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A note on dimension-like functions of the type Ind defined by big bases

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Abstract. This paper introduces new dimension-like functions of the type Ind defined by big bases. Relations between them are investigated. It is shown that these dimension-like functions satisfy subspace, partition, and sum theorems.

1. Introduction and preliminaries

The origin of a notion of the classical dimension Ind goes back to L. Brouwer and was formally defined for normal spaces by E. Čech. Its transfinite extension was introduced by Yu. Smirnov (see, for example, [1, 2, 10, 14, 15]). First of all, for the purpose of its reasonable usage in the broader than normal classes of spaces different dimension-like functions appeared. V. Filippov and M. Charalambous introduced dimension Ind_0, M. Charalambous uniform dimension μ−Ind, A. Chigogidze relative dimension I, S. Iliadis base-normal dimension I, S. Bogatyĭ and G. Himšiašvili uniform large dimension (see [3–5, 7, 8, 11]). The latter is based on the G. Toulmin's idea in the case of small inductive dimension: to fix a base on a space and examine dimensions of its closed subsets being equipped with the trace of this fixed base (see [16, 17]).

Another generalized approach to the investigation of inductive dimension-like functions belongs to A. Lelek (see [12]). It allows, for example, to examine dimension Ind and dimension-like invariant Cmp from one point of view. This approach is developed in the works of M. Charalambous, V. Chatyrko, Y. Hattory and others (see, for example, [6]). The paper is devoted to the investigation of dimension-like functions of the type Ind and generalizes both approaches of G. Toulmin and A. Lelek.

We denote by ω the first infinite cardinal, by $\mathcal{O}$ the class of all ordinals, and by (+) the natural sum of Hessenberg (see [13]). We also consider two extra symbols, "-1" and "∞" such that $-1 < \alpha < \infty$ for every $\alpha \in \mathcal{O}$, $-1(+)\alpha = \alpha(+)(-1) = \alpha$ for every $\alpha \in \mathcal{O} \cup \{-1, \infty\}$, and $\infty(+)\alpha = \alpha(+)\infty = \infty$ for every $\alpha \in \mathcal{O} \cup \{\infty\}$. We recall some properties of natural sum. Let α and β be ordinals. Then,
(1) $\alpha(+)\beta = \beta(+)\alpha$,
(2) if $\alpha_1 < \alpha_2$, then $\alpha_1(+)\beta < \alpha_2(+)\beta$, and
(3) $\alpha(+)n = \alpha + n$ for $n < \omega$.

Let U be a subset of a space X. We denote by $\mathrm{Cl}_X(U)$ and $\mathrm{Bd}_X(U)$ the closure and the boundary of U in X, respectively.

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Recall that a family B of open subsets of a space X is called a *big base for* X if for every pair (F, U) of subsets of X, where F is closed, U is open, and $F \subseteq U$, there exists $V \in B$ such that $F \subseteq V \subseteq U$.

The large inductive dimension of a space X (see for example [10] and [15]), denoted by $\mathrm{Ind}(X)$, is defined as follows:
(i) $\mathrm{Ind}(X) = -1$ if and only if $X = \emptyset$.
(ii) $\mathrm{Ind}(X) \leq \alpha$, where $\alpha \in O$, if and only if there exists a big base B for X such that for every $V \in B$ we have $\mathrm{Ind}(\mathrm{Bd}_X(V)) < \alpha$.
(iii) $\mathrm{Ind}(X) = \infty$ if and only if the inequality $\mathrm{Ind}(X) \leq \alpha$ does not hold for every $\alpha \in O \cup \{-1\}$.

By a *class of big bases* we mean a class consisting of pairs (B, X), where B is a big base for the space X containing the sets $\emptyset$ and X. Let $\mathbb{B}$ be a class of big bases. A big base B of a space X is said to be a $\mathbb{B}$-*big base* if $(B, X) \in \mathbb{B}$.

In [11] base dimension-like functions of the type Ind were introduced. In Section 2 we introduce and study new dimension-like functions of the type Ind. In Sections 3, 4, and 5 we give for these dimension-like functions subspace, partition, and sum theorems. Finally, in Section 6 we give some questions concerning these functions.

