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On some results of metrics induced by a fuzzy ultrametric

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Abstract. In this paper, we introduce a formula to induce metrics from a fuzzy ultrametric, and consider consistency of the metrics. We also show that completeness and precompactness of a fuzzy ultrametric space are, respectively, identical with completeness and totally bounded property of the metric spaces induced by the fuzzy ultrametric. Furthermore, we explore three types of Hausdorff fuzzy metrics in a fuzzy metric space, and prove that they are identical if the fuzzy metric space is a fuzzy ultrametric space. At last, we discuss consistency between the Hausdorff fuzzy metric in a fuzzy ultrametric space and the Hausdorff metric in the metric space induced by this fuzzy ultrametric.

1. Introduction

Fuzzy metric space, which was first constructed by Kramosil and Michalek in 1975 [9], is one of the important notions in the theory of fuzzy topology. To make the topology induced by a fuzzy metric space to be Hausdorff, George and Veeramani [3] modified the notion given by Kramosil and Michalek and presented a new notion with the help of continuous t-norms. The new version of fuzzy metric space is more restrictive, but it determines the class of spaces that are tightly connected with the class of metrizable topological spaces. So it is interesting to study the new version of a fuzzy metric space.

Our basic reference for general topology is [2].

Following [3], a continuous t-norm is a binary operation $* : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions:

(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, and $a, b, c, d \in [0,1]$.

Clearly, $a * b = \min\{a, b\}$ and $a * b = a \cdot b$ are two common examples of t-norms.

According to [3], a fuzzy metric space is 3-tuple $(X, M, *)$ such that X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy subset of $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t \in (0, \infty)$:

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(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(v) the function $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

In [3], George and Veeramani proved that every fuzzy metric M generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \frac{1}{n}, \frac{1}{n}) | n \in \mathbb{N}, x \in X\}$, where $B_M(x, \frac{1}{n}, \frac{1}{n}) = \{y \in X | M(x, y, \frac{1}{n}) > 1 - \frac{1}{n}\}$. They also proved that (X, τ_M) is Hausdorff and first countable.

Fuzzy ultrametric space, which is a special case of a fuzzy metric space, was introduced by Savchenko and Zarichnyi in [12]. A fuzzy ultrametric space is 3-tuple $(X, M, *)$ such that X is an arbitrary set, $* = \min$ and M is a fuzzy subset of $X \times X \times (0, \infty)$ satisfying conditions (i), (ii), (iii), (v) from the definition of a fuzzy metric space and the following condition for all $x, y, z \in X$ and $s, t \in (0, \infty)$:
(iv′) $M(x, y, t) * M(y, z, s) \leq M(x, z, \max\{t, s\})$.

Let us recall (see [6]) that a fuzzy metric space $(X, M, *)$ is said to be strong if it satisfies the following condition for each $x, y, z \in X$ and each $t > 0$:
(iv″) $M(x, y, t) * M(y, z, t) \leq M(x, z, t)$.

It is remarked in [10] that if $* = \min$, then condition (iv′) is equivalent to condition (iv″). Obviously, any fuzzy ultrametric space is just the same as a strong fuzzy metric space for the continuous t-norm min.

A simple but useful fact is that for every $x, y \in X$, the function $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is nondecreasing [12].

The topological space induced by a fuzzy metric is metrizable [7]. However, no authors gave a method to generate a metric from a fuzzy metric. Here, we introduce a formula to induce metrics from a fuzzy ultrametric, and consider consistency of the metrics. Then we prove that completeness and precompactness of a fuzzy ultrametric space are, respectively, identical with completeness and totally bounded property of the metric spaces induced by the fuzzy ultrametric. Furthermore, corresponding to the three type of Hausdorff metrics in a metric space [1], we explore three types of Hausdorff fuzzy metrics in a fuzzy metric space, and prove that they are identical if the fuzzy metric space is a fuzzy ultrametric space. At last, we discuss consistency between the Hausdorff fuzzy metric in a fuzzy ultrametric space and the Hausdorff metric in the metric space induced by this fuzzy ultrametric.

