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On homogeneity and homogeneity components in generalized topological spaces

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Abstract. In this paper, we extend the notion of homogeneous topological spaces to the setting of generalized topological (GT) spaces in the sense of Császár [6]. We introduce the notions of homogeneity and homogeneity components in GT spaces and obtain several results concerning these notions, in particular, product and sum theorems are obtained.

1. Introduction and preliminaries

A generalized topology (breifly, GT) [6] μ on a non-empty set X is a collection of subsets of X such that $\emptyset \in \mu$ and μ is closed under arbitrary unions. A set X with a GT μ on it is called a generalized topological space (briefly, GTS) and is denoted by (X, μ). Let (X, μ) be a GTS. The elements of μ are called μ-open sets. The union of all elements of μ will be denoted by M_μ. (X, μ) is said to be strong if $M_\mu = X$ [7]. (X, μ) is called a quasi-topological space [8] if μ is closed under finite intersections. For $A \subset X$ we denote by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [6, 9]).

Let $f : (X, \mu) \to (Y, \lambda)$ be a function on GTS's. f is said to be (μ, λ)-continuous [6] if $B \in \lambda$ implies that $f^{-1}(B) \in \mu$. f is said to be (μ, λ)-open if $A \in \mu$ implies that $f(A) \in \lambda$ [11]. f is called a (μ, λ)-homeomorphism [12] if f is bijective, (μ, λ)-continuous, and f^{-1} is (λ, μ)-continuous, equivalently if f is bijective, (μ, λ)-continuous, and (μ, λ)-open. If $f : (X, \mu) \to (Y, \lambda)$ is a (μ, λ)-homeomorphism, then we say that (X, μ) is homeomorphic to (Y, λ). If (X, μ) is a GTS, then the set of all (μ, μ)-homeomorphisms from (X, μ) onto itself will clearly form a group under composition.

Let X be a non-empty set and let $\mathcal{B}$ be a collection of subsets of X with $\emptyset \in \mathcal{B}$. Then the collection of all possible unions of elements of $\mathcal{B}$ forms a GT $\mu(\mathcal{B})$ on X and $\mathcal{B}$ is called a base for $\mu(\mathcal{B})$ [10].

Let $K \neq \emptyset$ be an index set, $(X_k, \mu_k), k \in K$ a family of GTS's and $X = \prod_{k\in K} X_k$ be the cartesian product of the sets X_k. Consider all sets of the form $\prod_{k\in K} A_k$, where $A_k \in \mu_k$ and with the exception of finite number of indices k, $A_k = M_{\mu_k}$. We denote by $\mathcal{B}$ the collection of all these sets. We call the GT $\mu = \mu(\mathcal{B})$ having $\mathcal{B}$ as a base the product [12] of the GTS's μ_k and denote it by $\mathbf{P}_{k\in K}\mu_k$. The GTS (X, μ) is called the product of the GTS's (X_k, μ_k). If (X, μ) is the product of the GTS's $(X_k, \mu_k), k \in K$, (Y, λ) is the product of the GTS's (Y_k, λ_k),

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and $f_k : (X_k, \mu_k) \to (Y_k, \lambda_k), k \in K$ is a family of functions, then the cartesian product function $f : X \to Y$, defined by $f((x_k)_{k\in K}) = (f_k(x_k))_{k\in K}$, is denoted by $\prod_{k\in K} f_k$. It is known that the product function is continuous if $f_k : (X_k, \mu_k) \to (Y_k, \lambda_k)$ is a continuous function on topological spaces for all $k \in K$.

A topological space (X, τ) is said to be homogeneous [18] if for any two points $x, y \in X$ there exists an autohomeomorphism f on (X, τ) such that $f(x) = y$. Seven years earlier, L. Brouwer had shown that if A and B are two countable dense subsets of the n-dimensional Euclidean space $\mathbb{R}^n$, then there is an autohomeomorphism on $\mathbb{R}^n$ that takes A to B. He needed this result in his development of dimension theory. Too many works related to homogeneity were appeared, we consult some of the recent of them [1,9-18]. One of the main goals of the present work is to show how the definition of homogeneous topological spaces can be modified in order to define homogeneous GTS's.

In Section 2, we introduce several results related to homeomorphisms in GTS, especially, we focus with those results that will be used in the proof of the main results in Section 3 such as product, disjoint sum, and minimality results.

In Section 3, we introduce homogeneity in GTS's, and we obtain several results regarding this concept.