2. New dimension-like functions of the type Ind

Definition 2.1. A class $\mathbb{L}$ of big bases is said to be b-*rim-hereditary* if for every $(A, X) \in \mathbb{L}$ and $U \in A$ we have

$$(\{\mathrm{Bd}_X(U) \cap V : V \in A\}, \mathrm{Bd}_X(U)) \in \mathbb{L}.$$

Definition 2.2. Let $\mathbb{L}$ be a b-rim-hereditary class of big bases. We denote by $\text{b-Ind}_{\mathbb{L}}$ the *base dimension-like function* with domain the class of all big bases and range the class $O \cup \{-1, \infty\}$ satisfying the following conditions:
(i) $\text{b-Ind}_{\mathbb{L}}(A, X) = -1$ if and only if $(A, X) \in \mathbb{L}$.
(ii) $\text{b-Ind}_{\mathbb{L}}(A, X) \leq \alpha$, where $\alpha \in O$, if and only if for every $U \in A$ we have

$$\text{b-Ind}_{\mathbb{L}}(\{\mathrm{Bd}_X(U) \cap V : V \in A\}, \mathrm{Bd}_X(U)) < \alpha.$$

(iii) $\text{b-Ind}_{\mathbb{L}}(A, X) = \infty$ if and only if the inequality $\text{b-Ind}_{\mathbb{L}}(A, X) \leq \alpha$ does not hold for every $\alpha \in O \cup \{-1\}$.

Definition 2.3. Let $\mathbb{B}$ be a class of big bases. A class $\mathbb{L}$ of big bases is said to be $\mathbb{B}$-b_0-*rim-hereditary* if for every $(A, X) \in \mathbb{L}$ there exists a $\mathbb{B}$-big base B for X such that for every $U \in B$ we have

$$(\{\mathrm{Bd}_X(U) \cap V : V \in A\}, \mathrm{Bd}_X(U)) \in \mathbb{L}.$$

Definition 2.4. Let $\mathbb{B}$ be a class of big bases and $\mathbb{L}$ a $\mathbb{B}$-b_0-rim-hereditary class of big bases. We denote by $\mathrm{b}_0\text{-Ind}_{\mathbb{L}}^{\mathbb{B}}$ the *base dimension-like function* with domain the class of all big bases and range the class $O \cup \{-1, \infty\}$ satisfying the following conditions:
(i) $\mathrm{b}_0\text{-Ind}_{\mathbb{L}}^{\mathbb{B}}(A, X) = -1$ if and only if $(A, X) \in \mathbb{L}$.
(ii) $\mathrm{b}_0\text{-Ind}_{\mathbb{L}}^{\mathbb{B}}(A, X) \leq \alpha$, where $\alpha \in O$, if and only if there exists a $\mathbb{B}$-big base B for X such that for every $U \in B$ we have

$$\mathrm{b}_0\text{-Ind}_{\mathbb{L}}^{\mathbb{B}}(\{\mathrm{Bd}_X(U) \cap V : V \in A\}, \mathrm{Bd}_X(U)) < \alpha.$$

(iii) $\mathrm{b}_0\text{-Ind}_{\mathbb{L}}^{\mathbb{B}}(A, X) = \infty$ if and only if the inequality $\mathrm{b}_0\text{-Ind}_{\mathbb{L}}^{\mathbb{B}}(A, X) \leq \alpha$ does not hold for every $\alpha \in O \cup \{-1\}$.

Remark 2.5. If $\mathbb{L} = \{(\{\emptyset\}, \emptyset)\}$, then the base dimension-like functions $\text{b-Ind}_{\mathbb{L}}$ and $\mathrm{b}_0\text{-Ind}_{\mathbb{L}}^{\mathbb{B}}$ are denoted by b-Ind and $\mathrm{b}_0\text{-Ind}^{\mathbb{B}}$, respectively. Moreover, if the class $\mathbb{B}$ consists of all pairs (B, X), where B is a big base for the space X containing the sets $\emptyset$ and X, then the base dimension-like function $\mathrm{b}_0\text{-Ind}^{\mathbb{B}}$ is denoted by b_0-Ind.

The proof of the following theorems are straightforward verifications of the inductive definitions.

Theorem 2.6. *For every big base A of a space X the following relations are true:*
(1) $\mathrm{Ind}(X) \leq \text{b-Ind}(A, X)$.
(2) $\mathrm{Ind}(X) = \text{b}_0\text{-Ind}(A, X)$.