2. Metrics induced by a fuzzy ultrametric

Let $(X, M, *)$ be a fuzzy ultrametric space and $t_0 \in (0, \infty)$. We define a mapping $d^X_{t_0} : X \times X \to [0, \infty]$ by

$$d^X_{t_0}(x, y) = \frac{t_0}{M(x, y, t_0)} - t_0.$$

Then we have

Theorem 2.1. *$d^X_{t_0}$ is a metric in X.*

Proof. Let $x, y, z \in X$.

(1) Since $0 < M(x, y, t) \leq 1$ for all $t \in (0, \infty)$, we have $d^X_{t_0}(x, y) \geq 0$. If $d^X_{t_0}(x, y) = 0$, then it follows from $\frac{t_0}{M(x,y,t_0)} - t_0 = 0$ that $M(x, y, t_0) = 1$. Hence $x = y$. Conversely, if $x = y$, then $d^X_{t_0}(x, y) = \frac{t_0}{M(x,y,t_0)} - t_0 = 0$.

(2) Obviously, $d^X_{t_0}(x, y) = d^X_{t_0}(y, x)$.

(3) We are going to prove $d^X_{t_0}(x, y) + d^X_{t_0}(y, z) \geq d^X_{t_0}(x, z)$.
Since $M(x, z, t_0) \geq M(x, y, t_0) * M(y, z, t_0) = \min\{M(x, y, t_0), M(y, z, t_0)\}$, we obtain
$\frac{1}{M(x,z,t_0)} \leq \max\{\frac{1}{M(x,y,t_0)}, \frac{1}{M(y,z,t_0)}\}$.
Without loss of generality, we assume that $\frac{1}{M(x,z,t_0)} \leq \frac{1}{M(x,y,t_0)}$, then
$d^X_{t_0}(x, y) + d^X_{t_0}(y, z) = \frac{t_0}{M(x,y,t_0)} - t_0 + \frac{t_0}{M(y,z,t_0)} - t_0$

$\geq \frac{t_0}{M(x,z,t_0)} - t_0 + \frac{t_0}{M(y,z,t_0)} - t_0$
$\geq \frac{t_0}{M(x,z,t_0)} - t_0$
$= d_{t_0}^X(x,z).$

Therefore, $d_{t_0}^X$ is a metric in X. □

In the following, we will use $d_{t_0}^X$ to denote the metric in (X, τ_M) induced by $t_0 \in (0, \infty)$. Furthermore, it is easy to see that $d_{t_0}^X$ is also an ultrametric in X. As a result, we can have the following three corollaries.

Corollary 2.2. *Let $(X, M, *)$ be a fuzzy ultrametric space. If $B_{d_{t_0}^X}(x, \varepsilon) \bigcap B_{d_{t_0}^X}(y, \varepsilon) \neq \emptyset$ for all $x, y \in X$ and $\varepsilon > 0$, then $B_{d_{t_0}^X}(x, \varepsilon) = B_{d_{t_0}^X}(y, \varepsilon)$.*

Corollary 2.3. *Let $(X, M, *)$ be a fuzzy ultrametric space. Then, for every $x \in X$ and $\varepsilon > 0$, $B_{d_{t_0}^X}(x, \varepsilon)$ is an open and closed set (i.e., a clopen set).*

Corollary 2.4. *Let $(X, M, *)$ be a fuzzy ultrametric space. Then $(X, d_{t_0}^X)$ is zero-dimensional.*

Theorem 2.5. *Let $(X, M, *)$ be a fuzzy ultrametric space. If $k(1 - M(x, y, kt)) \geq 1 - M(x, y, t)$ for all $x, y \in X, k \geq 1$, then τ_M coincides with the topology $\tau_{d_{t_0}^X}$ on X generated by $d_{t_0}^X$.*

Proof. Let $B_M(x, \frac{1}{n}, \frac{1}{n})$ be an open ball with center x and radius $\frac{1}{n}$ with respect to $\frac{1}{n}$, where $x \in X$ and $n > [\frac{1}{t_0}] + 1$. Then there exists a $\varepsilon = \frac{1}{2}(\frac{n^2 t_0^2}{n^2 t_0 - 1} - t_0)$ such that

