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A stochastic delay Gilpin-Ayala competition system under regime switching

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Abstract. In this paper, a stochastic delay Gilpin-Ayala competition system under regime switching is proposed and studied. We show that there is a unique global positive solution of the system for any given positive initial value. Moreover, we show that the solution is stochastically ultimately bounded under some conditions. Finally, asymptotic moment estimation of the solution with respect to a large time behavior is derived.

1. Introduction

A deterministic Gilpin-Ayala competition system with N interacting species is as follows:

$$\begin{aligned} dx_i(t) &= x_i(t)\Big[r_i(t) - \sum_{j=1}^{N} a_{ij}(t)x_j^{\alpha_{ij}}(t) - \sum_{j=1}^{N} b_{ij}(t)x_j^{\beta_{ij}}(t-\tau_{ij}) - \sum_{j=1}^{N} c_{ij}(t) \\ &\quad \times \int_{-\infty}^{0} K_{ij}(s)x_i^{\gamma_{ij}}(t+s)x_j^{\delta_{ij}}(t+s)ds\Big]dt, \quad 1 \le i \le N, \end{aligned} \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_N(t))^T$, $x_i(t)$ and $r_i(t)$ are the population density and intrinsic growth rate of the ith species at time t, respectively; $A(t) = (a_{ij}(t))_{N\times N}, B(t) = (b_{ij}(t))_{N\times N}, C(t) = (c_{ij}(t))_{N\times N}$, where $a_{ij}(t), b_{ij}(t), c_{ij}(t)$ stand for the effects of interspecific (for $i \neq j$) and intraspecific (for $i = j$) interaction at time t; $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij} \ge 0$ and they denote nonlinear measures of interspecific or intraspecific interferences; $\tau_{ij} \ge 0$ denote the maturation time of the jth species. We assume that $K_{ij} \in C((-\infty, 0]; \mathbb{R}_+^N)$ satisfying $\int_{-\infty}^{0} K_{ij}(s)ds = 1$. Furthermore, we suppose $r_i(t), a_{ij}(t), b_{ij}(t)$ and $c_{ij}(t)$ are positive continuous and bounded functions on $[0, \infty)$, $i, j = 1, 2, \dots, n$.

On the other hand, in the real world, populations systems are often affected by environmental noises. There exist a large number of papers which consider stochastic population systems, but most of them are Lotka-Volterra competition systems and just a few articles deal with Gilpin-Ayala competition systems (see e.g. [1–4]).

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In general, the intrinsic growth rate $r_i(t)$ of the ith species is estimated by an average value plus an error term. Then we can replace the rate $r_i(t)$ by an average growth rate plus a stochastic fluctuation term

$$r_i(t) \to r_i(t) + \sum_{j=1}^{N} \sigma_{ij}(t) x_j^{\theta_{ij}}(t) dB_j(t),$$

where $\sigma_{ij}(t)$ is the intensity of the white noise at time $t (i, j = 1, 2, \ldots, N)$ and $\sigma(t) = (\sigma_{ij}(t))_{N\times N}$ satisfies

$$\sigma_{ii}(t) > 0, \quad i = 1, 2, \ldots, N, \quad \sigma_{ij}(t) \geq 0, \quad i \neq j. \tag{$*$}$$

We choose $\theta_{ij}(\theta_{ij} \geq 0, i, j = 1, 2, \ldots, N)$ dependent on the parameters $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}$ such that the solution of the stochastic system has some nice properties. Then corresponding to the deterministic system (1), we have the following stochastic N-species Gilpin-Ayala competition system:

$$\begin{aligned} dx_i(t) &= x_i(t)\Big[r_i(t) - \sum_{j=1}^{N} a_{ij}(t) x_j^{\alpha_{ij}}(t) - \sum_{j=1}^{N} b_{ij}(t) x_j^{\beta_{ij}}(t-\tau_{ij}) - \sum_{j=1}^{N} c_{ij}(t) \int_{-\infty}^{0} K_{ij}(s) x_i^{\gamma_{ij}}(t+s) x_j^{\delta_{ij}}(t+s) ds\Big] dt \\ &\quad + \sum_{j=1}^{N} \sigma_{ij}(t) x_i(t) x_j^{\theta_{ij}}(t) dB_j(t), \quad i = 1, 2, \ldots, N. \end{aligned} \tag{2}$$

It is well known that in the real world there are several types of environmental noise. Besides the white noise, in this paper we consider a classical colored noise, i.e., telegraph noise. The telegraph noise can be demonstrated as a switching between two or more regimes of environment. Frequently, the switching among different environments is memoryless and the waiting time for the next switch is exponentially distributed. Consequently, we can take the random factors in the stochastic system by a continuous-time Markovian chain $\xi(t), t \geq 0$ with a finite state space $S = \{1, 2, \ldots, m\}$. Let $\xi(t)$ be generalized by $Q = (q_{ij})$, that is

$$P\Big\{\xi(t+\Delta t) = j | \xi(t) = i\Big\} = \begin{cases} q_{ij}\Delta t + o(\Delta t), & \text{if } j \neq i; \\ 1 + q_{ii}\Delta t + o(\Delta t), & \text{if } j = i, \end{cases} \tag{3}$$

where $q_{ij} \geq 0$ for $i, j = 1, 2, \ldots, m$ with $j \neq i$ and $\sum_{j=1}^{m} q_{ij} = 0$ for $i = 1, 2, \ldots, m$.

