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Fredholm Perturbations and Seminorms Related to Upper Semi-Fredholm Perturbations

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Abstract. In this paper, we investigate the stability in the set of Fredholm perturbations under composition with bounded operators. Moreover, we introduce the concept of a measure of non-Fredholm perturbations, which allows us to give a general approach to the question of obtaining perturbation theorems for semi-Fredholm operators. Finally, we prove some localization results about the Wolf, the Schechter and the Browder essential spectrum of bounded operators on a Banach space X.

1. Introduction

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X,Y)$ (respectively $\mathcal{K}(X,Y)$) the space of all bounded (respectively compact) linear operators from X into Y. If $T \in \mathcal{L}(X,Y)$, we write $\mathcal{N}(T) \subseteq X$ and $Ran(T) \subseteq Y$ for the null space and the range of T. We set $\alpha(T) := \dim \mathcal{N}(T)$ and $\beta(T) := \operatorname{codim} Ran(T)$. The sets of upper and lower semi-Fredholm operators in $\mathcal{L}(X,Y)$ are denoted by $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$. We use $\Phi_\pm(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y)$ for the set of semi-Fredholm operators, and $\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ for the set of Fredholm operators. If $T \in \Phi_\pm(X,Y)$, then $i(T) := \alpha(T) - \beta(T)$ is called the index of T. If $X = Y$, we simply write $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\pm(X)$ and $\Phi(X)$.

Set $\mathcal{N}^\infty(T) = \bigsqcup_n \mathcal{N}(T^n)$, $\mathcal{R}^\infty(T) = \bigcap_n Ran(T^n)$, and denote by $a(T)$ respectively $\delta(T)$, the ascent and the descent of $T \in \mathcal{L}(X)$. The sets of upper and lower semi-Browder operators are denoted by $\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$. The set of Browder operators on X is $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$.

In this paper we are concerned with the following essential spectra:

Wolf essential spectrum: $\sigma_e(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi(X)\}$,

Schechter essential spectrum: $\sigma_{ess}(T) := \mathbb{C}\backslash\{\lambda - T \in \Phi(X) \text{ such that } i(\lambda - T) = 0\}$,

Browder essential spectrum : $\sigma_b = \{\lambda \in \mathbb{C};\ \lambda - T \notin \mathcal{B}(X)\}$.

An operator $T \in \mathcal{L}(X,Y)$ is said to be left Atkinson if $T \in \Phi_+(X,Y)$ and $Ran(T)$ is complemented. The operator $T \in \mathcal{L}(X,Y)$ is said to be right Atkinson if $T \in \Phi_-(X,Y)$ and $\mathcal{N}(T)$ is complemented. The class of left Atkinson operators and right Atkinson operators will be denoted by $\Phi_l(X,Y)$ and $\Phi_r(X,Y)$, respectively.

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Clearly, we have $\Phi(X, Y) \subset \Phi_l(X, Y) \subset \Phi_+(X, Y)$ and $\Phi(X, Y) \subset \Phi_r(X, Y) \subset \Phi_-(X, Y)$. Moreover, $\Phi(X, Y) = \Phi_l(X, Y) \cap \Phi_r(X, Y)$.

An operator $A \in \mathcal{L}(X, Y)$ is called a Fredholm perturbation, if $A + B \in \Phi(X, Y)$ whenever $B \in \Phi(X, Y)$. The operator A is called upper (respectively lower) semi-Fredholm perturbation if $A+B \in \Phi_+(X, Y)$ (respectively $\Phi_-(X, Y)$) whenever $B \in \Phi_+(X, Y)$ (respectively $\Phi_-(X, Y)$). The sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted respectively by $\mathcal{F}(X, Y)$, $\mathcal{F}_+(X, Y)$ and $\mathcal{F}_-(X, Y)$.
An operator $A \in \mathcal{L}(X, Y)$ is called a left (respectively right) Fredholm perturbation, if $A + B \in \Phi_l(X, Y)$ (respectively $\Phi_r(X, Y)$) whenever $B \in \Phi_l(X, Y)$ (respectively $\Phi_r(X, Y)$). The sets of left Fredholm and right Fredholm perturbation are denoted respectively by $\mathcal{F}_l(X, Y)$ and $\mathcal{F}_r(X, Y)$. If $\Phi(X, Y) = \emptyset$, we shall agree that $\mathcal{F}(X, Y) = \mathcal{L}(X, Y)$. It is well known that, if $\Phi(X, Y) \neq \emptyset$, then $\mathcal{F}(X, Y) = \mathcal{F}_l(X, Y) = \mathcal{F}_r(X, Y)$ (see, for example, [2, Theorem 3.16]). If $X = Y$, we use $\mathcal{F}(X)$, $\mathcal{F}_+(X)$, $\mathcal{F}_-(X)$, $\mathcal{F}_l(X)$ and $\mathcal{F}_r(X)$.

