fact that

$$\sum_{k=1}^{\infty} |4b_{k+1,n}b_{k,n}| \leq \sum_{k=1}^{\infty} 2(b_{k+1,n}^2 + b_{k,n}^2) \leq 4.$$

References

[1] C.A. McCarthy, c_p, Israel J. Math. 5 (1967) 249–271.
[2] S. Mecheri and N.L. Braha, Spectral properties of n-perinormal operators, Oper. Matrices 6 (2012) 725–734.