Corresponding to system (2), in this paper we consider a stochastic delay Gilpin-Ayala system under regime switching:

$$\begin{aligned} dx_i(t) &= x_i(t)\Big[r_i(\xi(t)) - \sum_{j=1}^{N} a_{ij}(\xi(t)) x_j^{\alpha_{ij}}(t) - \sum_{j=1}^{N} b_{ij}(\xi(t)) x_j^{\beta_{ij}}(t-\tau_{ij}) - \sum_{j=1}^{N} c_{ij}(\xi(t)) \\ &\quad \times \int_{-\infty}^{0} K_{ij}(s) x_i^{\gamma_{ij}}(t+s) x_j^{\delta_{ij}}(t+s) ds\Big] dt + \sum_{j=1}^{N} \sigma_{ij}(\xi(t)) x_i(t) x_j^{\theta_{ij}}(t) dB_j(t), \quad 1 \leq i \leq N, \end{aligned} \tag{4}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_N(t))^T$. The initial condition is

$$x_i(\theta) = \varphi_i(\theta) > 0, \quad -\infty < \theta \leq 0; \quad \sup_{-\infty<\theta\leq 0} |\varphi(\theta)| < \infty, \tag{5}$$

where $\varphi_i(i = 1, 2, \ldots, N)$ are continuous functions on $(-\infty, 0]$. Suppose that the Markovian chain $\gamma(\cdot)$ is independent of $B_j(t), j = 1, 2, \ldots, N$. As the standard hypothesis, we assume that $\gamma(\cdot)$ has a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ which can be obtained by solving the following linear equation $\pi Q = 0$ subject to $\sum_{i=1}^{m} \pi_i = 1$ and $\pi_i > 0,\ i \in S$.

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $D(t) = (B_1(t), B_2(t), \ldots, B_N(t))^T$ be a N-dimensional Brownian motion defined on a filtered probability space and $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_i > 0 \text{ for all } i = 1, 2, \ldots, N\}$. Moreover, define

$\hat{\nu} = \max_{i\in S} \nu(i)$, $\check{\nu} = \min_{i\in S} \nu(i)$ and let $C((-\infty, 0]; \mathbb{R}^N)$ be the collection of continuous functions from $(-\infty, 0]$ to $\mathbb{R}^N_+$.

This paper is organized in the following way: In Section 2, we show that the solution of system (4) is global and positive. In Section 3, we obtain the stochastically ultimate boundedness of the solution. Finally, the asymptotic moment behavior of the solution is analyzed.

2. Positive and global solutions

Since $x_i(t)$ in system (4) represents the population size, it should be nonnegative. For further study, we firstly give some conditions under which system (4) has a global positive solution.

Theorem 1. In addition to assumption (*), let us suppose that

$$\theta_{ii} \geq \max_j\{\theta_{ij}\}, \quad i = 1, 2, \dots, N, \tag{6}$$

$$\max_i\{\theta_{ii}\} > \max_{i,j}\{\alpha_{ij}/2, \beta_{ij}, \gamma_{ij} + \delta_{ij}\}, \tag{7}$$

then for any system parameters $A(t), B(t), C(t) \in \mathbb{R}^{N\times N}$ and any given initial value $\{x(t) : -\infty < t \leq 0\} \in C((-\infty, 0]; \mathbb{R}^N_+)$, there exists a unique solution $x(t)$ to system (4) on $t \in \mathbb{R}$ and the solution will remain in $\mathbb{R}^N_+$ with probability 1.

Proof. The proof of this theorem is motivated by the idea of Theorem 1 in [5]. Because the coefficients of system (4) are locally Lipschitz continuous, for any given initial value $\{x(t) : -\infty < t \leq 0\} \in C((-\infty, 0]; \mathbb{R}^N_+)$, there exists a unique maximal local positive solution $x(t)$ defined on $t \in [0, \tau_e)$ (see e.g. [6, 7]), where τ_e is the explosion time. To show this solution is global, we only need to verify $\tau_e = \infty$ a.s. Let k_0 be so large that every component of $\{x(t) : -\infty < t \leq 0\} \in C((-\infty, 0]; \mathbb{R}^N_+)$ is lying within the interval $[1/k_0, k_0]$. For each integer $k \geq k_0$, let us define the stopping times

$$\tau_k = \inf\{t \in [0, \tau_e) | x_i(t) \notin (1/k, k) \quad \text{for some } \ i = 1, 2, \dots, N\},$$