In Section 4, we define homogeneity components in GTS's, and we introduce several related results.

2. Homeomorphisms in GTS's

For simplicity, in this section the product of the GTS's (X_k, μ_k) (resp. (Y_k, λ_k)), $k \in K$ will be denoted by (X, μ) (resp. (Y, λ)).

Proposition 2.1. *If K is finite and $f_k : (X_k, \mu_k) \to (Y_k, \lambda_k)$ is (μ_k, λ_k)-continuous for every $k \in K$, then the cartesian product function $f = \prod_{k\in K} f_k : (X, \mu) \to (Y, \lambda)$ is (μ, λ)-continuous.*

Proof. Let $G = \prod_{k\in K} B_k$, where $B_k \in \lambda_k$ for every $k \in K$, be a basic element of λ. Then

$$f^{-1}(G) = \prod_{k\in K} f_k^{-1}(B_k).$$

For every $k \in K$, f_k is (μ_k, λ_k)-continuous and so $f_k^{-1}(B_k) \in \mu_k$. Therefore, $f^{-1}(G) \in \mu$. □

The following example shows that the condition "K is finite" in Proposition 2.1 cannot be dropped:

Example 2.2. Let $K = \mathbb{N}$, for every $k \in K$, let $X_k = Y_k = \mathbb{R}, \mu_k = \{\emptyset, \mathbb{R}, \{k\}\}, \lambda_k = \{\emptyset, \{k\}\}$, and $f_k : (X_k, \mu_k) \to (Y_k, \lambda_k)$ defined by $f_k(x) = x, x \in \mathbb{R}$. Then it is easy to check that f_k is (μ_k, λ_k)-continuous for every $k \in K$. On the other hand, as $\prod_{k\in K} \{k\} \in \lambda$ and

$\left(\prod_{k\in K} f_k\right)^{-1}\left(\prod_{k\in K} \{k\}\right) = \prod_{k\in K} \{k\} \notin \mu$, then $\prod_{k\in K} f_k$ is not (μ, λ)-continuous.

Lemma 2.3. *Let $g : (X, \mu) \to (Y, \lambda)$ be (μ, λ)-continuous.*

1. *If (Y, λ) is strong, then $g^{-1}(M_\lambda) = M_\mu$.*
2. *If g is (μ, λ)-open, then $g^{-1}(M_\lambda) = M_\mu$.*

Proof. (1) Since $M_\lambda = Y$, $g^{-1}(M_\lambda) = X \subset M_\mu$. Therefore, $g^{-1}(M_\lambda) = M_\mu$.

(2) Since $M_\lambda \in \lambda$ and g is (μ, λ)-continuous, $g^{-1}(M_\lambda) \subset M_\mu$. Also, since $M_\mu \in \mu$ and g is (μ, λ)-open, $g\left(M_\mu\right) \subset M_\lambda$. Therefore,

$$M_\mu \subset g^{-1}\left(g\left(M_\mu\right)\right) \subset g^{-1}(M_\lambda) \subset M_\mu$$

and hence $g^{-1}(M_\lambda) = M_\mu$. □

Lemma 2.4. *If $f_k : (X_k, \mu_k) \to (Y_k, \lambda_k)$ is (μ_k, λ_k)-continuous and $f_k^{-1}(M_{\lambda_k}) = M_{\mu_k}$ for every $k \in K$, then the cartesian product function $f = \prod_{k\in K} f_k : (X, \mu) \to (Y, \lambda)$ is (μ, λ)-continuous.*

Proof. Let $G = \left(\prod_{k\in K_0} A_k\right) \times \left(\prod_{k\in K-K_0} M_{\lambda_k}\right)$ be a basic element of λ, where $K_0 \subset K, K_0$ is finite, and $A_k \in \lambda_k$ for every $k \in K_0$. Then

$$\begin{aligned} f^{-1}(G) &= \left(\prod_{k\in K_0} f_k^{-1}(A_k)\right) \times \left(\prod_{k\in K-K_0} f_k^{-1}(M_{\lambda_k})\right) \\ &= \left(\prod_{k\in K_0} f_k^{-1}(A_k)\right) \times \left(\prod_{k\in K-K_0} M_{\mu_k}\right) \end{aligned}$$

and hence $f^{-1}(G) \in \mu$. □

Proposition 2.5. *Let $f_k : (X_k, \mu_k) \to (Y_k, \lambda_k)$ be (μ_k, λ_k)-continuous for every $k \in K$ and let $f = \prod_{k\in K} f_k : (X, \mu) \to (Y, \lambda)$ be the cartesian product function.*

