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On the cardinality of the θ-closed hull of sets II

Filippo Cammaroto[a], Andrei Catalioto[b], Bruno Antonio Pansera[c], Jack Porter[d]

[a] *Department of Mathematics, University of Messina, Italy*
[b] *Department of Mathematics, University of Messina, Italy*
[c] *Department of Mathematics, University of Messina, Italy*
[d] *Department of Mathematics, University of Kansas, USA*

Abstract. The research in this paper is a continuation of the investigation of the cardinality of the θ-closed hull of subsets of spaces. This research obtains new upper bounds of the cardinality of the θ-closed hull of subsets using cardinal functions of θ-bitightness ([8]), finite θ-bitightness ([7]) and θ-bitightness small number ([7]) of spaces. In the final section, examples of spaces are presented including one that answers a question posed in [4] and [6].

1. Introduction, notation and terminology

Throughout this paper, X is used to denote a topological space and A is an arbitrary subset of X. All spaces are assume to be Hausdorff unless specifically mentioned otherwise. Also, the Greek letters α, β, γ,... are used to denote infinite ordinal numbers and κ, λ, μ,... infinite cardinal numbers. The family of open sets of X is denoted by $\tau(X)$, and $\mathcal{N}_x$ (resp. $c\mathcal{N}_x$) is used to denote the collection of open (resp. closed) neighborhoods of $x \in X$. Our notation and terminology are mainly as in [9] (for general topological notions) and [10] (for cardinal functions). We start by recalling some basic concepts that are used in the sequel.

The *semiregularization* of a space X, denoted by X_s (or $X(s)$), is the set X with the topology generated by the family $RO(X) = \{U \in \tau(X) : U = int_X(cl_X(U))\}$ of regular open sets of X. A space X is called *semiregular* when $X = X_s$.

The *θ-closure* of A, denoted by $cl_\theta(A)$, is the set of all elements of $x \in X$ such that $cl_X U \cap A \neq \emptyset$, whenever $x \in U \in \tau(X)$. A is said *θ-closed* if $A = cl_\theta(A)$. The *θ-closed hull of A*, denoted by $[A]_\theta$, is the smallest θ-closed subset of X containing A (i.e. $[A]_\theta := \bigcap\{C \subseteq X : A \subseteq C \text{ and } C = cl_\theta(C)\}$).

Recall that for $x \in X$, $\chi(x, X)$ denotes the smallest cardinality of a local base of X at x, and the character $\chi(X)$ of the space X is the maximum of $\aleph_0$ and $\sup_{x\in X}\chi(x, X)$. Also, for $x \in X$, $\chi_\theta(x, X)$ denotes the smallest cardinal κ for which there is a collection $\mathcal{V}_x \subseteq c\mathcal{N}_x$ such that $|\mathcal{V}_x| \leq \kappa$ and if $W \in c\mathcal{N}_x$, then W contains a member of $\mathcal{V}_x$, and the *closed character* $\chi_\theta(X)$ of the space X is the maximum of $\aleph_0$ and $\sup_{x\in X}\chi_\theta(x, X)$.

It follows that $\chi_\theta(X) = \chi(X_s)$ (see Proposition 1 below); thus $\chi_\theta(X) \leq \chi(X)$ and $\chi_\theta(X)$ may be strictly smaller than $\chi(X)$. In [1], it is shown that for a subset A of a Urysohn space X, $|cl_\theta(A)| \leq |A|^{\chi_\theta(X)}$.