where throughout this paper we set $\inf \emptyset = \infty$. It is easy to see that τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$. If we can prove $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $x(t) \in \mathbb{R}^N_+$ a.s. for all $t \geq 0$. In other words, to finish the proof, we need to show $\tau_\infty = \infty$ a.s. or for all $T > 0$, we should obtain $P(\tau_k \leq T) \to 0$ as $k \to \infty$. To show this assertion, define

$$V(x) = \sum_{i=1}^N (x_i^\gamma - 1 - \gamma \ln x_i),$$

where $0 < \gamma < 1$. Let $k \geq k_0$ and $T > 0$ be arbitrary. For $0 \leq t \leq \tau_k \wedge T$, applying Itô's formula to $V(x(t))$ yields

$$dV(x(t)) = LV(x(t))dt + \gamma \sum_{i,j=1}^N \sigma_{ij}(\xi(t))(x_i^\gamma(t) - 1)x_j^{\theta_{ij}}(t)dB_j(t),$$

where

$$\begin{aligned} LV(x(t)) \;\; &= \;\; \gamma \sum_{i=1}^N (x_i^\gamma(t) - 1)\Big[r_i(\xi(t)) - \sum_{j=1}^N a_{ij}(\xi(t))x_j^{\alpha_{ij}}(t) - \sum_{j=1}^N b_{ij}(\xi(t))x_j^{\beta_{ij}}(t - \tau_{ij}) \\ &\quad - \sum_{j=1}^N c_{ij}(\xi(t)) \int_{-\infty}^0 K_{ij}(s)x_i^{\gamma_{ij}}(t+s)x_j^{\delta_{ij}}(t+s)ds\Big] + \frac{\gamma}{2}\sum_{i,j=1}^N \sigma_{ij}^2(\xi(t))[1 - (1-\gamma)x_i^\gamma(t)]x_j^{2\theta_{ij}}(t). \end{aligned}$$

From the elementary inequality $2xy \leq x^2 + y^2$, it follows that

$$-\gamma \sum_{i,j=1}^N a_{ij}(\xi(t))(x_i^\gamma(t) - 1)x_j^{\alpha_{ij}}(t) \leq \gamma \sum_{i,j=1}^N \hat{a}_{ij}x_i^\gamma(t)x_j^{\alpha_{ij}}(t) + \gamma \sum_{i,j=1}^N \hat{a}_{ij}x_j^{\alpha_{ij}}(t),$$

$$-\gamma\sum_{i,j=1}^{N} b_{ij}(\xi(t))(x_i^{\gamma}(t)-1)x_j^{\beta_{ij}}(t-\tau_{ij}) \le \frac{\gamma N}{4}\sum_{i,j=1}^{N}\hat{b}_{ij}^2(x_i^{\gamma}(t)-1)^2+\frac{\gamma}{N}\sum_{i,j=1}^{N}x_j^{2\beta_{ij}}(t-\tau_{ij}),$$

$$\begin{aligned}
&-\gamma\sum_{i,j=1}^{N} c_{ij}(\xi(t))(x_i^{\gamma}(t)-1)\int_{-\infty}^{0}K_{ij}(s)x_i^{\gamma_{ij}}(t+s)x_j^{\delta_{ij}}(t+s)ds\\
&\le \gamma\sum_{i,j=1}^{N}\int_{-\infty}^{0}K_{ij}(s)\Big[\frac{N}{4}\hat{c}_{ij}^2(x_i^{\gamma}(t)-1)^2+\frac{1}{N}x_i^{2\gamma_{ij}}(t+s)x_j^{2\delta_{ij}}(t+s)\Big]ds\\
&= \frac{\gamma N}{4}\sum_{i,j=1}^{N}\hat{c}_{ij}^2(x_i^{\gamma}(t)-1)^2+\frac{\gamma}{N}\sum_{i,j=1}^{N}\int_{-\infty}^{0}K_{ij}(s)x_i^{2\gamma_{ij}}(t+s)x_j^{2\delta_{ij}}(t+s)ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
dV(x(t)) \le\ & \gamma\Big[\sum_{i=1}^{N}\hat{r}_i(x_i^{\gamma}(t)-1)+\sum_{i,j=1}^{N}\hat{a}_{ij}x_i^{\gamma}(t)x_j^{\alpha_{ij}}(t)+\sum_{i,j=1}^{N}\hat{a}_{ij}x_j^{\alpha_{ij}}(t)\\
&+\frac{N}{4}\sum_{i,j=1}^{N}(\hat{b}_{ij}^2+\hat{c}_{ij}^2)(x_i^{\gamma}(t)-1)^2+\frac{1}{N}\sum_{i,j=1}^{N}x_j^{2\beta_{ij}}(t-\tau_{ij})\\
&+\frac{1}{N}\sum_{i,j=1}^{N}\int_{-\infty}^{0}K_{ij}(s)x_i^{2\gamma_{ij}}(t+s)x_j^{2\delta_{ij}}(t+s)ds+\frac{1}{2}\sum_{i,j=1}^{N}\hat{\sigma}_{ij}^2x_j^{2\theta_{ij}}(t)\\
&-\frac{1-\gamma}{2}\sum_{i,j=1}^{N}\check{\sigma}_{ij}^2x_i^{\gamma}(t)x_j^{2\theta_{ij}}(t)\Big]dt+\gamma\sum_{i,j=1}^{N}\hat{\sigma}_{ij}(x_i^{\gamma}(t)-1)x_j^{\theta_{ij}}(t)dB_j(t).
\end{aligned}$$