Theorem 2.7. *Let* $\mathbb{B}$ *be a class of big bases. The following propositions are true:*
(1) For every big base A of a space X, $\mathrm{Ind}(X) \leq \text{b}_0\text{-Ind}^{\mathbb{B}}(A, X)$.
(2) For every $\mathbb{B}$*-big base A of a space X,* $\text{b}_0\text{-Ind}^{\mathbb{B}}(A, X) \leq \text{b-Ind}(A, X)$.

Example 2.8. (1) Let $\mathbb{Q}$ be the space of the rational numbers with the natural topology. It is known that $\mathrm{Ind}(\mathbb{Q}) = 0$ (see for example [10] and [15]). We consider the big base

$$A = \{\cup\{(a_n, b_n) \cap \mathbb{Q} : n = 1, 2, \ldots\} : a_n, b_n \in \mathbb{Q}\}$$

for $\mathbb{Q}$. Then, $\text{b-Ind}(A, \mathbb{Q}) \geq 1$. Indeed, for the element

$$U = \cup\{(\frac{1}{n+1}, \frac{1}{n}) \cap \mathbb{Q} : n = 1, 2, \ldots\} \in A$$

we have

$$\mathrm{Bd}_{\mathbb{Q}}(U) = \{\frac{1}{n} : n = 1, 2, \ldots\} \cup \{0\}.$$

Since $\mathrm{Bd}_{\mathbb{Q}}(U) \neq \emptyset$, we have

$$\text{b-Ind}(\{\mathrm{Bd}_{\mathbb{Q}}(U) \cap V : V \in A\}, \mathrm{Bd}_{\mathbb{Q}}(U)) \geq 0.$$

Thus, $\text{b-Ind}(A, \mathbb{Q}) \geq 1$ and, therefore, $\mathrm{Ind}(\mathbb{Q}) < \text{b-Ind}(A, \mathbb{Q})$. Also, if we consider as $\mathbb{B}$ the class of all pairs (B, X), where B is a big base for the space X containing the sets $\emptyset$ and X, then by Theorem 2.6(2) we have

$$\text{b}_0\text{-Ind}^{\mathbb{B}}(A, \mathbb{Q}) = \text{b}_0\text{-Ind}(A, \mathbb{Q}) = \mathrm{Ind}(\mathbb{Q}) = 0.$$

Thus, $\text{b}_0\text{-Ind}^{\mathbb{B}}(A, \mathbb{Q}) < \text{b-Ind}(A, \mathbb{Q})$.

(2) Let $\mathbb{B} = \{(\{\emptyset\}, \emptyset), (B, \mathbb{Q})\}$, where $\mathbb{Q}$ is the space of the rational numbers with the natural topology and

$$B = \{\cup\{(a_n, b_n) \cap \mathbb{Q} : n = 1, 2, \ldots\} : a_n, b_n \in \mathbb{R} \setminus \mathbb{Q}\}.$$

For every big base A for $\mathbb{Q}$ we have $\text{b}_0\text{-Ind}^{\mathbb{B}}(A, \mathbb{Q}) \geq 1$. Indeed, the only $\mathbb{B}$-big base for $\mathbb{Q}$ is B. For the element

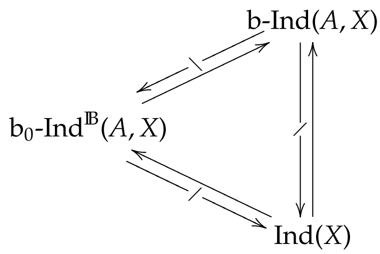
$$U = \cup\{(\frac{\pi}{n+1}, \frac{\pi}{n}) \cap \mathbb{Q} : n = 1, 2, \ldots\} \in B$$

we have $\mathrm{Bd}_{\mathbb{Q}}(U) = \{0\}$. Since $\mathrm{Bd}_{\mathbb{Q}}(U) \neq \emptyset$, we have

$$\text{b}_0\text{-Ind}^{\mathbb{B}}(\{\mathrm{Bd}_{\mathbb{Q}}(U) \cap V : V \in A\}, \mathrm{Bd}_{\mathbb{Q}}(U)) \geq 0.$$

Thus, $\text{b}_0\text{-Ind}^{\mathbb{B}}(A, \mathbb{Q}) \geq 1$ and, therefore, $\mathrm{Ind}(\mathbb{Q}) < \text{b}_0\text{-Ind}^{\mathbb{B}}(A, \mathbb{Q})$.