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Email addresses: camfil@unime.it (Filippo Cammaroto), acatalioto@unime.it (Andrei Catalioto), bpansera@unime.it (Bruno Antonio Pansera), porter@math.ku.edu (Jack Porter)

A space X is *Urysohn* if for two distinct points $x, y \in X$ there are neighborhoods U of x and V of y such that $cl_X U \cap cl_X V = \emptyset$. In [4] Bonanzinga, Cammaroto and Matveev defined the following cardinal function and used it to extend the definition of Urysohn.

$$U(X) = \min\{\kappa : \text{for every } A \in [X]^{\geq\kappa} \text{ one can find neighborhoods } U_a \text{ of } a \text{ for } a \in A \text{ such that } \bigcap_{a\in A} \overline{U}_a = \emptyset\}.$$

The cardinal function $U(X)$ is called the *Urysohn number* of X. For $2 \leq n < \omega$, the space X is *n-Urysohn* whenever $U(X) = n$; in particular, X is Urysohn if and only if $U(X) = 2$.

In 1993, Cammaroto and Kočinac ([8]; see also [11]) introduced the concept of *θ-bitightness* for a space X, denoted by $bt_\theta(X)$, as the smallest cardinal κ such that for each non-θ-closed set $A \subseteq X$ there are $x \in cl_\theta(A)\backslash A$ and $\mathcal{S} \in [[A]^{\leq\kappa}]^{\leq\kappa}$ such that $\{x\} = \bigcap_{S\in\mathcal{S}} cl_\theta(S)$. They showed, for Urysohn spaces, that $bt_\theta(X)$ is defined, $bt_\theta(X) \leq \chi(X)$, and the inequality is strict.

Recently, Cammaroto, Catalioto, Pansera and Tsaban ([7]) introduced

(1) the concept of *finite θ-bitightness* for a space X, denoted by $fbt_\theta(X)$, as the smallest cardinal κ such that for each non-θ-closed $A \subseteq X$, there is $\mathcal{S} \in [[A]^{\leq\kappa}]^{\leq\kappa}$ such that $\bigcap_{S\in\mathcal{S}} cl_\theta(S)\backslash A$ is finite nonempty, and

(2) the concept of *θ-bitightness small number* for any space X, denoted by $bts_\theta(X)$, as the smallest cardinal κ such that for each non-θ-closed $A \subseteq X$ that is not a singleton (true for Hausdorff spaces), there is $\mathcal{S} \in [[A]^{\leq\kappa}]^{\leq\kappa}$ such that $\bigcap_{S\in\mathcal{S}} cl_\theta(S)\backslash A$ is nonempty and $|\bigcap_{S\in\mathcal{S}} cl_\theta S| \leq |A|^\kappa$.

It is clear that when $bt_\theta(X)$ is defined, so is $fbt_\theta(X)$ and $bts_\theta(X) \leq fbt_\theta(X) \leq bt_\theta(X)$ and that $bts_\theta(X)$ is defined for any space.

In 1988, Bella and Cammaroto ([2]) proved that $|[A_\theta]| \leq |A|^{\chi(X)}$ for every subset A of an Urysohn space X. This result was improved, for Urysohn spaces, to $|[A]_\theta| \leq |A|^{bt_\theta(X)}$, in 1993, by Cammaroto and Kočinac ([8]). Recently, Bonanzinga, Cammaroto, Matveev and Pansera ([4, 6]) improved Bella and Cammaroto's result to $|[A]_\theta| \leq |A|^{\chi_\theta(X))}$ when $U(X)$ is finite, and Cammaroto, Catalioto, Pansera, and Tsaban ([7]) improved Cammaroto and Kočinac's result to $|[A]_\theta| \leq |A|^{bts_\theta(X)}$ for any space X.

In this paper we give some results concerning the relationship between $bt_\theta(X)$ and $fbt_\theta(X)$ and several examples are showed including a negative answer to a problem of Bonanzinga-Cammaroto-Matveev ([4]) and Bonanzinga-Pansera ([6]).

2. Cardinal properties of the θ-bitightness, the finite θ-bitightness and the θ-bitightness small number

In [7] the authors defined for any space X, a new topological cardinal invariant called the *θ-bitightness small number* of X, denoted as $bts_\theta(X)$, as defined above. They also proved that, for every topological space X, the cardinality of $[A]_\theta$ is at most $|A|^{bts_\theta(X)}$ where $A \subseteq X$.