Then define

$$V_1(x(t))=\frac{1}{N}\sum_{i,j=1}^{N}\int_{t-\tau_{ij}}^{t}x_j^{2\beta_{ij}}(s)ds,$$

$$V_2(x(t))=\frac{1}{N}\sum_{i,j=1}^{N}\int_{-\infty}^{0}K_{ij}(s)\int_{t+s}^{t}x_i^{2\gamma_{ij}}(u)x_j^{2\delta_{ij}}(u)duds$$

and compute that

$$dV_1(x(t))=\frac{1}{N}\sum_{i,j=1}^{N}\Big[x_j^{2\beta_{ij}}(t)-x_j^{2\beta_{ij}}(t-\tau_{ij})\Big],$$

$$\begin{aligned}
dV_2(x(t)) &= \frac{1}{N}\sum_{i,j=1}^{N}\int_{-\infty}^{0}K_{ij}(s)\Big[x_i^{2\gamma_{ij}}(t)x_j^{2\delta_{ij}}(t)-x_i^{2\gamma_{ij}}(t+s)x_j^{2\delta_{ij}}(t+s)\Big]ds\\
&= \frac{1}{N}\sum_{i,j=1}^{N}\Big[x_i^{2\gamma_{ij}}(t)x_j^{2\delta_{ij}}(t)-\int_{-\infty}^{0}K_{ij}(s)x_i^{2\gamma_{ij}}(t+s)x_j^{2\delta_{ij}}(t+s)ds\Big].
\end{aligned}$$

Hence,

$$d[V(x(t))+V_1(x(t))+V_2(x(t))]\le F(x(t))dt+\gamma\sum_{i,j=1}^{N}\hat{\sigma}_{ij}(x_i^{\gamma}(t)-1)x_j^{\theta_{ij}}(t)dB_j(t), \tag{8}$$

where

$$\begin{aligned}F(x(t)) &= \gamma\Big[\sum_{i=1}^{N}\hat{r}_i(x_i^{\gamma}(t)-1)+\sum_{i,j=1}^{N}\hat{a}_{ij}x_i^{\gamma}(t)x_j^{\alpha_{ij}}(t)+\sum_{i,j=1}^{N}\hat{a}_{ij}x_j^{\alpha_{ij}}(t)\\ &+\frac{N}{4}\sum_{i,j=1}^{N}(\hat{b}_{ij}^2+\hat{c}_{ij}^2)(x_i^{\gamma}(t)-1)^2+\frac{1}{N}\sum_{i,j=1}^{N}\left[x_j^{2\beta_{ij}}(t)+x_i^{2\gamma_{ij}}(t)x_j^{2\delta_{ij}}(t)\right]\\ &+\frac{1}{2}\sum_{i,j=1}^{N}\hat{\sigma}_{ij}^2x_j^{2\theta_{ij}}(t)-\frac{1-\gamma}{2}\sum_{i,j=1}^{N}\check{\sigma}_{ii}^2x_i^{\gamma+2\theta_{ii}}(t)\Big].\end{aligned} \tag{9}$$

However, $F(x(t))$ will be bounded if the extent of the term with negative coefficient is greater than any degree of the terms with positive coefficients. As conditions (6) and (7) are satisfied, for $\gamma < 2\max_i\{\theta_{ii}\}\wedge 1$, there is a positive constant K such that $F(x(t)) \le K$. Consequently, we have

$$d[V(x(t))+V_1(x(t))+V_2(x(t))] \le Kdt+\gamma\sum_{i,j=1}^{N}\hat{\sigma}_{ij}(x_i^{\gamma}(t)-1)x_j^{\theta_{ij}}(t)dB_j(t).$$

Integrating the above inequality from 0 to $\tau_k \wedge T$ and then taking expectation leads to

$$\begin{aligned}EV(x(\tau_k\wedge T)) &\le EV(x(\tau_k\wedge T))+EV_1(x(\tau_k\wedge T))+EV_2(x(\tau_k\wedge T))\\ &\le V(x(0))+V_1(x(0))+V_2(x(0))+KE(\tau_k\wedge T)\\ &\le V(x(0))+V_1(x(0))+V_2(x(0))+KT.\end{aligned}$$