These sets of operators are introduced and investigated in [7, 14]. In particular, it is shown that $\mathcal{F}(X, Y)$ is a closed linear subspace of $\mathcal{L}(X, Y)$ and $\mathcal{F}(X)$, $\mathcal{F}_+(X)$, $\mathcal{F}_-(X)$ are closed two-sided ideals of $\mathcal{L}(X)$. Moreover, $\mathcal{K}(X, Y) \subset \mathcal{SS}(X, Y) \subset \mathcal{F}_+(X, Y) \subset \mathcal{F}(X, Y)$, and $\mathcal{K}(X, Y) \subset \mathcal{CS}(X, Y) \subset \mathcal{F}_-(X, Y) \subset \mathcal{F}(X, Y)$, where $\mathcal{SS}(X, Y)$ and $\mathcal{CS}(X, Y)$ are respectively the classes of strictly singular operators and strictly cosingular operators.

In this paper, we are interested in the properties of the class of Fredholm perturbations. This class of operators has been subject of interest for several authors (see for instance, [4, 7, 8, 12, 14]). Let X_1, X_2, Y_1 and Y_2 be Banach spaces, and consider $U \in \mathcal{L}(X_2, Y_2)$ and $V \in \mathcal{L}(Y_1, X_1)$. It is familiar that if $S \in \mathcal{K}(X_1, X_2)$, respectively $\mathcal{SS}(X_1, X_2)$, then $USV \in \mathcal{K}(Y_1, Y_2)$, respectively $\mathcal{SS}(Y_1, Y_2)$. To study the stability problem in the class of Fredholm perturbations, it is natural to ask the following: For $S \in \mathcal{F}(X_1, X_2)$, under which conditions does $USV \in \mathcal{F}(Y_1, Y_2)$? To answer this question I. Gohberg and all in [7, pp. 69-70] have shown the following:

Proposition 1.1. *[7] Let X, Y, Z be Banach spaces. If at least one of the sets $\Phi(X, Y)$ or $\Phi(Y.Z)$ is not empty, then*

(i) $S \in \mathcal{F}(X, Y)$, $U \in \mathcal{L}(Y, Z)$, imply $US \in \mathcal{F}(X, Z)$.

(ii) $S \in \mathcal{F}(Y, Z)$, $V \in \mathcal{L}(X, Y)$, imply $SV \in \mathcal{F}(X, Z)$.

The purpose of this paper is to extend the above results and to give a positive answer in more general cases (see Theorem 2.4 in section 2).

The aim of section 3 is to treat the problem of stability in semi-Fredholm (respectively semi-Browder) operators set. For this, we construct a new measure called measure of non-upper semi-Fredholm perturbations and we prove some localization results about the essential spectra σ_e, σ_{ess} and σ_b of bounded operators on a Banach space X. These results provide, in particular, an extension of ones done by [1].

2. Fredholm perturbation

The purpose of this section is to extend the results of Proposition 1.1 in more general cases. First, we adopt the following definition:

Definition 2.1. *Let X and Y be two Banach spaces. We say that Y is essentially stronger than X and write $X \leq Y$, if there exists $R \in \Phi_l(X, Y)$.*

Remark 2.2. (i) *It is clear that, for X a Banach space, $X \leq X$.*

(ii) *Let X_1, X_2, X_3 be three Banach spaces such that $X_1 \leq X_2 \leq X_3$. Then there exists $R_1 \in \Phi_+(X_1, X_2)$ and $R_2 \in \Phi_+(X_2, X_3)$ with $Ran(R_1)$ complemented in X_2 and $Ran(R_2)$ complemented in X_3. By [11, Theorem 14, p. 160], there exists S_i such that $S_iR_i = I_{X_i} + K_i$ with $K_i \in \mathcal{K}(X_i)$, $i = 1, 2$. Hence, $(S_1S_2)(R_2R_1) = I_{X_1} + K_1 + S_1K_2R_1$. Thus, by [13, Theorem 5.37, p. 126], $R_2R_1 \in \Phi_+(X_1, X_3)$ with $Ran(R_2R_1)$ complemented in X_3 which implies that $X_1 \leq X_3$.*

(iii) *To deduce that " $\leq$ " is not antisymmetric, we notice that W.T. Gowers showed in [10] that there is a Banach space Z that is isomorphic to $Z \oplus Z \oplus Z$ but not isomorphic to $Z \oplus Z$.*