For completeness of our exposition, we provide a proof of the following widely accepted lemma.

Lemma 1. *For any space X and $A \subseteq X$, $cl^X_\theta(A) = cl^{X_s}_\theta(A)$ as sets.*

Proof. Let $p \in cl^X_\theta(A)$ and $p \in U \in \tau(X_s)$ and $U \in RO(X)$. Then, $cl_{X_s}U = cl_X U$ (see [12]), then $cl_{X_s}U \cap A \neq \emptyset$ and $p \in cl^{X_s}_\theta(A)$. Conversely, suppose $p \in cl^{X_s}_\theta(A)$ and $p \in U \in \tau(X)$. Then $p \in int_X cl_X U$ and $int_X cl_X U \in \tau(X_s)$. Now $cl_X int_X cl_X U = cl_X U$ and $\emptyset \neq cl_X int_X cl_X U \cap A = cl_X U \cap A$. So, $p \in cl^X_\theta(A)$. □

So, we also have the following result concerning the θ-closed hull.

Corollary 1. *If $A \subseteq X$, then $|[A]^X_\theta| = |[A]^{X_s}_\theta|$.*

Now, we will study the relationships among the cardinal functions.

Proposition 1. *For a space X, $\chi_\theta(X) = \chi_\theta(X_s) = \chi(X_s) \leq \chi(X)$.*

Proof. Let C be a family of closed neighborhoods of $p \in X$ such that $|C| = \chi_\theta(p, X)$. Then $\mathcal{R} = \{int_X A : A \in C\}$ is a family of regular open sets, each containing p and $|\mathcal{R}| \leq |C|$. To show that $\mathcal{R}$ is a $\tau(X_s)$-open base for p, let $p \in U \in \tau(X_s)$ where U is a regular open subset of X. Then, there is some $A \in C$ such that $A \subseteq cl_X U$. Now, $int_X A \subseteq int_X cl_X U = U$, and it follows that $\chi(p, X_s) \leq \chi_\theta(p, X)$ for each $p \in X$. Thus, $\chi(X_s) \leq \chi_\theta(X)$. Moreover, Let $\mathcal{R}$ be a $\tau(X_s)$-open base for $p \in X$ such that $|\mathcal{R}| = \chi(p, X_s)$ and each $U \in \mathcal{R}$ is regular open in X. Let $C = \{cl_X U : U \in \mathcal{R}\}$. To show that the family C of closed neighborhoods of p is a base for the closed neighborhoods of p, let A be a closed neighborhood of p in X. Then $int_X A$ is a regular open subset in X and contains p. There is some $U \in \mathcal{R}$ such that $p \in U \subseteq int_X A$. Thus, $cl_X U \subseteq A$ and $cl_X U \in C$. As C is a base for the closed neighborhoods of p in X, $\chi_\theta(p, X) \leq \chi(p, X_s)$. This completes the proof that $\chi_\theta(X) \leq \chi(X_s)$. By Lemma 1, we have that $\chi_\theta(X) = \chi(X_s)$ and $\chi_\theta(X_s) = \chi((X_s)_s) = \chi(X_s)$. □

The inequalities in Proposition 1 hold for any space and supplement the following inequalities established in [7] that hold in Urysohn spaces.

Proposition 2. *If X is Urysohn, then $bts_\theta(X) \leq fbt_\theta(X) \leq bt_\theta(X) \leq \chi_\theta(X) \leq \chi(X)$.*

The inequalities in Proposition 2 hold whenever $bt_\theta(X)$ is defined ($fbt_\theta(X)$ is defined whenever $bt_\theta(X)$ is defined and $bts_\theta(X), \chi_\theta(X)$, and $\chi(X)$ are defined for any space). As $bts_\theta(X)$ and $\chi_\theta(X)$ are defined for any space, it is natural to ask if $bts_\theta(X) \leq \chi_\theta(X)$ holds for any space. It is interesting that it is not true that $bts_\theta(X) \leq \chi_\theta(X)$ holds for any space. A counterexample is a part of Example 1.