Thus, for each $\omega \in \{\tau_k \le T\}$, there exists some i such that $x_i(\tau_k,\omega) \notin (1/k,k)$. So

$$V(x(\tau_k)) \ge x_i^{\gamma}(\tau_k)-1-\gamma\ln x_i(\tau_k) = (1/k^{\gamma}-1+\gamma\ln k)\wedge(k^{\gamma}-1-\gamma\ln k)$$

and then

$$\begin{aligned}\infty &> V(x(0))+V_1(x(0))+V_2(x(0))+KT \ge EV(x(\tau_k\wedge T))\\ &= P(\tau_k\le T)V(x(\tau_k))+P(\tau_k>T)V(x(T)) \ge P(\tau_k\le T)V(x(\tau_k))\\ &\ge P(\tau_k\le T)[(1/k^{\gamma}-1+\gamma\ln k)\wedge(k^{\gamma}-1-\gamma\ln k)].\end{aligned}$$

Since $(1/k^{\gamma}-1+\gamma\ln k)\wedge(k^{\gamma}-1-\gamma\ln k)$ prones to ∞ as $k\to\infty$, one can see that $\lim_{k\to\infty}P(\tau_k\le T)=0$ and hence $P(\tau_\infty \le T)=0$. Because $T>0$ is arbitrary, we conclude that

$$P(\tau_\infty<\infty)=0 \quad\text{and}\quad P(\tau_\infty=\infty)=1,$$

which finishes the proof of Theorem 1.

3. Stochastically ultimate boundedness

In this section, we will investigate how the solutions vary in $\mathbb{R}_+^N$. Firstly, we give the definition of stochastic ultimate boundedness. Then, we prove Lemma 1. Finally, we prove the solution of system (4) is stochastically ultimately bounded.

Definition 1. The solution of system (4) is said to be stochastically ultimately bounded if for any $\varepsilon \in (0,1)$, there exists a positive constant $H = H(\varepsilon)$ such that for any initial value $\{x(t) : -\infty < t \le 0\} \in C((-\infty,0];\mathbb{R}_+^N)$ satisfying (5), the solution $x(t)$ of system (4) satisfies

$$\limsup_{t\to\infty}\mathbb{P}\{|x(t)|\le H\} \ge 1-\varepsilon.$$

Lemma 1. Let us suppose that conditions of Theorem 1 hold and $\mu \in (0, 2\max_i\{\theta_{ii}\} \wedge 1)$. Moreover, we assume that there is a constant $\lambda > 0$ such that

$$\int_{-\infty}^{0} K_{ij}(s)e^{-\lambda s}ds = \bar{K}_{ij} < \infty, \quad i, j = 1, 2, \ldots, N, \tag{10}$$

then there exists a positive constant $C = C(\theta)$ which is independent of the initial value $\{x(t) : -\infty < t \le 0\} \in C((-\infty, 0]; \mathbb{R}^N_+)$ satisfying (5) such that the solution $x(t)$ of system (4) has the property

$$\limsup_{t\to\infty} E|x(t)|^{\mu} \le C.$$

Proof. The proof of this lemma is motivated by the idea of Lemma 1 in [5]. For $x \in \mathbb{R}^N_+$ and $0 < \mu < 1$, define

$$\bar{V}(x) = \sum_{i=1}^{N} x_i^{\mu}.$$

Using Itô's formula to $e^{\lambda t}\bar{V}(x(t))$ results in

$$d[e^{\lambda t}\bar{V}(x(t))] = e^{\lambda t}d\bar{V}(x(t)) + \lambda e^{\lambda t}\bar{V}(x(t))dt = L\bar{V}(x(t))dt + e^{\lambda t}\sum_{i,j=1}^{N} \mu\sigma_{ij}(\xi(t))x_i^{\mu}(t)x_j^{\theta_{ij}}(t)dB_j(t),$$

where

$$\begin{aligned} L\bar{V}(x(t)) &= e^{\lambda t}\sum_{i=1}^{N}\Big[(\lambda + \mu r_i(\xi(t)))x_i^{\mu}(t) - \mu x_i^{\mu}(t)\sum_{j=1}^{N}\Big(a_{ij}(\xi(t))x_j^{\alpha_{ij}}(t) + b_{ij}(\xi(t))x_j^{\beta_{ij}}(t-\tau_{ij}) + c_{ij}(\xi(t)) \\ &\quad \times \int_{-\infty}^{0} K_{ij}(s)x_i^{\gamma_{ij}}(t+s)x_j^{\delta_{ij}}(t+s)ds\Big) - \frac{\mu(1-\mu)}{2}\sum_{j=1}^{N}\sigma_{ij}^2(\xi(t))x_i^{\mu}(t)x_j^{2\theta_{ij}}(t)\Big]. \end{aligned}$$