An other important property is the following :

Lemma 2.3. *Let X and Y be two Banach spaces. Suppose that $X \leq Y$, then $X^* \leq Y^*$.*

Proof. We have $X \leq Y$, then there exist $R \in \Phi_+(X, Y)$ and $S \in \Phi_-(Y, X)$, such that $SR = I_X + K$, with $K \in \mathcal{K}(X)$. This implies that $R^*S^* = I_{X^*} + K^*$. Since $R^* \in \Phi_-(Y^*, X^*)$, $S^* \in \Phi_+(X^*, Y^*)$ and $K^* \in \mathcal{K}(X^*)$, then we get $X^* \leq Y^*$. Q.E.D.

Now, we are ready to state and prove the main result of this section.

Theorem 2.4. *Let (X_1, X_2) and (Y_1, Y_2) be two couples of Banach spaces satisfying $X_1 \leq Y_1$. Consider $U \in \mathcal{L}(X_2, Y_2)$ and $V \in \mathcal{L}(Y_1, X_1)$.*

(i) *Suppose that $\Phi(X_1, X_2) \neq \emptyset$. If $S \in \mathcal{F}(X_1, X_2)$, then $USV \in \mathcal{F}(Y_1, Y_2)$.*

(ii) *Suppose that $\Phi_l(X_1, X_2) \neq \emptyset$. If $S \in \mathcal{F}_l(X_1, X_2)$, then $USV \in \mathcal{F}_l(Y_1, Y_2)$.*

(iii) *Suppose that $\Phi_r(X_1, X_2) \neq \emptyset$. If $S \in \mathcal{F}_r(X_1, X_2)$, then $USV \in \mathcal{F}_r(Y_1, Y_2)$.*

Proof. (i) Remark that the result is trivial if $\Phi(Y_1, Y_2) = \emptyset$. So let us assume that $\Phi(Y_1, Y_2) \neq \emptyset$. Since $X_1 \leq Y_1$ and $\Phi(X_1, X_2) \neq \emptyset$, this yields $X_2 \leq Y_2$. Hence, there exists $R_i \in \Phi_+(X_i, Y_i)$ and a closed subspace Z_i such that $Y_i := Ran(R_i) \oplus Z_i$, $i = 1, 2$. Without loss of generality, we can suppose that R_1 and R_2 are injective. For $i = 1, 2$ denote the following invertible operator

$$\begin{aligned} R_{i0} : X_i &\longrightarrow Ran(R_i) \\ x &\longmapsto R_{i0}(x) = R_i(x). \end{aligned}$$

By [11, Theorem 14, p. 160], there exists $R'_i \in \mathcal{L}(Y_i, X_i)$ such that $R'_iR_i = I_{X_i} + K_i$, with $K_i \in \mathcal{K}(X_i)$. We can choose R'_1 as follows:

$$\begin{aligned} R'_1 : Y_1 = Ran(R_1) \oplus Z_1 &\longrightarrow X_1 \\ y = (y_1, z_1) &\longmapsto R_{10}^{-1}(y_1). \end{aligned}$$

Hence,

$$\begin{aligned} R_2SR'_1 : Y_1 = Ran(R_1) \oplus Z_1 &\longrightarrow Y_2 = Ran(R_2) \oplus Z_2 \\ y = (y_1, z_1) &\longmapsto (R_{20}SR_{10}^{-1}(y_1), 0) = \begin{pmatrix} R_{20}SR_{10}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}. \end{aligned}$$

Since $S \in \mathcal{F}(X_1, X_2)$, then, by [7, pp. 69-70], we get $R_{20}SR_{10}^{-1} \in \mathcal{F}(Ran(R_1), Ran(R_2))$. First, we claim that $F := R_2SR_1' \in \mathcal{F}(Y_1, Y_2)$.

Consider an arbitrary element $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Phi(Y_1, Y_2)$. It follows, by Atkinson theorem, that there exists $L_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} \in \Phi(Y_2, Y_1)$ and $K \in \mathcal{K}(Y_2)$ such that $LL_0 = I + K$ on Y_2. Then

$$(L+F)L_0 = I + K + FL_0 = K + \begin{pmatrix} I & R_{20}SR_{10}^{-1}B_0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I + R_{20}SR_{10}^{-1}A_0 & 0 \\ 0 & I \end{pmatrix}.$$