1. *If (Y_k, λ_k) is strong for for every $k \in K$, then f is (μ, λ)-continuous.*
2. *If f_k is (μ_k, λ_k)-open for every $k \in K$, then f is (μ, λ)-continuous.*

Proof. (1) This follows from Lemmas 2.3 (1) and 2.4.
(2) This follows from Lemmas 2.3 (2) and 2.4. □

Theorem 2.6. *If $f_k : (X_k, \mu_k) \to (Y_k, \lambda_k)$ is a (μ_k, λ_k)-homeomorphism for every $k \in K$, then the cartesian product function $f = \prod_{k\in K} f_k : (X, \mu) \to (Y, \lambda)$ is a (μ, λ)-homeomorphism.*

Proof. It is clear that f is a bijection. By Proposition 2.5 (2), f is a (μ, λ)-continuous. Since $f^{-1} = \prod_{k\in K} f_k^{-1}$, then again by Proposition 2.5 (2) f^{-1} is a (λ, μ)-continuous. This shows that f is a (μ, λ)-homeomorphism. □

Theorem 2.7. *Let $f : (X, \mu) \to (Y, \lambda)$ be a (μ, λ)-homeomorphism. Then*

1. $f\left(M_\mu\right) = M_\lambda$.
2. $f\left(X - M_\mu\right) = Y - M_\lambda$.
3. $f_{|M_\mu} : (M_\mu, \mu) \to (M_\lambda, \lambda)$ *is a (μ, λ)-homeomorphism.*

Proof. (1) Since $f^{-1} : (Y, \lambda) \to (X, \mu)$ is (λ, μ)-continuous and (λ, μ)-open, by Lemma 2.3 (2), $f\left(M_\mu\right) = \left(f^{-1}\right)^{-1}\left(M_\mu\right) = M_\lambda$.
(2) Since f is a bijection, by (1) $f\left(X - M_\mu\right) = f(X) - f\left(M_\mu\right) = Y - M_\lambda$.
(3) By (1) $f_{|M_\mu} : M_\mu \to M_\lambda$ is well defined. It is obvious that $f_{|M_\mu}$ is a bijection. Since $f : (X, \mu) \to (Y, \lambda)$ is a (μ, λ)-homeomorphism, then for every $A \subset M_\mu$, $f(A) \in \lambda$ if and only if $A \in \mu$. It follows that for every $A \subset M_\mu$, $f_{|M_\mu}(A) \in \lambda$ if and only if $A \in \mu$. □

Definition 2.8. Let $K \neq \emptyset$ be an index set and $\{(X_k, \mu_k) : k \in K\}$ be a collection of GTS's such that $X_k \cap X_s = \emptyset$ for all $k \neq s$. The GT on $\bigcup_{k\in K} X_k$ which has the collection $\{A : A \in \mu_k, k \in K\}$ as a base will be denoted by $\bigoplus_{k\in K} \mu_k$ and the pair $(\bigcup_{k\in K} X_k, \bigoplus_{k\in K} \mu_k)$ will be called the disjoint sum of the GTS's of $(X_k, \mu_k), k \in K$.

Proposition 2.9. *Every GTS is either strong or can be written as a disjoint sum of two GTS's one of which is strong.*

Proof. Let (X, μ) be a GTS and suppose (X, μ) is not strong. Let $X_1 = M_\mu, \mu_1 = \mu, X_2 = X - X_1$, and $\mu_2 = \{\emptyset\}$. Then (X_1, μ_1) is a strong GTS and (X_2, μ_2) is a GTS. Also, $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, and it is clear that $\bigoplus_{k\in\{1,2\}} \mu_k = \mu$. □

Proposition 2.10. *Let (X, μ) (resp. (Y, λ)) be the disjoint sum of the GTS's $(X_k, \mu_k), k \in K$ (resp. $(Y_k, \lambda_k), k \in K$). If $f_k : (X_k, \mu_k) \to (Y_k, \lambda_k)$ is (μ_k, λ_k) -continuous for every $k \in K$, then the function $f = \bigcup_{k\in K} f_k : (X, \mu) \to (Y, \lambda)$ is (μ, λ)-continuous.*

Proof. Let $k \in K$ and $B \in \lambda_k$ be a basic element of λ. Then $f^{-1}(B) = f_k^{-1}(B) \in \mu_k \subset \mu$. □

The following result follows directly from Proposition 2.10.