$x \in B_{d_{t_0}^X}(x, \varepsilon) \subset B_M(x, \frac{1}{n}, \frac{1}{n}).$

In fact, for each $z \in B_{d_{t_0}^X}(x, \varepsilon)$, we have

$d_{t_0}^X(x, z) = \frac{t_0}{M(x,z,t_0)} - t_0 < \varepsilon.$

Thus $M(x, z, t_0) > \frac{t_0}{t_0+\varepsilon}$. Since $nt_0 \geq 1$ and $k(1 - M(x, y, kt)) \geq 1 - M(x, y, t)$ for all $x, y \in X$, $k \geq 1$, we get

$\frac{1}{nt_0}(M(x, z, \frac{1}{n}) - 1) + 1 \geq M(x, z, nt_0 \cdot \frac{1}{n}) = M(x, z, t_0) > \frac{t_0}{t_0+\varepsilon}.$

Hence

$M(x, z, \frac{1}{n}) > nt_0(\frac{t_0}{t_0+\varepsilon} - 1) + 1 > nt_0(\frac{t_0}{t_0+2\varepsilon} - 1) + 1 = nt_0(\frac{n^2 t_0 - 1}{n^2 t_0} - 1) + 1 = 1 - \frac{1}{n},$

which implies that $z \in B_M(x, \frac{1}{n}, \frac{1}{n})$. So

$x \in B_{d_{t_0}^X}(x, \frac{1}{2}(\frac{n^2 t_0^2}{n^2 t_0 - 1} - t_0)) \subset B_M(x, \frac{1}{n}, \frac{1}{n}).$

Conversely, let $B_{d_{t_0}^X}(x, \varepsilon)$ be an open ball in $(X, \tau_{d_{t_0}^X})$ for each $x \in X$ and $\varepsilon > 0$. Put $n = \max\{[\frac{1}{t_0}], [\frac{t_0+\varepsilon}{\varepsilon}]\} + 1$. Then

$x \in B_M(x, \frac{1}{n}, \frac{1}{n}) \subset B_{d_{t_0}^X}(x, \varepsilon).$

In fact, for each $z \in B_M(x, \frac{1}{n}, \frac{1}{n})$, since $\frac{1}{n} < \frac{\varepsilon}{t_0+\varepsilon}$ and $t_0 > \frac{1}{n}$, we obtain

$M(x, z, t_0) \geq M(x, z, \frac{1}{n}) > 1 - \frac{1}{n} > 1 - \frac{\varepsilon}{t_0+\varepsilon} = \frac{t_0}{t_0+\varepsilon}.$

Then $(t_0 + \varepsilon)M(x, z, t_0) > t_0$. Hence $\frac{t_0}{M(x,z,t_0)} - t_0 < \varepsilon$, that is, $d_{t_0}^X(x, z) < \varepsilon$, which means that $z \in B_{d_{t_0}^X}(x, \varepsilon)$. Consequently, we deduce that

$$x \in B_M(x, \frac{1}{n}, \frac{1}{n}) \subset B_{d_{t_0}^X}(x, \varepsilon).$$

We are done. □

With the above theorem we obtain immediately

Corollary 2.6. *Let $(X, M, *)$ be a fuzzy ultrametric space, and $t_1, t_2 \in (0, \infty)$ with $t_1 \neq t_2$. If $k(1 - M(x, y, kt)) \geq 1 - M(x, y, t)$ for all $x, y \in X$, $k \geq 1$, then the topology $\tau_{d_{t_1}^X}$ coincides with the topology $\tau_{d_{t_2}^X}$.*

Example 2.7. Let $\varphi(t) = 1 - \frac{1}{t+1}(t \in (0,\infty))$, and D an ultrametric in X with $D(x,y) < 1$ for all $x, y \in X$. Define $M_D : X \times X \times (0,\infty) \to [0,1]$ as follows:

$$M_D(x,y,t) = 1 - D(x,y) + D(x,y)\varphi(t).$$

By Proposition 3.5 from [12], we know that $(X, M_D, *)$ is a fuzzy ultrametric space. It is easy to see that $d^X_{t_0}(x,y) = \frac{t_0 D(x,y)}{t_0+1-D(x,y)}$ for every $t_0 \in (0,\infty)$. Note that

$$k(1 - M_D(x,y,kt)) = \frac{kD(x,y)}{kt+1} \geq \frac{D(x,y)}{t+1} = 1 - M_D(x,y,t)$$

for every $k \geq 1$. According to Theorem 2.5, we conclude that $\tau_{M_D} = \tau_{d^X_{t_0}}$.