Notice that $\Phi(X_1, X_2) \neq \emptyset$ implies $\Phi(Ran(R_1), Ran(R_2)) \neq \emptyset$. According to [7, pp. 69-70], $R_{20}SR_{10}^{-1} \in \mathcal{F}(Ran(R_1), Ran(R_2))$, yields that $R_{20}SR_{10}^{-1}A_0 \in \mathcal{F}(Ran(R_2))$. Thus, $I + R_{20}SR_{10}^{-1}A_0 \in \Phi(Ran(R_2))$ which implies that $\begin{pmatrix} I + R_{20}SR_{10}^{-1}A_0 & 0 \\ 0 & I \end{pmatrix}$ is Fredholm. Observing that $\begin{pmatrix} I & R_{20}SR_{10}^{-1}B_0 \\ 0 & I \end{pmatrix}$ is invertible, with inverse $\begin{pmatrix} I & -R_{20}SR_{10}^{-1}B_0 \\ 0 & I \end{pmatrix}$, we get $(L+F)L_0 \in \Phi(Y_2)$. Hence, $(L+F) \in \Phi(Y_1, Y_2)$ and therefore $F \in \mathcal{F}(Y_1, Y_2)$. Our claim is proved.

Now, since $R_i'R_i = I_{X_i} + K_i$, then

$$\begin{aligned} USV &= U(R_2'R_2 - K_2)S(R_1'R_1 - K_1)V \\ &= UR_2'(R_2SR_1')R_1V + K', \end{aligned}$$

with $K' \in \mathcal{K}(Y_1, Y_2)$. Finally, the result follows by Proposition 1.1, since $UR_2' \in \mathcal{L}(Y_2)$ and $R_1V \in \mathcal{L}(Y_1)$.

(*ii*)-(*iii*) The proof is analogous to the previous one. Q.E.D.

Remark 2.5. *Notice that if $\Phi(X_1, X_2) = \emptyset$ and $\Phi(Y_1, Y_2) \neq \emptyset$, then the results in the above theorem need not hold. Consider the Banach space Z constructed by Gowers in [10] (see Remark 2.2(iii)). If we take $X_1 = Z$ and $X_2 = Y_1 = Y_2 = Z \oplus Z$, then $F := id_{X_2} \circ i \circ p \notin \mathcal{F}(Y_1, Y_2)$, where i and p are respectively the natural embedding and the projection.*

As a consequence of Theorem 2.4, we have:

Corollary 2.6. *Let (X_1, X_2) and (Y_1, Y_2) be two couples of Banach spaces such that $X_1 \leq Y_1$. Assume that $\Phi(X_1, X_2) \neq \emptyset$.*
(i) If $\mathcal{F}(Y_1, Y_2) = \mathcal{K}(Y_1, Y_2)$, then $\mathcal{F}(X_1, X_2) = \mathcal{K}(X_1, X_2)$.
(ii) If $\mathcal{F}(Y_1, Y_2) = \mathcal{SS}(Y_1, Y_2)$, then $\mathcal{F}(X_1, X_2) = \mathcal{S}(X_1, X_2)$.

Proof. We shall prove (*i*), the proof of (*ii*) is similar. Since $X_i \leq Y_i$, $i = 1, 2$, then there exists $R_1 \in \Phi_+(X_1, Y_1)$ and $R_2 \in \Phi_+(X_2, Y_2)$ with $Ran(R_1)$ complemented in Y_1 and $Ran(R_2)$ complemented in Y_2. By [11, Theorem 14, p. 160], there exists R_i' such that $R_i'R_i = I_{X_i} + K_i$ with $K_i \in \mathcal{K}(X_i)$, $i = 1, 2$. Let $T \in \mathcal{F}(X_1, X_2)$. The use of Theorem 2.4 leads to $R_2TR_1' \in \mathcal{F}(Y_1, Y_2) = \mathcal{K}(Y_1, Y_2)$. Thus, $R_2'R_2TR_1'R_1 = T + K_2T + K_2TK_1 \in \mathcal{K}(X_1, X_2)$ and therefore $T \in \mathcal{K}(X_1, X_2)$. Q.E.D.

Corollary 2.7. *Let X be a Banach space satisfying $X \leq L_p(\mu)$, for some $p \geq 1$, then $\mathcal{F}(X) = \mathcal{SS}(X) = \mathcal{CS}(X)$.*

Proof. By [16] we have $\mathcal{F}(L_p(\mu)) = \mathcal{SS}(L_p(\mu))$. Since $X \leq L_p(\mu)$ then by Corollary 2.6 (*ii*), we get $\mathcal{F}(X) = \mathcal{SS}(X)$.
Now, consider $F \in \mathcal{F}(X)$, then $F^* \in \mathcal{F}(X^*)$. Since $X \leq L_p(\mu)$ then by Lemma 2.3, $X^* \leq L_p^*(\mu) = L_q(\mu)$ for some $q \geq 1$. Again by Corollary 2.6, $\mathcal{F}(X^*) = \mathcal{SS}(X^*)$. Hence $F^* \in \mathcal{SS}(X^*)$, and therefore $F \in \mathcal{CS}(X)$. This yields $\mathcal{F}(X) \subset \mathcal{CS}(X)$, and we get the result since we have $\mathcal{CS}(X) \subset \mathcal{F}(X)$. Q.E.D.