Theorem 2.11. *Let (X, μ) (resp. (Y, λ)) be the disjoint sum of the GTS's $(X_k, \mu_k), k \in K$ (resp. $(Y_k, \lambda_k), k \in K$). If $f_k : (X_k, \mu_k) \to (Y_k, \lambda_k)$ is a (μ_k, λ_k)-homeomorphism for every $k \in K$, then the function $f = \bigcup_{k\in K} f_k : (X, \mu) \to (Y, \lambda)$ is a (μ, λ)-homeomorphism.*

Theorem 2.12. *Let (X, μ) be a GTS and $f : (X, \mu) \to (X, \mu)$ be a function. If μ is finite, then the following are equivalent:*

1. *f is a (μ, μ)-homeomorphism.*
2. *f is bijective and (μ, μ)-continuous.*

Proof. (1) ⇒ (2). Obvious.

(2) ⇒ (1). Let $G \in \mu$. Since f is (μ, μ)-continuous, then the function $\Psi : \mu \to \mu$ defined by $\Psi(A) = f^{-1}(A), A \in \mu$ is well defined. Since f is a bijection then Ψ is an injection, and also, since μ is finite then Ψ is surjective. So, there exists $B \in \mu$ such that $\Psi(B) = G$ and hence $f(G) = B$. This shows that f is (μ, μ)-open and hence f is a (μ, μ)-homeomorphism. □

Corollary 2.13. *Let (X, μ) be a GTS and $f : (X, \mu) \to (X, \mu)$ be a function. If X is finite, then the following are equivalent:*

1. *f is a (μ, μ)-homeomorphism.*
2. *f is an injection and (μ, μ)-continuous.*

Definition 2.14. A non-empty μ-open subset A of a GTS (X, μ) is called a minimal μ-open set if the only non-empty μ-open set which is contained in A is A. The collection of all minimal μ-open sets in a GTS (X, μ) will be denoted by $\min(X, \mu)$.

Proposition 2.15. *Let (X, μ) be a quasi-topological space. If $A, B \in \min(X, \mu)$, then either $A = B$ or $A \cap B = \emptyset$.*

Proof. Suppose $A \cap B \neq \emptyset$. Since (X, μ) is a quasi-topological space, then $A \cap B \in \mu$. Therefore, $A \cap B = A$ and $A \cap B = B$. Hence $A = B$. □

The condition 'quasi-topological space' in Proposition 2.15 cannot be replaced by 'GTS' in general, as the following example shows:

Example 2.16. Let $X = \mathbb{R}$ and $\mu = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. Then $\{1, 2\}, \{1, 3\} \in \min(X, \mu)$, but neither $\{1, 2\} = \{1, 3\}$ nor $\{1, 2\} \cap \{1, 3\} = \emptyset$.

Proposition 2.17. *Let (X, μ) be a GTS with $\mu \neq \{\emptyset\}$. Then*

1. *For every base $\mathcal{B}$ of μ we have $\min(X, \mu) \subset \mathcal{B}$.*

2. $\min(X,\mu) = \min(M_\mu,\mu)$.
3. *If μ is finite, then* $\min(X,\mu) \neq \emptyset$.
4. *If X is finite, then* $\min(X,\mu) \neq \emptyset$.

Proof. (1) Suppose $A \in \min(X,\mu)$. Choose $x \in A$. Then there exists $B \in \mathcal{B}$ such that $x \in B \subset A$. Since $A \in \min(X,\mu)$, then $A = B$ and hence $A \in \mathcal{B}$.
(2) Obvious.
(3) By assumption $M_\mu \neq \emptyset$. If M_μ is minimal μ-open then we are done. If M_μ is not minimal μ-open then there exists a non-empty $A_1 \in \mu$ such that $A_1 \subset M_\mu$. If A_1 is minimal μ-open then we are done, if not, then there exists a non-empty $A_2 \in \mu$ such that $A_2 \subset A_1$. Since μ is finite, if we go on this way we will reach to minimal μ-open, say A_n.
(4) If X is finite, then μ is finite and by (3) we get the result. □

Proposition 2.18. *Let $f : (X,\mu) \to (Y,\lambda)$ be injective, (μ,λ)-continuous and (μ,λ)-open. If A is a minimal μ-open set, then $f(A)$ is a minimal λ-open set.*