An immediate consequence of Proposition 2 and the Cammaroto, Catalioto, Pansera, and Tsaban's inequality for a subset A of an Urysohn space X ($|[A]_\theta| \leq |A|^{bts_\theta(X)}$) is that $|[A]_\theta| \leq |A|^{fbt_\theta(X)}$. A direct proof of this consequence is provided by the next result.

Theorem 1. *If X is a Urysohn space and $A \subseteq X$, then $|[A]_\theta| \leq |A|^{fbt_\theta(X)}$.*

Proof. Let $fbt_\theta(X) = \kappa$. It suffices to consider only non-θ-closed subsets A of X. Let $|A| \leq \mu$. By induction, we will construct an increasing sequence $\{A_\alpha : \alpha < \kappa^+\}$ of subsets of X such that $A_0 = A$, $|A_\alpha| \leq \mu^\kappa$, and for $\alpha < \beta < \kappa^+$, $A_\alpha \subseteq A_\beta$. Suppose $\beta < \kappa^+$ and A_α is defined for $\alpha < \beta$.
If β is a limit ordinal, then define $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ and A_β has the desired properties. Suppose $\beta = \gamma + 1$ is a successor ordinal. Define $A_\beta = A_\gamma \cup C_\gamma$ where $C_\gamma = \{x \in X \setminus A_\gamma : \exists\, \mathcal{S} \in [[A_\gamma]^{\leq\kappa}]^{\leq\kappa}$ such that $\{x\} = \bigcap_{S\in\mathcal{S}} cl_\theta(S) \setminus A_\gamma\}$. For each $x \in C_\gamma$, select one $\mathcal{S}_x \in [[A_\gamma]^{\leq\kappa}]^{\leq\kappa}$ such that $\{x\} = \bigcap_{S\in\mathcal{S}_x} cl_\theta(S) \setminus A_\gamma$. Note that $|[[A_\gamma]^{\leq\kappa}]^{\leq\kappa}| \leq (\mu^\kappa)^\kappa = \mu^\kappa$. Thus, $|A_\beta| \leq \mu^\kappa$. This completes the induction. For $B = \bigcup_{\alpha\in\kappa^+} A_\alpha$, we have that $|B| \leq \mu^\kappa$. The final step is to show that B is θ-closed. Suppose $cl_\theta B \setminus B \neq \emptyset$. Then B is non-θ-closed and, by [7, Prop. 2.4], there is $\mathcal{S} \in [[B]^{\leq\kappa}]^{\leq\kappa}$ such that $\{x\} = \bigcap_{S\in\mathcal{S}} cl_\theta(S) \setminus B$. So, there is $\beta < \kappa^+$ such that $\bigcup \mathcal{S} \subseteq A_\beta$. By the construction of $A_{\beta+1}$, $x \in A_{\beta+1} \subseteq B$, a contradiction. Thus, B is θ-closed. □

Now, we provide a lower and upper bound of $bt_\theta(X)$ in terms of $fbt_\theta(X)$.

Theorem 2. *If X is Urysohn, then $fbt_\theta(X) \leq bt_\theta(X) \leq 2^{fbt_\theta(X)}$.*