Remark 2.9. The relations between base dimension-like functions of the type Ind are summarized in the following diagram, where for dimension-like functions df_1, df_2 "$df_1 \to df_2$" stands for $df_1 \leq df_2$ and "$df_1 \nrightarrow df_2$" stands for $df_1 \nleq df_2$.



Definition 2.10. Let A_1 be a big base of a space X_1 and A_2 a big base of a space X_2. The pairs (A_1, X_1) and (A_2, X_2) are *homeomorphic* if there exists a homeomorphism $h : X_1 \to X_2$ such that $A_2 = \{h(U) : U \in A_1\}$.

Definition 2.11. A class $\mathbb{B}$ of big bases is said to be *topological* if for every homeomorphism $h : X \to Y$ the condition $(B, X) \in \mathbb{B}$ implies that $(\{h(U) : U \in B\}, Y) \in \mathbb{B}$.

Theorem 2.12. *Let* $\mathbb{L}$ *be a* b*-rim-hereditary topological class of big bases. If the pairs* (A_1, X_1) *and* (A_2, X_2) *are homeomorphic, then* $\text{b-Ind}_{\mathbb{L}}(A_1, X_1) = \text{b-Ind}_{\mathbb{L}}(A_2, X_2)$.

Proof. Let (A_1, X_1) and (A_2, X_2) be two homeomorphic pairs. We prove that

$$\text{b-Ind}_{\mathbb{L}}(A_1, X_1) \leq \text{b-Ind}_{\mathbb{L}}(A_2, X_2).$$

Let $h : X_1 \to X_2$ be a homeomorphism such that $A_2 = \{h(U) : U \in A_1\}$ and $\text{b-Ind}_{\mathbb{L}}(A_2, X_2) = \alpha \in O \cup \{-1, \infty\}$. The inequality is clear if $\alpha = -1$ or $\alpha = \infty$. We suppose that $\alpha \in O$ and the inequality is true for every homeomorphic pairs (A^X, X) and (A^Y, Y) with $\text{b-Ind}_{\mathbb{L}}(A^Y, Y) < \alpha$. Since $\text{b-Ind}_{\mathbb{L}}(A_2, X_2) = \alpha$, for every $U \in A_2$ we have

$$\text{b-Ind}_{\mathbb{L}}(\{\text{Bd}_{X_2}(U) \cap V : V \in A_2\}, \text{Bd}_{X_2}(U)) < \alpha.$$

We prove that

$$\text{b-Ind}_{\mathbb{L}}(\{\text{Bd}_{X_1}(h^{-1}(U)) \cap h^{-1}(V) : V \in A_2\}, \text{Bd}_{X_1}(h^{-1}(U))) < \alpha$$

for every $U \in A_2$. Indeed, let $U \in A_2$. Since

$$\text{Bd}_{X_1}(h^{-1}(U)) = \text{Cl}_{X_1}(h^{-1}(U)) \setminus h^{-1}(U) = h^{-1}(\text{Cl}_{X_2}(U) \setminus U) = h^{-1}(\text{Bd}_{X_2}(U)),$$

we have $h(\text{Bd}_{X_1}(h^{-1}(U))) = h(h^{-1}(\text{Bd}_{X_2}(U))) = \text{Bd}_{X_2}(U)$.

Moreover, for $V \in A_2$ we have

$$h(\text{Bd}_{X_1}(h^{-1}(U)) \cap h^{-1}(V)) = h(h^{-1}(\text{Bd}_{X_2}(U)) \cap h^{-1}(V)) = \text{Bd}_{X_2}(U) \cap V.$$

Thus, the pairs

$$(\{\text{Bd}_{X_2}(U) \cap V : V \in A_2\}, \text{Bd}_{X_2}(U))$$

and

$$(\{\text{Bd}_{X_1}(h^{-1}(U)) \cap h^{-1}(V) : V \in A_2\}, \text{Bd}_{X_1}(h^{-1}(U)))$$

are homeomorphic. By inductive assumption, we have

$$\text{b-Ind}_{\mathbb{L}}(\{\text{Bd}_{X_1}(h^{-1}(U)) \cap h^{-1}(V) : V \in A_2\}, \text{Bd}_{X_1}(h^{-1}(U))) \leq \text{b-Ind}_{\mathbb{L}}(\{\text{Bd}_{X_2}(U) \cap V : V \in A_2\}, \text{Bd}_{X_2}(U)) < \alpha.$$

The following theorem is proved similar to Theorem 2.12.