Proof. Since A is a minimal μ-open set, then $A \neq \emptyset$ and so $f(A) \neq \emptyset$. Also, since A is μ-open, $f(A)$ is λ-open. Suppose $H \subset f(A)$, where $H \in \lambda - \{\emptyset\}$. Then $f^{-1}(H) \subset f^{-1}(f(A))$. Since f is injective, $f^{-1}(f(A)) = A$. Also, since $H \subset f(A)$, $f^{-1}(H) \neq \emptyset$. Since f is (μ,λ)-continuous, $f^{-1}(H) \in \mu$. Since A is minimal μ-open, $f^{-1}(H) = A$. Thus, $f(A) = f\left(f^{-1}(H)\right) \subset H$ and hence $f(A) = H$. □

The following result follows directly from Proposition 2.18.

Theorem 2.19. *Let $f : (X,\mu) \to (Y,\lambda)$ be a (μ,λ)-homeomorphism. Then a subset $A \subset X$ is a minimal μ-open set if and only if $f(A)$ is a minimal λ-open set.*

3. Homogeneous GTS's

Definition 3.1. A GTS (X,μ) is said to be homogeneous if for any two points $x, y \in M_\mu$ there exists a (μ,μ)-homeomorphism $f : (X,\mu) \to (X,\mu)$ such that $f(x) = y$.

The following result follows directly:

Proposition 3.2. *Let (X,μ) be a GTS. Then*

1. *If $\mu = \{\emptyset\}$, then (X,μ) is homogeneous.*
2. *If (X,μ) is a topological space, then (X,μ) is homogeneous as a GTS if and only if (X,μ) is homogeneous as a topological space.*

Example 3.3. Let $X = \mathbb{R}$ and $\mu = \{\emptyset, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. Then (X,μ) is a homogeneous GTS that is neither strong nor quasi.

Theorem 3.4. *Let (X,μ) be a GTS. Then the following are equivalent:*

1. *(X,μ) is homogeneous.*
2. *(M_μ,μ) is homogeneous.*

Proof. (1) $\Rightarrow$ (2). Let $x, y \in M_\mu$. Then there exists a (μ,μ)-homeomorphism $f : (X,\mu) \to (X,\mu)$ such that $f(x) = y$. Theorem 2.7 (3) completes the proof.
(2) $\Rightarrow$ (1). We may assume that $M_\mu \neq X$. Let $x, y \in M_\mu$. Then there exists a (μ,μ)-homeomorphism $f : (M_\mu,\mu) \to (M_\mu,\mu)$ such that $f(x) = y$. Define $g : (X - M_\mu, \{\emptyset\}) \to (X - M_\mu, \{\emptyset\})$ by $g(t) = t, t \in X - M_\mu$. Then it is clear that g is (μ,μ)-homeomorphism. By Theorem 2.11, $f \cup g : (X,\mu) \to (X,\mu)$ is a (μ,μ)-homeomorphism. Moreover, $(f \cup g)(x) = f(x) = y$. □

Theorem 3.5. *Let (X, μ) be the product of the GTS's $(X_k, \mu_k), k \in K$. If every (X_k, μ_k) is homogeneous, then (X, μ) is homogeneous.*

Proof. Let $(x_k)_{k\in K}, (y_k)_{k\in K} \in M_\mu$. Then for every $k \in K, x_k, y_k \in M_{\mu_k}$ and so there exists a (μ_k, μ_k)-homeomorphism $f_k : (X_k, \mu_k) \to (X_k, \mu_k)$ such that $f_k(x_k) = y_k$. By Theorem 2.6, it follows that the cartesian product function $f : (X, \mu) \to (X, \mu)$ is (μ, μ)-homeomorphism. Moreover, $f((x_k)_{k\in K}) = (f_k(x_k))_{k\in K} = (y_k)_{k\in K}$. □

Theorem 3.6. *Let (X, μ) be the disjoint sum of the GTS's $(X_k, \mu_k), k \in K$. If every (X_k, μ_k) is homogeneous and (X_k, μ_k) is homeomorphic to (X_s, μ_s) for all $k, s \in K$, then (X, μ) is homogeneous.*

Proof. Let $x, y \in M_\mu$. Then we have two cases.