Proof. Let $fbt_\theta(X) = \kappa$ and A be a non-θ-closed subset of X. By [7, Prop. 2.4], there is $x \in cl_\theta(A) \setminus A$ and $\mathcal{S} \in [[A]^{\leq\kappa}]^{\leq\kappa}$ such that $\{x\} = \bigcap_{S\in\mathcal{S}} cl_\theta(S) \setminus A$. We can write $\mathcal{S} = \{S_\alpha : \alpha < \kappa\}$ and assume that $S_0 \supseteq S_\alpha$ for all $\alpha < \kappa$. By Theorem 1, $|A \cap cl_\theta S_0| \leq |cl_\theta S_0| \leq |S_0|^\kappa = 2^\kappa$. For each $y \in A$, let $U_y, V_y \in \tau(X)$ such that $x \in U_y$, $y \in V_y$, and $clU_y \cap clV_y = \emptyset$. Note that $x \in cl_\theta(S_0 \setminus cl(V_y)) \subseteq X \setminus \{y\}$. For $\mathcal{S}' = \mathcal{S} \cup \{S_0 \setminus cl(V_y) : y \in A \cap cl_\theta S_0\}$, we have that $\{x\} = \bigcap_{S\in\mathcal{S}'} cl_\theta(S)$ and $|\mathcal{S}'| \leq 2^\kappa$. So, $bt_\theta(X) \leq 2^\kappa$. □

Remark 1. In Example 3, we provide a Hausdorff space X such that $U(X) = \omega$ and $|[A]_\theta| > |A|^{\chi(X)}$; however, we know that $|[A]_\theta| \leq |A|^{\chi_\theta(X)}$ when X is Hausdorff and finitely Urysohn. The future research goal is to identify those spaces X for which $U(X)$ is infinite and $|[A]_\theta| \leq |A|^{\chi_\theta(X)}$. This research project is simplified by using that $U(X) = U(X_s)$ for any space X and then applying corollary 1: to obtain that $|[A]_\theta| \leq |A|^{\chi_\theta(X)}$ is reduced to verifying $|[A]_\theta| \leq |A|^{\chi_\theta(X)}$ for a semiregular Hausdorff space X for which $U(X)$ is infinite.

3. Examples

Spaces are presented to show the existence of non-Urysohn spaces, where $bt_\theta(X)$ is not defined, $fbt_\theta(X) = \omega$, and $bts_\theta(X) > \chi_\theta(X)$ (Example 1) and where $bt_\theta(X)$ and $fbt_\theta(X)$ are defined and $fbt_\theta(X) = bt_\theta(X) = \omega$ (Example 2). Finally, Example 3 gives a negative answer to a question present in [4] and in [6]. What remains open is the existence of a space X where $bt_\theta(X)$ and $fbt_\theta(X)$ are defined and $fbt_\theta(X) < bt_\theta(X)$.

Example 1. A first countable, Hausdorff space X for which $bt_\theta(X)$ is not defined, $fbt_\theta(X) = \omega$, and $bts_\theta(X) > \chi_\theta(X)$.

Let $\mathbb{Q} = \{r_n : n \in \omega\}$ denote the space of rational numbers with the usual topology and $\mathbb{D} = \mathbb{Q} + \sqrt{2}$ denote the dense subspace of irrational numbers. Let Λ be nonempty set and $X(\Lambda) = \mathbb{Q} \cup (\mathbb{D} \times \Lambda)$. A set $U \subseteq X(\Lambda)$ is defined to be open if:

(1) $p \in U \cap \mathbb{Q}$ implies there is $\epsilon > 0$ such that $((p-\epsilon, p+\epsilon) \cap \mathbb{Q}) \cup ((p-\epsilon, p+\epsilon) \cap \mathbb{D}) \times \Lambda) \subseteq U$, and

(2) $(p, \alpha) \in U \cap (\mathbb{D} \times \{\alpha\})$ for some $\alpha \in \Lambda$ implies there is $\epsilon > 0$ such that $((p-\epsilon, p+\epsilon) \cap \mathbb{D}) \times \{\alpha\} \subseteq U$.