3. Properties of the induced metrics

In this section we will study some properties of metrics induced by a fuzzy ultrametric.

Definition 3.1. ([11]) Let $(X_i, M_i, *_i)(i = 1,2)$ be two fuzzy metric spaces.

(1) A mapping $f : X_1 \to X_2$ is called an *isometry mapping*, if for every $x, y \in X_1$ and $t > 0, M_1(x,y,t) = M_2(f(x), f(y), t)$.

(2) $(X_i, M_i, *_i)(i = 1,2)$ are called *isometric*, if there is an isometry mapping from X_1 onto X_2.

(3) A complete fuzzy metric space $(X_2, M_2, *_2)$ is called a *completion* of $(X_1, M_1, *_1)$, if $(X_1, M_1, *_1)$ is isometric to a dense subspace of X_2.

Theorem 3.2. *Let $(X_i, M_i, *)(i = 1,2)$ be two fuzzy ultrametric spaces, and f a mapping from X_1 to X_2. Then f is an isometry mapping from $(X_1, M_1, *)$ to $(X_2, M_2, *)$ if and only if f is an isometry mapping from $(X_1, d^{X_1}_{t_0})$ to $(X_2, d^{X_2}_{t_0})$ for every $t_0 \in (0,\infty)$.*

Proof. Let $x, y \in X_1$, $t_0 \in (0,\infty)$. Then $M_1(x,y,t_0) = M_2(f(x), f(y), t_0)$ if and only if $\frac{t_0}{M_1(x,y,t_0)} - t_0 = \frac{t_0}{M_2(f(x),f(y),t_0)} - t_0$, that is, $d^{X_1}_{t_0}(x,y) = d^{X_2}_{t_0}(f(x), f(y))$. □

Definition 3.3. ([5]) Let $(X, M, *)$ be a fuzzy metric space. $\{x_n\}$ is called a *Cauchy sequence* in X, if for each $\varepsilon \in (0,1)$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ whenever $n, m \geq n_0$.

Definition 3.4. ([5]) A fuzzy metric space $(X, M, *)$ is *complete* provided that each Cauchy sequence in X is convergent.

Theorem 3.5. *Let $(X, M, *)$ be a fuzzy ultrametric space. If $k(1 - M(x,y,kt)) \geq 1 - M(x,y,t)$ for all $x, y \in X, k \geq 1$, then $(X, M, *)$ is complete if and only if $(X, d^X_{t_0})$ is complete.*

Proof. Sufficiency. Let $\{x_n\}$ be a Cauchy sequence in $(X, M, *)$. Then, for each $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t_0) > 1 - \frac{\varepsilon}{t_0+\varepsilon}$ whenever $n, m \geq n_0$. Hence $\frac{t_0}{M(x_n,x_m,t_0)} - t_0 < \varepsilon$, that is, $d^X_{t_0}(x_n, x_m) < \varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence in $(X, d^X_{t_0})$. Since $(X, d^X_{t_0})$ is complete, $\{x_n\}$ is convergent, which implies that $(X, M, *)$ is complete.

Necessity. Let $\{x_n\}$ be a Cauchy sequence in $(X, d^X_{t_0})$. For each $r_1 \in (0,1)$ and $t_1 \in (0, t_0)$, put $\varepsilon = \frac{r_1 t_1 t_0}{2(t_0 - t_1 r_1)}$. Then there exists an $m_0 \in \mathbb{N}$ such that

$$d^X_{t_0}(x_n, x_m) = \frac{t_0}{M(x_n,x_m,t_0)} - t_0 < \varepsilon$$

whenever $n, m \geq m_0$. So $M(x_n, x_m, t_0) > \frac{t_0}{t_0+\varepsilon}$. Since $\frac{t_0}{t_1} \geq 1$ and $k(1 - M(x,y,kt)) \geq 1 - M(x,y,t)$ for all $x, y \in X$, $k \geq 1$, we obtain

$$\frac{t_1}{t_0}(M(x_n, x_m, t_1) - 1) + 1 \geq M(x_n, x_m, \tfrac{t_0}{t_1} \cdot t_1) = M(x_n, x_m, t_0) > \frac{t_0}{t_0+\varepsilon}.$$