3. Seminorm related to upper semi-Fredholm perturbations

3.1. Basic construction and properties

Let X be a Banach space. We write M_X for the family of all nonempty and bounded subset of X. Given α the Kuratowski measure of noncompactness, we define, for $T \in \mathcal{L}(X)$, the two non-negative quantities associated with T by:

$$\alpha(T) = \sup\left\{\frac{\alpha(T(A))}{\alpha(A)};\ A \in M_X,\ \alpha(A) > 0\right\} \text{ and } \beta(T) = \inf\left\{\frac{\alpha(T(A))}{\alpha(A)};\ A \in M_X,\ \alpha(A) > 0\right\}.$$

For more detail of some fundamental properties satisfied by α and β we refer to [1, 6].

Definition 3.1. *For $T \in \mathcal{L}(X)$, we define the non-negative quantity:*

$$\varphi(T) = \sup\{\beta(T + S),\ \beta(S) = 0\}.$$

In what follows, we give some fundamental properties satisfied by $\varphi(.)$.

Proposition 3.2. *(i) $\varphi(T) = 0$ if and only if $T \in \mathcal{F}_+(X)$.*

(ii) $\varphi(T + S) = \varphi(S)$, for all $T \in \mathcal{F}_+(X)$.

(iii) $\varphi(\lambda T) = |\lambda|\varphi(T)$.

(iv) $\beta(T) \leq \varphi(T) \leq \alpha(T)$.

(v) $\varphi(T) - \alpha(S) \leq \varphi(T + S) \leq \varphi(T) + \alpha(S)$.

(vi) $\varphi(T) \leq \|T\|_{\mathcal{F}_+} \leq \|T\|_{\mathcal{K}}$, where $\|T\|_{\mathcal{K}} = \inf\{\|T - K\|;\ K \in \mathcal{K}(X)\}$ and

$\|T\|_{\mathcal{F}_+} = \inf\{\|T - K\|;\ K \in \mathcal{F}_+(X)\}$.

(vii) $\varphi(ST) \geq \varphi(T)\beta(S)$, for all $S \in \mathcal{L}(X)$.

(viii) If $\varphi(T) = 0$, then, for all $S \in \mathcal{L}(X)$, $\varphi(TS) = \varphi(ST) = 0$.

Proof. (*i*) It follows immediately from the fact that $\beta(T) > 0$ if and only if $T \in \Phi_+(X)$.

(*ii*) Due to the fact that for $T \in \mathcal{F}_+(X)$, $\beta(S) = 0$ if and only if $\beta(T + S) = 0$.

(*iii*) − (*v*) Follow from the definition of $\varphi(.)$ and the fact that $\beta(T) \leq \alpha(T)$.

(*vi*) Deduction of (*ii*) and (*iv*).

(*vii*) Let $S, S_1 \in \mathcal{L}(X)$. According to [2, Theorem 1.46], we have

$$\beta(S_1) = 0 \Longrightarrow \beta(SS_1) = 0.$$

On the other hand, we have

$$\beta(ST + SS_1) = \beta(S(T + S_1)) \geq \beta(T + S_1)\beta(S).$$

Hence,

$$\sup_{\beta(SS_1)=0} \beta(ST + SS_1) \geq \sup_{\beta(S_1)=0} \beta(T + S_1)\beta(S)$$

and therefore, $\varphi(ST) \geq \varphi(T)\beta(S)$.

(*viii*) The fact that $\mathcal{F}_+(X)$ is a two-sided ideal of $\mathcal{L}(X)$ together with (*i*) gives immediately the assertion (*viii*). Q.E.D.