Theorem 2.13. *Let* $\mathbb{B}$ *be a topological class of big bases and* $\mathbb{L}$ *a* $\mathbb{B}$-b_0*-rim-hereditary topological class of big bases. If the pairs* (A_1, X_1) *and* (A_2, X_2) *are homeomorphic, then* $\text{b}_0\text{-Ind}^{\mathbb{B}}_{\mathbb{L}}(A_1, X_1) = \text{b}_0\text{-Ind}^{\mathbb{B}}_{\mathbb{L}}(A_2, X_2)$.

3. Subspace theorems

Theorem 3.1. (The first subspace theorem) *Let* $\mathbb{B}$ *be a class of bases and* A_1, A_2 *two bases of a space* X *with* $A_1 \subseteq A_2$*. Then, we have*
(1) $\text{b-Ind}(A_1, X) \leq \text{b-Ind}(A_2, X)$,
(2) $\text{b}_0\text{-Ind}^{\mathbb{B}}(A_1, X) \leq \text{b}_0\text{-Ind}^{\mathbb{B}}(A_2, X)$.

Proof. (1) Let $\text{b-Ind}(A_2, X) = \alpha \in \mathcal{O} \cup \{-1, \infty\}$. The inequality is clear if $\alpha = -1$ or $\alpha = \infty$. We suppose that $\alpha \in \mathcal{O}$ and the inequality is true if $\text{b-Ind}(A_2, X) < \alpha$. Since $\text{b-Ind}(A_2, X) = \alpha$, for every $U \in A_2$ we have

$$\text{b-Ind}(\{\text{Bd}_X(U) \cap V : V \in A_2\}, \text{Bd}_X(U)) < \alpha.$$

Also, for every $U \in A_1$ we have

$$\{\text{Bd}_X(U) \cap V : V \in A_1\} \subseteq \{\text{Bd}_X(U) \cap V : V \in A_2\}.$$

Hence, by inductive assumption, for every $U \in A_1$ we have

$$\text{b-Ind}(\{\text{Bd}_X(U) \cap V : V \in A_1\}, \text{Bd}_X(U)) \leq \text{b-Ind}(\{\text{Bd}_X(U) \cap V : V \in A_2\}, \text{Bd}_X(U)).$$

Thus, $\text{b-Ind}(A_1, X) \leq \alpha$.

Similar we can prove the relation (2).

Definition 3.2. A class $\mathbb{B}$ of big bases is said to be *closed with respect to the subspaces* if for every $(A, X) \in \mathbb{B}$ and for every closed subset X_1 of X we have $(A_1, X_1) \in \mathbb{B}$, where $A_1 = \{X_1 \cap U : U \in A\}$.

The following theorem is proved similar to Theorem 3.1.

Theorem 3.3. (The second subspace theorem) *Let $\mathbb{B}$ be a class of big bases, closed with respect to the subspaces, X_1 a closed subspace of a space X, A a big base for X, and $A_1 = \{X_1 \cap U : U \in A\}$. Then, we have*
(1) $\text{b-Ind}(A_1, X_1) \leq \text{b-Ind}(A, X)$,
(2) $\text{b}_0\text{-Ind}^{\mathbb{B}}(A_1, X_1) \leq \text{b}_0\text{-Ind}^{\mathbb{B}}(A, X)$.

4. Partition theorems

Definition 4.1. (See [9]) Let A and B be two disjoint subsets of a space X. A subset L of X is said to be *a partition between A and B* if there exist two open subsets O_1 and O_2 of X such that $A \subseteq O_1$, $B \subseteq O_2$, $O_1 \cap O_2 = \emptyset$, and $X \setminus L = O_1 \cup O_2$.