Hence $M(x_n, x_m, t_1) > 1 - \frac{t_0 \varepsilon}{t_1(t_0+\varepsilon)}$. Note that

$$\varepsilon = \frac{r_1 t_1 t_0}{2(t_0 - t_1 r_1)} < \frac{r_1 t_1 t_0}{t_0 - t_1 r_1}.$$

It follows that $\frac{t_0 \varepsilon}{t_1(t_0+\varepsilon)} < r_1$. Therefore,

$$M(x_n, x_m, t_1) > 1 - \frac{t_0 \varepsilon}{t_1(t_0+\varepsilon)} > 1 - r_1.$$

So $\{x_n\}$ is a Cauchy sequence in $(X, M, *)$. Since $(X, M, *)$ is complete, $\{x_n\}$ is convergence, which means that $(X, d^X_{t_0})$ is complete. This proof is finished. □

Definition 3.6. ([7]) A fuzzy ultrametric space $(X, M, *)$ is *precompact*, if for every $r \in (0, 1)$ and $t > 0$, there exists a finite set $A \subset X$ such that $X = \bigcup_{a \in A} B_M(a, r, t)$.

Theorem 3.7. *Let $(X, M, *)$ be a fuzzy ultrametric space, If $k(1 - M(x, y, kt)) \geq 1 - M(x, y, t)$ for all $x, y \in X$, $k \geq 1$, then $(X, M, *)$ is precompact if and only if $(X, d_{t_0}^X)$ is totally bounded.*

Proof. Sufficiency. For each $x \in X$, $r_1 \in (0, 1)$ and $t_1 \in (0, t_0)$, we have

$$x \in B_{d_{t_0}^X}(x, \frac{t_1 r_1 t_0}{2(t_0 - t_1 r_1)}) \subset B_M(x, r_1, t_1).$$

Since $(X, d_{t_0}^X)$ is totally bounded, there exists a finite set $C \subset X$ such that

$$X = \bigcup_{x \in C} B_{d_{t_0}^X}(x, \frac{t_1 r_1 t_0}{2(t_0 - t_1 r_1)}).$$

Hence

$$X = \bigcup_{x \in C} B_M(x, r_1, t_1).$$

Necessity. For each $x \in X$ and $\varepsilon > 0$, we obtain

$$x \in B_M(x, \frac{\varepsilon}{2(t_0 + \varepsilon)}, \frac{t_0}{2}) \subset B_{d_{t_0}^X}(x, \varepsilon).$$

Since $(X, M, *)$ is precompact, there exists a finite set $A \subset X$ such that

$$X = \bigcup_{x \in A} B_M(x, \frac{\varepsilon}{2(t_0 + \varepsilon)}, \frac{t_0}{2}).$$

So

$$X = \bigcup_{x \in A} B_{d_{t_0}^X}(x, \varepsilon).$$

We complete the proof. □

4. The Hausdorff fuzzy metric

Let $(X, M, *)$ be a fuzzy metric space, $\emptyset \neq C \subset X$. For every $a \in X$ and $t > 0$, $M(a, C, t) := \sup\{M(a, c, t) : c \in C\}$, $M(C, a, t) := \sup\{M(c, a, t) : c \in C\}$ (see [10]). It is easy to see that $M(a, C, t) = M(C, a, t)$.

Definition 4.1. Let $(X, M, *)$ be a fuzzy metric space. We denote by Comp(X) the set of all nonempty compact subsets of (X, τ_M). We define three functions $H_M, H_M^{'}, H_M^{''}$: Comp$(X) \times$ Comp$(X) \times (0, \infty) \to [0, 1]$ as follows, respectively: $\forall A, C \in$ Comp(X) and $t > 0$,

$$H_M(A, C, t) = \min\{\inf_{a \in A} M(a, C, t), \inf_{c \in C} M(A, c, t)\} \text{ (see [11])};$$

$$H_M^{'}(A, C, t) = 1 - \inf\{r | C \subset B_M(A, r, t), A \subset B_M(C, r, t)\} \text{ (see [12])};$$

$$H_M^{''}(A, C, t) = \inf_{x \in X} \frac{M(x, A, t) M(x, C, t)}{|M(x, A, t) - M(x, C, t)| + M(x, A, t) M(x, C, t)}.$$