Remark 3.3. (i) *Notice that φ is not a non-compactness measure. Endeed, using the Rademcher functions, the author in [3] constructs a strictly singular operator $T \in \mathcal{L}(L_p[-1,1])$, $(p \geq 1)$, hence $\varphi(T) = 0$. The proof of Theorem X.5.2 in [3] shows that $\alpha(T) = 1$.*

(ii) *The assertion* ($viii$) *is equivalent to say that $\mathcal{F}_+(X)$ is a two-sided ideal of $\mathcal{L}(X)$. Moreover by* (v), $| \varphi(T) - \varphi(S) | \leq \alpha(T - S) \leq \| T - S \|$. *This implies that the measure φ is continuous. Hence, it follows from* (i) *that $\mathcal{F}_+(X)$ is closed.*

3.2. Applications

In the following theorems we establish a stability properties in the upper semi-Fredholm and semi-Browder operators sets. In [12], V. Rakočević proves that the upper (lower) semi-Fredholm operators with finite ascent (descent) is closed under commuting operator perturbations that belongs to the perturbation class associated with the set of upper (lower) semi-Fredholm operators. In the following theorem we extend in some way the result of Rakočević.

Theorem 3.4. *Let T, S be two bounded operators on X.*

(i) *If $\varphi(T) < \beta(S)$, then $T + S \in \Phi_+(X)$ and $i(T + S) = i(S)$.*

Suppose moreover that $ST = TS$.

(ii) *If $\varphi(T) < \beta(S)$, then $(a(S) < \infty \Rightarrow a(T + S) < \infty)$.*

(iii) *Suppose that there exists $n \in \mathbb{N}^*$ such that $\varphi(T^n) < \beta(S^n)$, then we get*

(a) If $S \in \mathcal{B}_+(X)$, then $T + S \in \mathcal{B}_+(X)$.

(b) If $S \in \mathcal{B}(X)$, then $T + S \in \mathcal{B}(X)$.

Proof. (i) Suppose that $\beta(T + S) = 0$. Then $\beta(S) = \beta(T - (T + S)) < \varphi(T)$. Hence, if $\varphi(T) < \beta(S)$, then $\beta(T + S) > 0$ and therefore $T + S \in \Phi_+(X)$. Let $t \in [0, 1]$, then $\varphi(tT) < \beta(S)$, and so, by what we have just proved, $tT + S \in \Phi_+(X)$. Thus, by the continuity of the index on $\Phi_+(X)$, we get $i(T + S) = i(S)$.

(ii) For $t \in [0, 1]$, we have $\varphi(tT) < \beta(S)$, then, by (i), $tT + S \in \Phi_+(X)$. Since S and T are commuting, then according to [9, Theorem 3],

$$\overline{\mathcal{N}^\infty(tT + S)} \cap \mathcal{R}^\infty(tT + S) = \overline{\mathcal{N}^\infty(sT + S)} \cap \mathcal{R}^\infty(sT + S),$$

for all s in some open disk with center t. Hence, $\overline{\mathcal{N}^\infty(tT + S)} \cap \mathcal{R}^\infty(tT + S)$ is locally constant function of t on the interval $[0, 1]$. This yields that for all $t \in [0, 1]$,

$$\overline{\mathcal{N}^\infty(tT + S)} \cap \mathcal{R}^\infty(tT + S) = \overline{\mathcal{N}^\infty(S)} \cap \mathcal{R}^\infty(S).$$

Now, since $a(S) < \infty$, then from [17, Proposition 1.6(i)] :

$$\mathcal{N}^\infty(S) \cap \mathcal{R}^\infty(S) = \overline{\mathcal{N}^\infty(S)} \cap \mathcal{R}^\infty(S) = \{0\}.$$

Hence,

$$\overline{\mathcal{N}^\infty(T + S)} \cap \mathcal{R}^\infty(T + S) = \{0\}.$$

Thus,

$$\mathcal{N}^\infty(T + S) \cap \mathcal{R}^\infty(T + S) = \{0\},$$

and again by [17, Proposition 1.6(i)], it follows that $a(T + S) < \infty$.

(iii)(a) Let $t \in [0, 1]$. Since $\varphi((tT)^n) < \beta(S^n)$, by ($i$), $tT + S \in \Phi_+(X)$. Arguing as in the proof of (ii), we get the result.

(iii)(b) Since $S \in \mathcal{B}(X)$, then $i(S) = 0$. Arguing as in the proof of (i), we get $i(T + S) = 0$. On the other hand, (ii) yields $a(T + S) < \infty$. According to [15, Theorem 4.5 (d)], we get $\delta(T + S) < \infty$. Q.E.D.