Case 1. $x, y \in M_{\mu_{k_0}}$ for some $k_0 \in K$. Since (X_k, μ_k) is homogeneous, there exists a (μ_{k_0}, μ_{k_0})-homeomorphism $f_{k_0} : (X_{k_0}, \mu_{k_0}) \to (X_{k_0}, \mu_{k_0})$ such that $f_{k_0}(x) = y$. For every $k \in K - \{k_0\}$, the function $f_k : (X_k, \mu_k) \to (X_k, \mu_k)$ where $f_k(t) = t, t \in X_k$ is a (μ_k, μ_k)-homeomorphism. By Theorem 2.11, $\bigcup_{k\in K} f_k : (X, \mu) \to (X, \mu)$ is a (μ, μ)-homeomorphism. Also, $\left(\bigcup_{k\in K} f_k\right)(x) = y$.

Case 2. $x \in M_{\mu_k}$ and $y \in M_{\mu_s}$ for $k, s \in K$ with $k \neq s$. Since (X_k, μ_k) is homeomorphic to (X_s, μ_s), there exists a (μ_k, μ_s)-homeomorphism $g : (X_k, \mu_k) \to (X_s, \mu_s)$. Since (X_s, μ_s) is homogeneous, there exists a (μ_s, μ_s)-homeomorphism $h : (X_s, \mu_s) \to (X_s, \mu_s)$ such that $h(g(x)) = y$. Define $f : (X, \mu) \to (X, \mu)$ by

$$f(t) = \begin{cases} (h \circ g)(t) & \text{if} \quad t \in X_k \\ (h \circ g)^{-1}(t) & \text{if} \quad t \in X_s \\ t & \text{if} \quad t \in X - (X_s \cup X_k) \end{cases}$$

Then f is a bijection and $f(x) = y$. Also, by Theorem 2.11, f is a (μ, μ)-homeomorphism. □

In Theorem 3.6, the condition on the GTS's to be homeomorphic cannot be dropped as we will see in the following example.

Example 3.7. The GTS's $((-\infty, 1), \mu_1)$ and $([1, \infty), \mu_2)$ where $\mu_1 = \{\emptyset, \{0\}\}$ and $\mu_2 = \{\emptyset, \{1, 2\}\}$ are homogeneous. The disjoint sum of the GTS's $((-\infty, 1), \mu_1)$ and $([1, \infty), \mu_2)$ is $(\mathbb{R}, \mu)$, where $\mu = \{\emptyset, \{0\}, \{1, 2\}, \{0, 1, 2\}\}$ is not homogeneous.

A family $\mathcal{A}$ of sets is called a regular cover of a non-empty set Y if $Y = \bigcup \mathcal{A}$ and for every $A, B \in \mathcal{A}$, $|A| = |B|$.

Theorem 3.8. *Let (X, μ) be a homogeneous GTS. If A is a minimal μ-open set, then $\{B \in \min(X, \mu) : |B| = |A|\}$ is a regular cover of M_μ.*

Proof. It suffices to see that $M_\mu \subset \{B \in \min(X, \mu) : |B| = |A|\}$. Let $x \in M_\mu$. Pick $a \in A$. Since (X, μ) is homogeneous, there exists a (μ, μ)-homeomorphism $f : (X, \mu) \to (X, \mu)$ such that $f(a) = x$. By Theorem 2.19, $f(A)$ is a minimal μ-open set. Also, since f is a bijection, $|f(A)| = |A|$. This completes the proof. □

Corollary 3.9. *Let (X, μ) be a homogeneous GTS. If for some $x \in X$, $\{x\}$ is μ-open, then μ is the discrete topology on M_μ.*

Question 3.10. *Let (X, μ) be a homogeneous GTS such that $\min(X, \mu) \neq \emptyset$. Is it true that $\min(X, \mu)$ is a regular cover of M_μ?*

Corollary 3.9 answers Question 3.10 partially.

The authors in [17] had obtained the following result:

Proposition 3.11. *Let (X, τ) be a topological space which contains a minimal open set. Then the following are equivalent.*

1. (X, τ) *is a homogeneous topological space.*
2. (X, τ) *is a disjoint union of indiscrete topological spaces all of which are homeomorphic to one another.*

Using Proposition 3.11, the following result follows easily:

Theorem 3.12. *Let* (X, τ) *be a topological space for which* $\min(X, \tau) \neq \emptyset$*. Then the following are equivalent:*

1. (X, τ) *is a homogeneous space.*
2. $\min(X, \tau)$ *forms a regular cover of X and a partition of X, and* $\min(X, \tau) \cup \{\emptyset\}$ *forms a base for* τ*.*