For $|\Lambda| \geq 2$, the space $X(\Lambda)$ is Hausdorff, semiregular, and first countable but not Urysohn. Points in $\mathbb{Q}$ have clopen neighborhoods and for each $\alpha \in \Lambda$, a pair of points in $\mathbb{D} \times \{\alpha\}$ are contained in disjoint closed neighborhoods. In particular, it follows that if $|\Lambda| \in \omega$, $U(X(\Lambda)) = |\Lambda| + 1$. To compute $U(X(\Lambda))$ when $|\Lambda| \geq \omega$, let $p \in \mathbb{D}$ and $B = \{(p, \alpha_n) : n \in \omega\}$. For each $n \in \omega$, choose $\epsilon_n > 0$ such that $r_n \notin [p-\epsilon_n, p+\epsilon_n]$. Let $U_n = ((p-\epsilon_n, p+\epsilon_n) \cap \mathbb{D}) \times \{\alpha_n\}$. Then $\bigcap_{n\in\omega} cl_X U_n = \emptyset$. Thus, $U(X(\Lambda)) = \omega$.

Let $B = \{r_n : n \in \omega\}$ be a sequence in $\mathbb{Q}$ that converges to $\sqrt{2}$ and $C \subseteq B$ be an infinite subset. Note that $cl_\theta(C) = C \cup (\{\sqrt{2}\} \times \Lambda)$. If $\mathcal{S} \in [[B]^{\leq\kappa}]^\kappa$ for some cardinal κ, then $\{\sqrt{2}\} \times \Lambda \subseteq \bigcap_{S\in\mathcal{S}} cl_\theta S \subseteq B \cup (\{\sqrt{2}\} \times \Lambda)$. It follows that if $|\Lambda| \in \omega$, $bts_\theta(X(\Lambda)) = \omega$ and if $|\Lambda| \geq \omega$, $bts_\theta(X(\Lambda)) = \log_2(|\Lambda|)$. In particular, if $|\Lambda| = 2^{\mathfrak{c}}$, then $bts_\theta(X(\Lambda)) = \mathfrak{c} > \chi_\theta(X(\Lambda))$.

For $|\Lambda| = 2$ (i.e. $\Lambda = \{0, 1\}$), $U(X) = 3$ and the set $\mathbb{Q}$ is not θ-closed and $cl_\theta(\mathbb{Q}) = X$. If fact, the points $\{(\sqrt{2}, 0), (\sqrt{2}, 1)\}$ can not be separated by disjoint closed neighborhoods. Again, let $B = \{r_n : n \in \omega\}$ is a sequence in $\mathbb{Q}$ that converges to $\sqrt{2}$ and $C \subseteq B$ be an infinite subset. As $cl_\theta(C) = C \cup \{(\sqrt{2}, 0), (\sqrt{2}, 1)\}$, $bt_\theta(X)$ is not defined. On the other hand, it is easy to show that $fbt_\theta(X) = \omega$. [It is straightforward to show that if Y is the irrational slope space, then $U(Y) = 3$, $fbt_\theta(X) = \omega$, and $bt_\theta(X)$ is not defined.]

For each $n \in \mathbb{N}$, let Λ_n be a set with n elements and $X_n = X(\Lambda_n)$. The topological sum space $Y = \bigsqcup_{n\in\mathbb{N}} X_n$ is Hausdorff but not n-Urysohn for any $n \in \mathbb{N}$ even though $U(Y) = \omega$. However, $fbt_\theta(Y) = \omega$ and $bt_\theta(Y)$ is not defined.

Example 2. ([CH]) A Urysohn space X for which $fbt_\theta(X) = bt_\theta(X) = \omega$.

This example is like Example 2.3 in [8]. Let $\tau(\mathbb{R})$ be the usual topology on $\mathbb{R}$ and let the underlying set of X be $\mathbb{R}$ with this finer topology:

$$\tau(X) \text{ is generated by } \{U \setminus C : U \in \tau(\mathbb{R}), C \in [\mathbb{R}]^{\leq\omega_1}\}.$$

Now, we have $C \in [\mathbb{R}]^{\leq\omega_1}$ in the above definition whereas, the example in [8], it is $C \in [\mathbb{R}]^{\leq\omega}$. So, we need that $\mathfrak{c} > \omega_1$ (i.e., $\neg$**CH**).