Theorem 4.2. *Let $(X, M, *)$ be a fuzzy ultrametric space. Then $H_M(A, C, t) = H_M^{'}(A, C, t) = H_M^{''}(A, C, t)$.*

Proof. Firstly, we will prove $H_M(A, C, t) = H_M^{'}(A, C, t)$.

Put $H_M'(A, C, t) = 1 - r_0$. Take $r \in (r_0, 1)$. Then we have

$$C \subset B_M(A, r, t), \quad A \subset B_M(C, r, t).$$

This shows that

$$M(A, c, t) > 1 - r, \quad M(a, C, t) > 1 - r$$

for every $c \in C$ and $a \in A$. Hence

$$\inf_{c \in C} M(A, c, t) \geq 1 - r, \quad \inf_{a \in A} M(a, C, t) \geq 1 - r.$$

So

$$\min\{\inf_{a \in A} M(a, C, t), \inf_{c \in C} M(A, c, t)\} \geq 1 - r,$$

i.e., $H_M(A, C, t) \geq 1 - r$. Passing to the limit as $r \to r_0$, we conclude that

$$H_M(A, C, t) \geq 1 - r_0 = H_M'(A, C, t).$$

Suppose that $H_M(A,C,t) > H'_M(A,C,t)$. Put $H_M(A,C,t) = 1 - r_1$. Then we can find an $r_2 \in (r_1, r_0)$, which means that $1 - r_1 > 1 - r_2 > 1 - r_0$. Since $\inf_{a\in A} M(a,C,t) \geq 1 - r_1$ and $\inf_{c\in C} M(A,c,t) \geq 1 - r_1$, we immediately deduce that

$$M(a,C,t) \geq 1 - r_1 > 1 - r_2$$

and

$$M(A,c,t) \geq 1 - r_1 > 1 - r_2$$

for every $c \in C$ and $a \in A$. Hence

$$C \subset B_M(A, r_2, t), \quad A \subset B_M(C, r_2, t).$$

So

$$H'_M(A,C,t) \geq 1 - r_2 > 1 - r_0 = H'_M(A,C,t),$$

which is a contradiction.

Next, we are going to prove $H_M(A,C,t) = H''_M(A,C,t)$.

If $x \in A$, we have

$$\frac{M(x,A,t)M(x,C,t)}{|M(x,A,t) - M(x,C,t)| + M(x,A,t)M(x,C,t)} = \frac{M(x,C,t)}{|1 - M(x,C,t)| + M(x,C,t)} = M(x,C,t).$$

If $x \in C$, we have

$$\frac{M(x,A,t)M(x,C,t)}{|M(x,A,t) - M(x,C,t)| + M(x,A,t)M(x,C,t)} = \frac{M(x,A,t)}{|1 - M(x,A,t)| + M(x,A,t)} = M(x,A,t).$$

Hence $H_M(A,C,t) \geq H''_M(A,C,t)$. On the other hand, for each $x \in X$ and $\varepsilon > 0$, there exists an $a \in A$ such that

$$\frac{1}{M(x,a,t)} < \frac{1}{M(x,A,t)} + \frac{\varepsilon}{2}.$$

Also, there exists a $c \in C$ such that

$$\frac{1}{M(a,c,t)} < \frac{1}{M(a,C,t)} + \frac{\varepsilon}{2} \leq \frac{1}{\inf_{a\in A} M(a,C,t)} + \frac{\varepsilon}{2}.$$

Since $M(x,c,t) \geq M(x,a,t) * M(a,c,t) = \min\{M(x,a,t), M(a,c,t)\}$, we obtain

$$\frac{1}{M(x,c,t)} \leq \max\{\frac{1}{M(x,a,t)}, \frac{1}{M(a,c,t)}\} \leq \frac{1}{M(x,a,t)} + \frac{1}{M(a,c,t)} - 1.$$