Corollary 3.13. *Let* (X, μ) *be a quasi-topological space for which* $\min(X, \mu) \neq \emptyset$*. Then the following are equivalent:*

1. (X, μ) *is homogeneous.*
2. $\min(X, \mu)$ *forms a regular cover of* M_μ *and a partition of* M_μ*, and* $\min(X, \mu) \cup \{\emptyset\}$ *forms a base for* μ*.*

Proof. Since (X, μ) is a quasi-topological space, (M_μ, μ) is a topological space. Also, by Proposition 2.17 (2), (M_μ, μ) must contain a minimal open set. Therefore, Theorems 3.4 and 3.12 complete the proof. □

Corollary 3.13 answers Question 3.10 for quasi-topological space's.

Corollary 3.14. *Let* (X, μ) *be a quasi-topological space for which* μ *is finite and* $\mu \neq \{\emptyset\}$*. Then the following are equivalent:*

1. (X, μ) *is homogeneous.*
2. $\min(X, \mu)$ *forms a regular cover of* M_μ *and a partition of* M_μ*, and* $\min(X, \mu) \cup \{\emptyset\}$ *forms a base for* μ*.*

Proof. This follows from Proposition 2.17 (3) and Corollary 3.13. □

Corollary 3.15. *Let* (X, μ) *be a quasi-topological space for which X is finite. Then the following are equivalent:*

1. (X, μ) *is homogeneous.*
2. $\min(X, \mu)$ *forms a regular cover of* M_μ *and a partition of* M_μ*, and* $\min(X, \mu) \cup \{\emptyset\}$ *forms a base for* μ*.*

Proof. This follows from Proposition 2.17 (4) and Corollary 3.13. □

In each of Corollaries 3.13, 3.14, and 3.15, the condition 'quasi' cannot be dropped even when (X, μ) is strong as the following example says:

Example 3.16. Let $X = \{1, 2, 3\}$ and $\mu = \{\emptyset, X, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$. It is not difficult to see that (X, μ) is a homogeneous GTS. Also, it is clear that $\min(X, \mu)$ is not a partition of M_μ.

Question 3.17. *Let X be a finite non-empty set with* $|X| = n$*. Is there a clear formula to give the number of homogeneous GTS's on X (up to homoemorphism)?*

The authors in [17] had obtained the following result:

Proposition 3.18. *If X is a finite non-empty set with* $|X| = n$*, then there are precisely* $\tau(n)$ *homogeneous topological spaces on X (up to homeomorphism), where* $\tau(n)$ *is the number of positive divisors of n.*

The following result answers Question 3.17 partially.

Theorem 3.19. *If X is a finite non-empty set with* $|X| = n$*, then there are precisely* $\sum_{i=1}^{n} \mu(i)$ *homogeneous quasi-topological spaces on X (up to homeomorphism), where* $\mu(i)$ *is the number of positive divisors of i.*

Proof. Let (X, μ) be a homogeneous quasi-topological space for which $|X| = n$. Then by Theorem 3.4 (M_μ, μ) is a homogeneous topological space. As $|M_\mu| = 1, 2, ...,$or n, and as (M_μ, μ_1) and (M_μ, μ_2) are not homeomorphic for $|M_{\mu_1}| \neq |M_{\mu_2}|$, then applying Proposition 3.18, we get the result. □

4. Homogeneous components

In this section, for a GTS (X, μ), we assume $\mu \neq \{\emptyset\}$.

Definition 4.1. Let (X, μ) be a GTS. We define the relation $*$ on X as follows: For $x_1, x_2 \in M_\mu$, $x_1 * x_2$ if there exists a (μ, μ)-homeomorphism $f : (X, \mu) \to (X, \mu)$ such that $f(x_1) = x_2$.

Remark 4.2. Relation $*$ in Definition 4.1 is an equivalence relation.

Definition 4.3. Let (X, μ) be a GTS. A subset of M_μ which has the form $\mu\text{-}C(x) = \left\{y \in M_\mu : x * y\right\}$ is called a homogeneous component determined by $x \in M_\mu$.

Remark 4.4. A GTS (X, μ) is homogeneous iff it has exactly one homogeneous component.

Example 4.5. Let $X = \mathbb{R}$ and $\mu = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. Then $\mu\text{-}C(1) = \{1\}$ and $\mu\text{-}C(2) = \mu\text{-}C(3) = \{2, 3\}$.