Anyway, let's look at the example where $\kappa < \mathfrak{c}$. That is, X is $\mathbb{R}$ with this finer topology:

$$\tau(X) \text{ is generated by } \{U \setminus C : U \in \tau(\mathbb{R}), C \in [\mathbb{R}]^{\leq\kappa}\}.$$

Let $A \subseteq X$. Now $x \in cl_{\theta,X}A$ if and only if for $x \in U \in \tau(X)$, $cl_X U \cap A \neq \emptyset$ if and only if for $x \in U \setminus C$, where $U \in \tau(\mathbb{R})$ and $C \in [\mathbb{R}]^{\leq\kappa}$, $cl_X(U \setminus C) \cap A \neq \emptyset$. Note that $cl_X(U \setminus C) = cl_\mathbb{R} U$ as $X(s) = \mathbb{R}$. That is, $x \in cl_{\theta,X}A$ if and only if $x \in cl_{\theta,\mathbb{R}}A$ if and only if $x \in cl_\mathbb{R} A$.

Let A be a non-θ-closed subset of X and $x \in cl_{\theta,X}A \setminus A = cl_\mathbb{R} A \setminus A$. Let $(x_n)_{n\in\mathbb{N}} \subseteq A$ such that $(x_n)_{n\in\mathbb{N}} \to x$ in $\mathbb{R}$ and for $m \in \mathbb{N}$, let $S_m = \{x_n : n \geq m\}$. Then $cl_{\theta,X} S_m = cl_\mathbb{R} S_m = S_m \cup \{x\}$ and $\bigcap_{n\in\mathbb{N}} S_m = \{x\}$. So, $bt_\theta(X) = \omega$ and it follows, by the above fact, that $fbt_\theta(X) = bt_\theta(X) = \omega$.

The question asked in both [4, 6] is whether $|[A]_\theta| \leq |A|^{\chi_\theta(X)} \cdot U(X)$ is true for all Hausdorff spaces X, i.e., when $U(X)$ is infinite. A negative answer is presented in the next example using a space described in Example 1.

Example 3. *A Hausdorff space X with $U(X) = \chi(X) = \omega$ for which $|[A]_\theta| > |A|^{\chi(X)} \cdot U(X)$ and $|[A]_\theta| > |A|^{\chi(X)U(X)}$.*

Let Λ be a set such that $|\Lambda| > \mathfrak{c}$ and $X(\Lambda)$ be defined as in Example 1. As noted in Example 1, $X(\Lambda)$ is a first countable Hausdorff space with $U(X) = \omega$. As $cl_\theta\mathbb{Q} = X(\Lambda)$, $|cl_\theta\mathbb{Q}| = |\Lambda| > \mathfrak{c}$. However, $|\mathbb{Q}|^{\chi(X(\Lambda))} \cdot U(X(\Lambda)) = \omega^\omega \cdot \omega = 2^\omega$. Thus, $|cl_\theta\mathbb{Q}| > |\mathbb{Q}|^{\chi(X(\Lambda))} \cdot U(X(\Lambda))$. Analogously, as $|\mathbb{Q}|^{\chi(X(\Lambda))\cdot U(X(\Lambda))} = \omega^{\omega\cdot\omega} = 2^\omega$, we also have that $|cl_\theta\mathbb{Q}| > |\mathbb{Q}|^{\chi(X(\Lambda))\cdot U(X(\Lambda))}$.

4. Open problems

Here are two interesting research problems still open:

Question 1. *An unsolved problem is to characterize those Hausdorff spaces X for which $bt_\theta(X)$ and $fbt_\theta(X)$ are defined?*

Question 2. *Does there exist a Hausdorff (or Urysohn) space X for which $bt_\theta(X)$ and $fbt_\theta(X)$ are defined and $fbt_\theta(X) < bt_\theta(X)$?*

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