Applying the elementary inequality $2xy \le x^2 + y^2$ and condition (10) imply

$$-e^{\lambda t}\sum_{i,j=1}^{N}\mu a_{ij}(\xi(t))x_i^{\mu}(t)x_j^{\alpha_{ij}}(t) \le e^{\lambda t}\sum_{i,j=1}^{N}\mu\hat{a}_{ij}x_i^{\mu}(t)x_j^{\alpha_{ij}}(t),$$

$$-e^{\lambda t}\sum_{i,j=1}^{N}\mu b_{ij}(\xi(t))x_i^{\mu}(t)x_j^{\beta_{ij}}(t-\tau_{ij}) \le e^{\lambda t}\frac{\mu^2 N}{4}\sum_{i,j=1}^{N}\hat{b}_{ij}^2x_i^{2\mu}(t) + \frac{e^{\lambda t}}{N}\sum_{i,j=1}^{N}x_j^{2\beta_{ij}}(t-\tau_{ij}),$$

$$\begin{aligned} &-e^{\lambda t}\sum_{i,j=1}^{N}\mu c_{ij}(\xi(t))x_i^{\mu}(t)\int_{-\infty}^{0}K_{ij}(s)x_i^{\gamma_{ij}}(t+s)x_j^{\delta_{ij}}(t+s)ds \\ &\le e^{\lambda t}\sum_{i,j=1}^{N}\int_{-\infty}^{0}K_{ij}(s)\Big[\frac{N}{4}\mu^2\hat{c}_{ij}^2x_i^{2\mu}(t) + \frac{1}{N}x_i^{2\gamma_{ij}}(t+s)x_j^{2\delta_{ij}}(t+s)\Big]ds \\ &= e^{\lambda t}\frac{\mu^2 N}{4}\sum_{i,j=1}^{N}\hat{c}_{ij}^2x_i^{2\mu}(t) + \frac{e^{\lambda t}}{N}\sum_{i,j=1}^{N}\int_{-\infty}^{0}K_{ij}(s)x_i^{2\gamma_{ij}}(t+s)x_j^{2\delta_{ij}}(t+s)ds. \end{aligned}$$

Consequently,

$$\begin{aligned} d\bar{V}(x(t)) \quad &\leq \quad e^{\lambda t}\sum_{i=1}^{N}\Big[(\lambda+\mu\hat{r}_i)x_i^{\mu}(t)+\mu\sum_{j=1}^{N}\hat{a}_{ij}x_i^{\mu}(t)x_j^{\alpha_{ij}}(t)+\frac{\mu^2 N}{4}\sum_{j=1}^{N}(\hat{b}_{ij}^2+\hat{c}_{ij}^2)x_i^{2\mu}(t)\Big] \\ &+\frac{1}{N}\sum_{j=1}^{N}\Big(x_j^{2\beta_{ij}}(t-\tau_{ij})+\int_{-\infty}^{0}K_{ij}(s)x_i^{2\gamma_{ij}}(t+s)x_j^{2\delta_{ij}}(t+s)ds\Big) \\ &-\frac{\mu(1-\mu)}{2}\breve{\sigma}_{ii}^2 x_i^{\mu+2\theta_{ii}}(t)\Big]dt+e^{\lambda t}\mu\sum_{i,j=1}^{N}\hat{\sigma}_{ij}x_i^{\mu}(t)x_j^{\theta_{ij}}(t)dB_j(t). \end{aligned}$$

Then define

$$\bar{V}_1(x(t))=\frac{1}{N}\sum_{i,j=1}^{N}\int_{t-\tau_{ij}}^{t}e^{\lambda(s+\tau_{ij})}x_j^{2\beta_{ij}}(s)ds,$$

$$\bar{V}_2(x(t))=\frac{1}{N}\sum_{i,j=1}^{N}\int_{-\infty}^{0}K_{ij}(s)\int_{t+s}^{t}e^{\lambda(u-s)}x_i^{2\gamma_{ij}}(u)x_j^{2\delta_{ij}}(u)duds,$$

then

$$d\bar{V}_1(x(t))=\frac{1}{N}\sum_{i,j=1}^{N}\Big[e^{\lambda(t+\tau_{ij})}x_j^{2\beta_{ij}}(t)-e^{\lambda t}x_j^{2\beta_{ij}}(t-\tau_{ij})\Big],$$

$$d\bar{V}_2(x(t))=\frac{1}{N}\sum_{i,j=1}^{N}\int_{-\infty}^{0}K_{ij}(s)\Big[e^{\lambda(t-s)}x_i^{2\gamma_{ij}}(t)x_j^{2\delta_{ij}}(t)-e^{\lambda t}x_i^{2\gamma_{ij}}(t+s)x_j^{2\delta_{ij}}(t+s)\Big]ds.$$