Theorem 4.6. *If $f : (X, \mu) \to (X, \mu)$ is a (μ, μ)-homeomorphism then for $x \in M_\mu$, $f(\mu\text{-}C(x)) = \mu\text{-}C(x)$.*

Proof. Since f and f^{-1} are (μ, μ)-homeomorphisms, it is sufficient to see that $f(\mu\text{-}C(x)) \subset \mu\text{-}C(x)$. Let $y \in f(\mu\text{-}C(x))$. Then there exists $w \in \mu\text{-}C(x)$ such that $y = f(w)$. Thus we have $w * x$ and $w * y$. Since $*$ is an equivalence relation, it follows that $y * x$ and hence $y \in \mu\text{-}C(x)$. □

Theorem 4.7. *In a GTS (X, μ), for every $x \in M_\mu$, $\mu\text{-}C(x)$ is μ-open or $i_\mu(\mu\text{-}C(x)) = \emptyset$.*

Proof. Suppose for some $x \in M_\mu$, $i_\mu(\mu\text{-}C(x)) \neq \emptyset$. We are going to show that $\mu\text{-}C(x) \subset i_\mu(\mu\text{-}C(x))$. Let $w \in \mu\text{-}C(x)$ and choose $y \in i_\mu(\mu\text{-}C(x)) \subset \mu\text{-}C(x)$. Then $y, w \in \mu\text{-}C(x)$ and so there exists a (μ, μ)-homeomorphism $f : (X, \mu) \to (X, \mu)$ such that $w = f(y)$. Therefore, by Theorem 4.6 $w \in f\left(i_\mu(\mu\text{-}C(x))\right) \subset f(\mu\text{-}C(x)) = \mu\text{-}C(x)$ and hence $f\left(i_\mu(\mu\text{-}C(x))\right) \subset i_\mu(\mu\text{-}C(x))$. It follows that $\mu\text{-}C(x) \subset i_\mu(\mu\text{-}C(x))$. □

Note that in Example 4.5, $i_\mu(\mu\text{-}C(x)) = \emptyset$ for all $x \in M_\mu$, which is compatible with Theorem 4.7.

Definition 4.8. Suppose (X, μ) is a GTS and Y is a non-empty subset of X. The subspace generalized topology of Y on X is generalized topology $\mu_Y = \{Y \cap U : U \in \mu\}$ on Y. The pair (Y, μ_Y) is called a subspace GTS of (X, μ).

Lemma 4.9. *If $f : (X, \mu) \to (Y, \lambda)$ is (μ, λ)-continuous and A is a non-empty subset of X then $f_{|A} : (A, \mu_A) \to \left(f(A), \lambda_{f(A)}\right)$ is $\left(\mu_A, \lambda_{f(A)}\right)$-continuous.*

Proof. Let $V \cap f(A) \in \lambda_{f(A)}$, where $V \in \lambda$. Since f is (μ, λ)-continuous, $f^{-1}(V) \in \mu$ and hence $(f_{|A})^{-1}(V \cap f(A)) = f^{-1}(V) \cap A \in \mu_A$. □

Theorem 4.10. *If (X, μ) is a GTS and $\mu\text{-}C(x)$ is a homogeneous component of (X, μ), then the subspace GTS $\left(\mu\text{-}C(x), \mu_{\mu\text{-}C(x)}\right)$ is homogeneous.*

Proof. Let $x_1, x_2 \in \mu\text{-}C(x)$. Then there exist two (μ, μ)-homeomorphisms $f_1, f_2 : (X, \mu) \to (X, \mu)$ such that $f_1(x) = x_1$ and $f_2(x) = x_2$. By Theorem 4.6, $\left(f_2 \circ f_1^{-1}\right)(\mu\text{-}C(x)) = \mu\text{-}C(x)$. Hence the function $f : \left(\mu\text{-}C(x), \mu_{\mu\text{-}C(x)}\right) \to \left(\mu\text{-}C(x), \mu_{\mu\text{-}C(x)}\right)$, where $f(t) = \left(\left(f_2 \circ f_1^{-1}\right)_{|\mu\text{-}C(x)}\right)(t)$ for all $t \in \mu\text{-}C(x)$ is well defined. Applying Lemma 4.9, we conclude that f is a $(\mu_{\mu\text{-}C(x)}, \mu_{\mu\text{-}C(x)})$-homeomorphism. Also, $f(x_1) = x_2$. Therefore, $\left(\mu\text{-}C(x), \mu_{\mu\text{-}C(x)}\right)$ is homogeneous. □

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