Theorem 4.2. *Let $\mathbb{L}$ be a* b*-rim-hereditary class of big bases and A a big base of a normal space X. If* $\text{b-Ind}_{\mathbb{L}}(A, X) \leq \alpha$, *where $\alpha \in \mathcal{O}$, then for every pair (F, K) of disjoint closed subsets of X there exists $U \in A$ such that the set* $\text{Bd}_X(U)$ *is a partition between F and K and* $\text{b-Ind}_{\mathbb{L}}(\{\text{Bd}_X(U) \cap V : V \in A\}, \text{Bd}_X(U)) < \alpha$.

Proof. Let $\text{b-Ind}_{\mathbb{L}}(A, X) \leq \alpha$, where $\alpha \in \mathcal{O}$, and (F, K) be a pair of disjoint closed subsets of X. Since the space X is normal, there exists an open subset W of X such that $F \subseteq W \subseteq \text{Cl}_X(W) \subseteq X \setminus K$. Therefore, there exists $U \in A$ such that

$$F \subseteq U \subseteq W \subseteq \text{Cl}_X(W) \subseteq X \setminus K$$

and

$$\text{b-Ind}_{\mathbb{L}}(\{\text{Bd}_X(U) \cap V : V \in A\}, \text{Bd}_X(U)) < \alpha.$$

We observe that the set $\text{Bd}_X(U)$ is the required partition between F and K.

The following theorem is proved similar to Theorem 4.2.

Theorem 4.3. *Let $\mathbb{B}$ be a class of big bases, $\mathbb{L}$ a $\mathbb{B}$-*b_0*-rim-hereditary class of big bases, and A a big base of a normal space X. If* $\text{b}_0\text{-Ind}^{\mathbb{B}}_{\mathbb{L}}(A, X) \leq \alpha$, *where $\alpha \in \mathcal{O}$, then for every pair (F, K) of disjoint closed subsets of X there exist a $\mathbb{B}$-base B for X and $U \in B$ such that the set* $\text{Bd}_X(U)$ *is a partition between F and K and* $\text{b}_0\text{-Ind}^{\mathbb{B}}_{\mathbb{L}}(\{\text{Bd}_X(U) \cap V : V \in A\}, \text{Bd}_X(U)) < \alpha$.

5. Sum theorems

Definition 5.1. A class $\mathbb{B}$ of big bases is said to be *closed with respect to the unions* if we have $(A_1 \cup A_2, X) \in \mathbb{B}$ for every $(A_1, X) \in \mathbb{B}$ and $(A_2, X) \in \mathbb{B}$.

Theorem 5.2. *Let $\mathbb{B}$ be a class of big bases, closed with respect to the unions and subspaces, and A_1, A_2 two big bases of a space X. Then, we have*
(1) $\text{b-Ind}(A_1 \cup A_2, X) \leq \text{b-Ind}(A_1, X)(+)\text{b-Ind}(A_2, X)$,
(2) $\text{b}_0\text{-Ind}^{\mathbb{B}}(A_1 \cup A_2, X) \leq \text{b}_0\text{-Ind}^{\mathbb{B}}(A_1, X)(+)\text{b}_0\text{-Ind}^{\mathbb{B}}(A_2, X)$.

Proof. (1) If $\text{b-Ind}(A_1, X) = \infty$ or $\text{b-Ind}(A_2, X) = \infty$, then the inequality holds. Also, if $\text{b-Ind}(A_1, X) = -1$ or $\text{b-Ind}(A_2, X) = -1$, then $X = \emptyset$ and, therefore, $\text{b-Ind}(A_1 \cup A_2, X) = -1$. We suppose that the inequality is true for every pairs (B_1, Y) and (B_2, Y) with $\text{b-Ind}(B_1, Y)(+)\text{b-Ind}(B_2, Y) < \alpha$, where α is a fixed ordinal and let (A_1, X) and (A_2, X) be two pairs with $\text{b-Ind}(A_1, X)(+)\text{b-Ind}(A_2, X) = \alpha$. We need to prove that b-Ind$(A_1 \cup A_2, X) \leq \alpha$. Let $\text{b-Ind}(A_1, X) = \alpha_1$ and $\text{b-Ind}(A_2, X) = \alpha_2$, where $\alpha_1, \alpha_2 \in O$. Since $\text{b-Ind}(A_1, X) = \alpha_1$, for every $U \in A_1$ we have

$$\text{b-Ind}(\{\text{Bd}_X(U) \cap V : V \in A_1\}, \text{Bd}_X(U)) < \alpha_1.$$