Hence

$$\frac{1}{M(x,C,t)} \leq \frac{1}{M(x,c,t)} \leq \frac{1}{M(x,a,t)} + \frac{1}{M(a,c,t)} - 1 < \frac{1}{M(x,A,t)} + \frac{1}{\inf_{a\in A} M(a,C,t)} + \varepsilon - 1.$$

It follows that

$$\frac{1}{M(x,C,t)} - \frac{1}{M(x,A,t)} < \frac{1}{\inf_{a\in A} M(a,C,t)} + \varepsilon - 1.$$

By the arbitrariness of ε, we have

$$\sup_{x\in X}(\frac{1}{M(x,C,t)} - \frac{1}{M(x,A,t)}) = \sup_{x\in X}|\frac{1}{M(x,C,t)} - \frac{1}{M(x,A,t)}| \leq \frac{1}{\inf_{a\in A} M(a,C,t)} - 1.$$

Therefore,

$$\inf_{x\in X} \frac{M(x,A,t)M(x,C,t)}{|M(x,A,t) - M(x,C,t)| + M(x,A,t)M(x,C,t)} \geq \inf_{a\in A} M(a,C,t).$$

Using the same method as above, we can get

$$\inf_{x\in X}\frac{M(x,A,t)M(x,C,t)}{|M(x,A,t)-M(x,C,t)|+M(x,A,t)M(x,C,t)}\geq \inf_{c\in C}M(A,c,t).$$

So

$$\inf_{x\in X}\frac{M(x,A,t)M(x,C,t)}{|M(x,A,t)-M(x,C,t)|+M(x,A,t)M(x,C,t)}\geq \min\{\inf_{a\in A}M(a,C,t),\inf_{c\in C}M(A,c,t)\},$$

i.e., $H''_M(A,C,t)\geq H_M(A,C,t)$. We are done. □

Lemma 4.3. ([12]) *Let $(X,M,*)$ be a fuzzy ultrametric space. Then (Comp(X), H'_M, $*$) is a fuzzy ultrametric space.*

According to Theorem 4.2 and Lemma 4.3, we obtain

Corollary 4.4. *Let $(X,M,*)$ be a fuzzy ultrametric space. Then (Comp(X), H_M, $*$) and (Comp(X), H''_M, $*$) are fuzzy ultrametric spaces.*

We call H_M a *Hausdorff fuzzy metric*.

Proposition 4.5. *Let $(X,M,*)$ be a fuzzy ultrametric space. Then, for each $x\in X$ and $A\in$ Comp(X), we have $d^X_{t_0}(x,A)=\frac{t_0}{M(x,A,t_0)}-t_0$.*

Proof. $d^X_{t_0}(x,A)=\inf_{a\in A}d^X_{t_0}(x,a)=\inf_{a\in A}(\frac{t_0}{M(x,a,t_0)}-t_0)=\frac{t_0}{\sup_{a\in A}M(x,a,t_0)}-t_0=\frac{t_0}{M(x,A,t_0)}-t_0$. □

Proposition 4.6. *Let $(X,M,*)$ be a fuzzy ultrametric space, and $A\in$ Comp(X). If $k(1-M(x,y,kt))\geq 1-M(x,y,t)$ for all $x,y\in X$, $k\geq 1$, then $k(1-M(x,A,kt))\geq 1-M(x,A,t)$.*

Proof. For every $a\in A$, $k(1-M(x,a,kt))\geq 1-M(x,a,t)$. Then
$\inf_{a\in A}k(1-M(x,a,kt))\geq \inf_{a\in A}(1-M(x,a,t))$.
Hence
$k(1-\sup_{a\in A}M(x,a,kt))\geq 1-\sup_{a\in A}M(x,a,t)$,
that is, $k(1-M(x,A,kt))\geq 1-M(x,A,t)$. □

Remark. From Theorem 2.1 and Corollary 4.4, we can see that $d^{Comp(X)}_{t_0}$ is the metric in Comp(X) induced by t_0.

Definition 4.7. ([1]) Let (X,d) be a metric space. For every $A,C\in$ Comp(X), let H_d:Comp(X)× Comp(X)→ $[0,\infty)$ be a mapping defined by

$$H_d(A,C)=\max\{\sup_{a\in A}d(a,C),\sup_{c\in C}d(A,c)\}.$$

H_d is a metric in Comp(X), which is called *Hausdorff metric*.