Recall the essential spectral radius $r_e(T) := \max\{|\lambda|;\ T - \lambda I \notin \Phi(X)\}$, defined for $T \in \mathcal{L}(X)$. According to [5, Section 1.4], we have $r_e(T) = \lim\limits_{n \to +\infty} \|T^n\|_{\mathcal{K}}^{\frac{1}{n}}$. By Theorem 3.4 and Proposition 3.2, we can deduce:

Corollary 3.5. $r_e(T) = \lim_{n\to+\infty} (\varphi(T^n))^{\frac{1}{n}}$

Proof. From Proposition 3.2(*vi*) it follows $r_e(T) \geq \lim_{n\to+\infty} (\varphi(T^n))^{\frac{1}{n}}$. To prove the opposite inequality, let $\lambda \in \mathbb{C}$ be such that $\mid \lambda \mid > (\varphi(T^n))^{\frac{1}{n}}$ for some $n \in \mathbb{N}$, then by Theorem 3.4 (*i*) it follows that $\lambda - T \in \Phi(X)$. Hence $r_e(T) \leq (\varphi(T^n))^{\frac{1}{n}}$ for every $n \in \mathbb{N}$.

Q.E.D.

For $T \in \mathcal{L}(X)$, define $\beta_0(T)$ to be the limit of the sequence $(\beta(T^n))^{\frac{1}{n}}$. For the existence of the quantity $\beta_0(T)$ see [11, Lemma 1.21]. As an application of Theorem 3.4, we prove some localization results about the essential spectra σ_e, σ_{ess} and σ_b of bounded operators on X. We use $\mathbb{D}(0,r)$ for the disc with center 0 and radius r and $\overline{\mathbb{D}}(0,r)$ for the closure of $\mathbb{D}(0,r)$. We write $C[r_1,r_2] = \overline{\mathbb{D}}(0,r_2) \setminus \mathbb{D}(0,r_1)$, for $r_1 \leq r_2$.

Corollary 3.6. *Let T be a bounded operator on X, we have :*

(*i*) $\sigma_{ess}(T) \subset \overline{\mathbb{D}}(0,r_e(T))$.

(*ii*) *If* $T \in \Phi_-(X)$, *then* $\sigma_e(T) \subset C[\beta_0(T), r_e(T)]$.

(*iii*) *If* $0 \notin \sigma_{ess}(T)$, *then* $\sigma_{ess}(T) \subset C[\beta_0(T), r_e(T)]$.

(*iv*) $\sigma_b(T) \subset \mathbb{D}(0,r_e(T))$.

(*v*) *If* $0 \notin \sigma_b(T)$, *then* $\sigma_b(T) \subset C[\beta_0(T), r_e(T)]$.

Proof. Let $n \in \mathbb{N}^*$ and suppose that $|\lambda|^n > \varphi(T^n)$, then, by Theorem 3.4(*i*), we have $\lambda - T \in \Phi(X)$ and $i(\lambda - T) = 0$. Hence, if $|\lambda| > r_e(T)$, then $\lambda \notin \sigma_{ess}(T)$, this proves (*i*).

Notice that if $\beta(T) = 0$, then $\beta_0(T) = 0$ and the results are all trivial. Suppose that $\beta(T) > 0$. For $|\lambda| < \beta_0(T)$, there exists $n \in \mathbb{N}^*$ such that $|\lambda|^n < \beta(T^n)$. Then, by Theorem 3.4(*i*), we have $\lambda - T \in \Phi_+(X)$ and $i(\lambda - T) = i(T)$. Hence, we get easily (*ii*) and (*iii*).

(*iv*) For $|\lambda| > r_e(T)$, there exists $n \in \mathbb{N}^*$ such that $|\lambda|^n > \varphi(T^n)$. By Theorem 3.4, we have $\lambda - T \in \mathcal{B}(X)$. The result follows since we can choose n arbitrary large.

(*v*) Since $0 \notin \sigma_b(T)$, then $T \in \Phi(X)$ and hence $\beta(T) > 0$. For $|\lambda| < \beta_0(T)$, there exists $n \in \mathbb{N}^*$ such that $|\lambda|^n < \beta(T^n)$. Theorem 3.4 implies that $\lambda - T \in \mathcal{B}(X)$ since $T \in \mathcal{B}(X)$. Q.E.D.

3.3. Weighted shift operators

Let $\omega = (\omega_n)_{n\in\mathbb{N}}$ be a bounded complex sequence. Consider the unilateral backward weighted shift operator $W(\omega,p)$ defined on $X = l_r(\mathbb{N},\mathbb{C})$, $r \geq 1$, by :

$$W(\omega,p)(x_0,x_1,...) = (\omega_p x_p, \omega_{p+1}x_{p+1},...).$$

Lemma 3.7. *If* 0 *is a cluster point of the sequence* $(\omega_n)_n$, *then* $\beta(W(\omega,p)) = 0$.