Since $\text{b-ind}(A_2, X) = \alpha_2$, for every $U \in A_2$ we have

$$\text{b-Ind}(\{\text{Bd}_X(U) \cap V : V \in A_2\}, \text{Bd}_X(U)) < \alpha_2.$$

Let $U \in A_1 \cup A_2$. Without loss of generality we can assume that $U \in A_1$. Then,

$$\text{b-Ind}(\{\text{Bd}_X(U) \cap V : V \in A_1\}, \text{Bd}_X(U)) < \alpha_1.$$

Also, by Theorem 3.3(1) we have

$$\text{b-Ind}(\{\text{Bd}_X(U) \cap V : V \in A_2\}, \text{Bd}_X(U)) \leq \text{b-Ind}(A_2, X) = \alpha_2.$$

Thus,

$$\text{b-Ind}(\{\text{Bd}_X(U) \cap V : V \in A_1\}, \text{Bd}_X(U))(+)\text{b-Ind}(\{\text{Bd}_X(U) \cap V : V \in A_2\}, \text{Bd}_X(U)) < \alpha_1 + \alpha_2 = \alpha.$$

Therefore, by inductive assumption, we have

$$\text{b-Ind}(\{\text{Bd}_X(U)\cap V : V \in A_1\cup A_2\}, \text{Bd}_X(U)) = \text{b-Ind}(\{\text{Bd}_X(U)\cap V : V \in A_1\}\cup\{\text{Bd}_X(U)\cap V : V \in A_2\}, \text{Bd}_X(U)) < \alpha.$$

This means that $\text{b-Ind}(A_1 \cup A_2, X) \leq \alpha$.
Similar we can prove the relation (2).

Definition 5.3. A class $\mathbb{B}$ of big bases is said to be *closed with respect to the free unions* if we have $(A_1 \cup A_2, X_1 \uplus X_2) \in \mathbb{B}$ for every $(A_1, X_1) \in \mathbb{B}$ and $(A_2, X_2) \in \mathbb{B}$, where the symbol $\uplus$ denotes the free union of topological spaces.

The following two theorems are straightforward verifications of the inductive definitions.

Theorem 5.4. *Let A_1 be a big base of a space X_1 and A_2 a big base of a space X_2. If* $\text{b-Ind}(A_1, X_1) \leq \alpha$ *and* $\text{b-Ind}(A_2, X_2) \leq \alpha$, *where* $\alpha \in O \cup \{-1, \infty\}$, *then*

$$\text{b-Ind}(A_1 \cup A_2, X_1 \uplus X_2) \leq \alpha.$$

Theorem 5.5. *Let $\mathbb{B}$ be a class of big bases, closed with respect to the free unions, A_1 a big base of X_1, and A_2 a big base of X_2. If* $\text{b}_0\text{-Ind}^{\mathbb{B}}(A_1, X_1) \leq \alpha$ *and* $\text{b}_0\text{-Ind}^{\mathbb{B}}(A_2, X_2) \leq \alpha$, *where* $\alpha \in O \cup \{-1, \infty\}$, *then*

$$\text{b}_0\text{-Ind}^{\mathbb{B}}(A_1 \cup A_2, X_1 \uplus X_2) \leq \alpha.$$

Theorem 5.6. *Let A be a big base of a space X, X_1 and X_2 two closed subsets of X, $A_1 = \{X_1 \cap U : U \in A\}$, and $A_2 = \{X_2 \cap U : U \in A\}$ such that $X = X_1 \cup X_2$, $\text{b-Ind}(A_1, X_1) \leq \alpha$, and $\text{b-Ind}(A_2, X_2) \leq \alpha$, where $\alpha \in \mathcal{O} \cup \{-1, \infty\}$. Then, $\text{b-Ind}(A, X) \leq \alpha$.*