Theorem 4.8. *Let $(X,M,*)$ be a fuzzy ultrametric space. Then for each $A,C\in$ Comp(X), $d^{Comp(X)}_{t_0}(A,C)=H_{d^X_{t_0}}(A,C)$.*

Proof.
$$\begin{aligned}d^{Comp(X)}_{t_0}(A,C)&=\frac{t_0}{H_M(A,C,t_0)}-t_0\\&=\max\{\frac{t_0}{\inf_{a\in A}M(a,C,t)}-t_0,\frac{t_0}{\inf_{c\in C}M(A,c,t)}-t_0\}\\&=\max\{\sup_{a\in A}(\frac{t_0}{M(a,C,t)}-t_0),\sup_{c\in C}(\frac{t_0}{M(A,c,t)}-t_0)\}\\&=\max\{\sup_{a\in A}d^X_{t_0}(a,C),\sup_{c\in C}d^X_{t_0}(A,c)\}=H_{d^X_{t_0}}(A,C).\end{aligned}$$
□

Theorem 4.9. *Let* $(X, M, *)$ *be a fuzzy ultrametric space. If* $k(1 - M(x, y, kt)) \geq 1 - M(x, y, t)$ *for all* $x, y \in X$, $k \geq 1$, *then (Comp*$(X), \tau_{H_M})$ *coincides with (Comp*$(X), \tau_{H_{d^X_{t_0}}})$.

Proof. By Theorem 4.8, (Comp$(X), \tau_{H_{d^X_{t_0}}}$) coincides with (Comp$(X), \tau_{d^{Comp(X)}_{t_0}}$). According to Theorem 2.5, we know that (Comp$(X), \tau_{H_M}$) coincides with (Comp$(X), \tau_{d^{Comp(X)}_{t_0}}$). Consequently, (Comp$(X), \tau_{H_M}$) coincides with (Comp$(X), \tau_{H_{d^X_{t_0}}}$). □

Lemma 4.10. ([11]) *Let* $(X, M, *)$ *be a fuzzy metric space. Then (Comp*$(X), H_M, *)$ *is complete if and only if* $(X, M, *)$ *is complete.*

Lemma 4.11. ([11]) *Let* $(X, M, *)$ *be a fuzzy metric space. Then (Comp*$(X), H_M, *)$ *is precompact if and only if* $(X, M, *)$ *is precompact.*

Definition 4.12. ([7]) A fuzzy metric space $(X, M, *)$ is called *compact*, if (X, τ_M) is a compact topological space.

Lemma 4.13. ([2]) *A metric space* (X, d) *is compact if and only if it is complete and totally bounded.*

Lemma 4.14. [7] *A fuzzy metric space* $(X, M, *)$ *is compact if and only if it is complete and totally bounded.*

According to Theorems 3.5, 3.7,4.9, Lemmas 4.10, 4.11, 4.13, 4.14 and Definition 4.12, we immediately deduce the following.

Theorem 4.15. *Let* $(X, M, *)$ *be a fuzzy ultrametric space. If* $k(1-M(x, y, kt)) \geq 1-M(x, y, t)$ *for all* $x, y \in X$, $k \geq 1$, *then (Comp*$(X), H_{d^X_{t_0}})$ *is complete if and only if* $(X, M, *)$ *is complete.*

Theorem 4.16. *Let* $(X, M, *)$ *be a fuzzy ultrametric space. If* $k(1-M(x, y, kt)) \geq 1-M(x, y, t)$ *for all* $x, y \in X$, $k \geq 1$, *then (Comp*$(X), H_{d^X_{t_0}})$ *is totally bounded if and only if* $(X, M, *)$ *is precompact.*

Theorem 4.17. *Let* $(X, M, *)$ *be a fuzzy ultrametric space. If* $k(1-M(x, y, kt)) \geq 1-M(x, y, t)$ *for all* $x, y \in X$, $k \geq 1$, *then (Comp*$(X), H_{d^X_{t_0}})$ *is compact if and only if* $(X, M, *)$ *is compact.*

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