By (10), we obtain

$$\begin{aligned} \frac{1}{N}\sum_{i,j=1}^{N}\int_{-\infty}^{0}K_{ij}(s)e^{\lambda(t-s)}x_i^{2\gamma_{ij}}(t)x_j^{2\delta_{ij}}(t)ds \quad &= \quad \frac{e^{\lambda t}}{N}\sum_{i,j=1}^{N}x_i^{2\gamma_{ij}}(t)x_j^{2\delta_{ij}}(t)\int_{-\infty}^{0}K_{ij}(s)e^{-\lambda s}ds \\ &= \quad \frac{e^{\lambda t}}{N}\sum_{i,j=1}^{N}\bar{K}_{ij}x_i^{2\gamma_{ij}}(t)x_j^{2\delta_{ij}}(t), \end{aligned}$$

it shows that

$$d\bar{V}_2(x(t))=\frac{e^{\lambda t}}{N}\sum_{i,j=1}^{N}\Big(\bar{K}_{ij}x_i^{2\gamma_{ij}}(t)x_j^{2\delta_{ij}}(t)-\int_{-\infty}^{0}K_{ij}(s)x_i^{2\gamma_{ij}}(t+s)x_j^{2\delta_{ij}}(t+s)ds\Big).$$

Then

$$d[e^{\lambda t}\bar{V}(x(t))+\bar{V}_1(x(t))+\bar{V}_2(x(t))]\leq e^{\lambda t}\Big(\bar{F}(x(t))dt+\sum_{i,j=1}^{N}\mu\hat{\sigma}_{ij}x_i^{\mu}(t)x_j^{\theta_{ij}}(t)dB_j(t)\Big),$$

where

$$\begin{aligned} \bar{F}(x(t)) \quad &= \quad \sum_{i=1}^{N}\Big\{(\mu\hat{r}_i+\lambda)x_i^{\mu}(t)+\sum_{j=1}^{N}\Big[\mu\hat{a}_{ij}x_i^{\mu}(t)x_j^{\alpha_{ij}}(t)+\frac{\mu^2 N}{4}(\hat{b}_{ij}^2+\hat{c}_{ij}^2)x_i^{2\mu}(t) \\ &+\frac{1}{N}\Big(e^{\lambda\tau}x_j^{2\beta_{ij}}(t)+\bar{K}x_i^{2\gamma_{ij}}(t)x_j^{2\delta_{ij}}(t)\Big)\Big]-\frac{\mu(1-\mu)}{2}\breve{\sigma}_{ii}^2 x_i^{\mu+2\theta_{ii}}(t)\Big\} \end{aligned}$$

and $\tau = \max_{i,j}\{\tau_{ij}\}, \bar{K} = \max_{i,j}\{\bar{K}_{ij}\}$. Because condition (7) is satisfied, for $\mu \in (0, 2\max_i\{\theta_{ii}\} \wedge 1)$ there is a positive constant K such that $\bar{F}(x(t)) \leq K$. Therefore,

$$d[e^{\lambda t}\bar{V}(x(t)) + \bar{V}_1(x(t)) + \bar{V}_2(x(t))] \leq e^{\lambda t}\Big(Kdt + \sum_{i,j=1}^{N} \mu\hat{\sigma}_{ij}x_i^{\mu}(t)x_j^{\theta_{ij}}(t)dB_j(t)\Big).$$

Integrating the above inequality from 0 to t and then taking expectation on both sides result in

$$\begin{aligned} e^{\lambda t}E\bar{V}(x(t)) &\leq e^{\lambda t}E\bar{V}(x(t)) + E\bar{V}_1(x(t)) + E\bar{V}_2(x(t)) \\ &\leq \bar{V}(x(0)) + \bar{V}_1(x(0)) + \bar{V}_2(x(0)) + \int_0^t Ke^{\lambda s}ds \\ &\leq \bar{V}(x(0)) + \bar{V}_1(x(0)) + \bar{V}_2(x(0)) + K\lambda^{-1}e^{\lambda t} \end{aligned}$$

and hence,

$$\limsup_{t\to\infty} E\bar{V}(x(t)) \leq K\lambda^{-1}.$$

Since

$$|x(t)|^{\mu} = \Big|\sum_{i=1}^{N} x_i^2(t)\Big|^{\frac{\mu}{2}} \leq N^{\frac{\mu}{2}} \max_{1\leq i\leq N} x_i^{\mu}(t) \leq N^{\frac{\mu}{2}} \sum_{i=1}^{N} x_i^{\mu}(t) = N^{\frac{\mu}{2}}\bar{V}(x(t)),$$

we get

$$\limsup_{t\to\infty} E|x(t)|^{\mu} \leq \limsup_{t\to\infty} E(N^{\frac{\mu}{2}}\bar{V}(x(t))) \leq N^{\frac{\mu}{2}}K\lambda^{-1}.$$

Thus, by taking $C = N^{\frac{\mu}{2}}K\lambda^{-1}$, we obtain the desired assertion of this lemma.

Theorem 2. Assume that conditions of Lemma 1 are fulfilled, then the solution of system (4) is stochastically ultimately bounded.