Proof. Obviously, the theorem is true if $\alpha = -1$ or $\alpha = \infty$. Let $\alpha \in \mathcal{O}$. We suppose that the theorem is true for every ordinal less than α and we prove the theorem for the ordinal α. Let $\text{b-Ind}(A_1, X_1) \leq \alpha$ and $\text{b-Ind}(A_2, X_2) \leq \alpha$. We prove that $\text{b-Ind}(A, X) \leq \alpha$. Since $\text{b-Ind}(A_1, X_1) \leq \alpha$, for every $U \in A$ we have

$$\text{b-Ind}(\{\text{Bd}_{X_1}(U \cap X_1) \cap V : V \in A\}, \text{Bd}_{X_1}(U \cap X_1)) = \beta_1 < \alpha.$$

Since $\text{b-Ind}(A_2, X_2) \leq \alpha$, for every $U \in A$ we have

$$\text{b-Ind}(\{\text{Bd}_{X_2}(U \cap X_2) \cap V : V \in A\}, \text{Bd}_{X_2}(U \cap X_2)) = \beta_2 < \alpha.$$

Without loss of generality we can suppose that $\beta_1 \leq \beta_2$. Let $U \in A$. Then,

$$\text{Bd}_X(U) = \text{Bd}_X((U \cap X_1) \cup (U \cap X_2)) \subseteq \text{Bd}_{X_1}(U \cap X_1) \cup \text{Bd}_{X_2}(U \cap X_2).$$

Therefore, by Theorem 3.3(1), we have

$\text{b-Ind}(\{\text{Bd}_X(U) \cap V : V \in A\}, \text{Bd}_X(U)) = \text{b-Ind}(\{\text{Bd}_X((U \cap X_1) \cup (U \cap X_2)) \cap V : V \in A\},$
$\text{Bd}_X((U \cap X_1) \cup (U \cap X_2))) \leq$
$\text{b-Ind}(\{(\text{Bd}_{X_1}(U \cap X_1) \cup \text{Bd}_{X_2}(U \cap X_2)) \cap V : V \in A\}, \text{Bd}_{X_1}(U \cap X_1) \cup \text{Bd}_{X_2}(U \cap X_2)) =$
$\text{b-Ind}(\{\text{Bd}_{X_1}(U \cap X_1) \cap V : V \in A\} \cup \{\text{Bd}_{X_2}(U \cap X_2)) \cap V : V \in A\}, \text{Bd}_{X_1}(U \cap X_1) \cup \text{Bd}_{X_2}(U \cap X_2)).$

Also, by inductive assumption, we have

$$\text{b-Ind}(\{\text{Bd}_{X_1}(U \cap X_1) \cap V : V \in A\} \cup \{\text{Bd}_{X_2}(U \cap X_2)) \cap V : V \in A\}, \text{Bd}_{X_1}(U \cap X_1) \cup \text{Bd}_{X_2}(U \cap X_2)) \leq \beta_2 < \alpha.$$

Thus, $\text{b-Ind}(A, X) \leq \alpha$.

6. Questions

(1) Is it true the converse of theorems 4.2 and 4.3?
(2) Is it true the sum theorem (Theorems 5.6) for the base dimension-like function $\text{b}_0\text{-Ind}_{\mathbb{L}}^{\mathbb{B}}$?
(3) Is it true the following product theorem:
Let A^X be a big base of a space X and A^Y a big base of a space Y such that the family

$$A^{X \times Y} = \{U \times V : U \in A^X, V \in A^Y\}$$

is a big base for $X \times Y$. Then, $\text{b-Ind}(A^{X\times Y}, X \times Y) \leq \text{b-Ind}(A^X, X)(+)\text{b-Ind}(A^Y, Y)$.
(4) Let df be one of the following base dimension like functions $\text{b-Ind}_{\mathbb{L}}$ and $\text{b}_0\text{-Ind}_{\mathbb{L}}^{\mathbb{B}}$. For every space X we consider the class of ordinals

$$Sp_{df}(X) = \{df(A, X) : A \text{ is a big base for X}\}.$$

(a) Find the class of all spaces X such that $Sp_{df}(X) = \{0, 1, 2, \ldots, n\}$, where $n \in \omega$.
(b) Find the class of all spaces X such that $Sp_{df}(X) = \{\infty\}$.
(c) Find the class of all spaces X such that $Sp_{df}(X) = \omega$.

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