Proof. By hypothesis, there exists $(\omega_{\rho(n)})_n$ such that $\lim_{n\to+\infty} \omega_{\rho(n)} = 0$. Let $\lambda = (\lambda_n)_n$ be the sequence defined by:

$$\begin{cases} \lambda_{\rho(n)} = \omega_{\rho(n)} \\ \lambda_n = 0 \quad \text{if } n \notin \mathrm{Ran}(\rho) \end{cases}$$

For $n \geq p$, define the operator of finite rank on X:

$$K_n : (x_k)_k \to (\lambda_p x_p, ..., \lambda_n x_n, 0, 0, ...).$$

Since $\|W(\lambda,p) - K_n\| = \sup_{k\geq n} |\lambda_k| \to 0$ when $n \to 0$, then $W(\lambda,p)$ is a compact operator. On the other hand, $\dim N(W(\omega,p) - W(\lambda,p)) = \infty$. Hence, $W(\omega,p) \notin \Phi_+(X)$ and therefore, $\beta(W(\omega,p)) = 0$. Q.E.D.

The following proposition extends the results of [1, Proposition 2.2] where one has shown that

$$\alpha(W(\omega,p)) \leq \limsup_{n\to+\infty} |\omega_n| \text{ and } \beta(W(\omega,p)) \geq \liminf_{n\to+\infty} |\omega_n|. \tag{1}$$

Proposition 3.8. *(i)* $\alpha(W(\omega,p)) = \limsup_{n\to+\infty} |\omega_n|$.

(ii) $\beta(W(\omega,p)) = \liminf_{n\to+\infty} |\omega_n|$.

(iii) $\varphi(W(\omega,p)) = \limsup_{n\to+\infty} |\omega_n|$.

Proof. Consider

$$\omega_+ := \limsup_{n\to+\infty} |\omega_n|,\ \omega_- := \liminf_{n\to+\infty} |\omega_n|.$$

There exists $(\omega_{\rho_+(n)})_n$ and $(\omega_{\rho_-(n)})_n$ such that $|\omega_{\rho_+(n)}| \to \omega_+$ and $|\omega_{\rho_-(n)}| \to \omega_-$ when $n \to +\infty$. Let c_+ (respectively c_-) be a cluster point of $(\omega_{\rho_+(n)})_n$ (respectively $(\omega_{\rho_-(n)})_n$). We have $|c_+| = \omega_+$ and $|c_-| = \omega_-$. There exists $(\omega_{\psi_+(n)})_n$ and $(\omega_{\psi_-(n)})_n$ such that $\omega_{\psi_+(n)} \to c_+$ and $\omega_{\psi_-(n)} \to c_-$ when $n \to +\infty$. Let

$$W_+(\omega,p) = c_+(x_p, x_{p+1}, ...),\ W_-(\omega,p) = c_-(x_p, x_{p+1}, ...).$$

Observe that

$$\alpha(W_+(\omega,p)) = \beta(W_+(\omega,p)) = |c_+| \text{ and } \alpha(W_-(\omega,p)) = \beta(W_-(\omega,p)) = |c_-|.$$

Since, 0 is a cluster point of the sequences $(\omega_n - c_+)_n$ and $(\omega_n - c_-)_n$, then by Lemma 3.7, $\beta(W(\omega,p) - W_+(\omega,p)) = \beta(W(\omega,p) - W_-(\omega,p)) = 0$.

(i) According to [1, Proposition 2.1(vi)], we have

$$\beta(W_+(\omega,p)) - \alpha(W(\omega,p)) \leq \beta(W(\omega,p) - W_+(\omega,p)) = 0$$

which implies that $\alpha(W(\omega,p)) \geq \omega_+$. The result follows from (1).

(ii) Since $\beta(W(\omega,p)) \leq \beta(W(\omega,p) - W_-(\omega,p)) + \alpha(W_-(\omega,p))$, then $\beta(W(\omega,p)) \leq \omega_-$. The result follows from (1).

(iii) Let $S = W(c_+ - \omega, p)$, then $\beta(S) = 0$. We have:

$$\begin{aligned}\varphi(W(\omega,p)) &\geq \beta(W(\omega,p) + S)\\ &\geq \beta(W(c_+,p)) = \alpha(W(\omega,p))\end{aligned}$$

Hence, the result follows since $\alpha(W(\omega,p)) \geq \varphi(W(\omega,p))$. Q.E.D.

As an immediate result from Proposition 3.8, we obtain the following :

Corollary 3.9. *$W(\omega,p) \in \mathcal{F}_+(l_r(\mathbb{N},\mathbb{C}))$ if and only if $W(\omega,p) \in \mathcal{K}(l_r(\mathbb{N},\mathbb{C}))$ if and only if ω converges to 0.*

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