Proof. The proof of this theorem is motivated by the idea of Theorem 2 in [5]. By Lemma 1, we can see that if we take some fixed $\theta \in (0, 2\min_i\{\theta_{ii}\} \wedge 1)$, then there is $C > 0$ such that

$$\limsup_{t\to\infty} E|x(t)|^{\theta} \leq C.$$

It follows from Markov's inequality that

$$\mathbb{P}\{|x(t)| > H\} \leq \frac{E|x(t)|^{\theta}}{H^{\theta}}.$$

For any $\varepsilon > 0$, set $H = \Big(\frac{C}{\varepsilon}\Big)^{1/\theta}$. Therefore,

$$\limsup_{t\to\infty} \mathbb{P}\{|x(t)| > H\} \leq \varepsilon,$$

which implies

$$\limsup_{t\to\infty} \mathbb{P}\{|x(t)| \leq H\} \geq 1 - \varepsilon.$$

Thus we complete the proof.

4. Asymptotic moment estimation

As no explicit solution of system (4) has yet been found, it is reasonable to investigate an asymptotic moment estimation of the solution. The following theorem shows that the average in time of the pth moment of the solution will be bounded.

Theorem 3. Suppose that conditions of Theorem 1 are fulfilled and $p \in (0, 2\max_i\{\theta_{ii}\})$, then there exists a positive constant C, which is independent of the initial value $\{x(t) : -\infty < t \le 0\}$ satisfying (5) such that the solution $x(t)$ of system (4) satisfies

$$\limsup_{t\to\infty} \frac{1}{t} E \int_0^t \sum_{i=1}^N x_i^p(s)ds \le C.$$

Proof. The proof of this theorem is motivated by the idea of Theorem 3 in [5]. We start from the relation (8), where $F(x(t))$ is given by (9),

$$d[V(x(t)) + V_1(x(t)) + V_2(x(t))] \le F(x(t))dt + \gamma \sum_{i,j=1}^N \hat{\sigma}_{ij}(x_i^\gamma(t) - 1)x_j^{\theta_{ij}}(t)dB_j(t). \tag{11}$$

Then we take

$$F_1(x(t)) = F(x(t)) + \sum_{i=1}^N x_i^p(t).$$

Because conditions (6) and (7) hold and $p \in (0, 2\max_i\{\theta_{ii}\})$, by the same way of Theorem 1, we can deduce that $F_1(x)$ is bounded. So there is a positive constant C such that $F_1(x) \le C$. In other words, we have

$$F(x(t)) \le C - \sum_{i=1}^N x_i^p(t).$$

If we employ this estimation, (11) becomes

$$d[V(x(t)) + V_1(x(t)) + V_2(x(t))] \le \Big[C - \sum_{i=1}^N x_i^p(t)\Big]dt + \gamma \sum_{i,j=1}^N \hat{\sigma}_{ij}(x_i^\gamma(t) - 1)x_j^{\theta_{ij}}(t)dB_j(t).$$

Integrating the above inequality from 0 to $\tau_k \wedge t$ and then taking expectation lead to

$$\begin{aligned} EV(x(\tau_k \wedge t)) \quad &+ \quad EV_1(x(\tau_k \wedge t)) + EV_2(x(\tau_k \wedge t)) \\ &\le \quad V(x(0)) + V_1(x(0)) + V_2(x(0)) + CE(\tau_k \wedge t) - E\int_0^{\tau_k\wedge t} \sum_{i=1}^N x_i^p(s)ds. \end{aligned}$$

By letting $k \to \infty$, we have

$$E\int_0^t \sum_{i=1}^N x_i^p(s)ds \le V(x(0)) + V_1(x(0)) + V_2(x(0)) + Ct,$$

which shows the required statement.

Remark. M. Vasilova and M. Jovanović [5] studied the following stochastic Gilpin-Ayala competition model with infinite delay:

$$\begin{aligned} dx_i(t) \quad &= \quad x_i(t)\Big[r_i - \sum_{j=1}^d a_{ij}x_j^{\alpha_{ij}}(t) - \sum_{j=1}^d b_{ij}x_j^{\beta_{ij}}(t - \tau_{ij}) - \sum_{j=1}^d c_{ij}\int_{-\infty}^0 K_{ij}(s)x_i^{\gamma_{ij}}(t+s)x_j^{\delta_{ij}}(t+s)ds\Big]dt \\ &\quad + \sum_{j=1}^d \sigma_{ij}x_i(t)x_j^{\theta_{ij}}(t)dw_j(t), \quad i = 1, 2, \dots, d \end{aligned}$$

with initial value

$$x_i(\theta) = \varphi_i(\theta) > 0, \quad -\infty < \theta \le 0; \quad \sup_{-\infty<\theta\le 0} |\varphi(\theta)| < \infty,$$

where $\varphi_i, i = 1, 2, \ldots, d$ are continuous functions on $(-\infty, 0]$. Under some conditions, the authors obtained some asymptotic properties of the positive solutions to the above system. It is easy to see that our model generalizes the above system into more complicated and realistic case.

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