On almost z-supercontinuity

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Abstract. Two new classes of functions between topological spaces have been defined recently, and their basic properties have been studied. They are called almost z-supercontinuous functions and almost D_δ-supercontinuous functions. We consider these two classes of functions from the perspective of changes of topologies. In particular, we show that each of these variants of continuity coincides with the classical notion of continuity when the domain and codomain of the function under consideration have been retopologized appropriately. Some of the consequences of this situation are examined in this paper.

1. Introduction

One of the fundamental ideas in all of mathematics is the notion of continuity. So much so that there has been a movement in recent years to categorize mathematics into two main parts, namely discrete mathematics and continuous mathematics. In topology there have been many variants of continuity considered in the literature. In a recent paper [8], Kohli, Singh and Kumar have introduced two new classes of functions *almost z-supercontinuous functions* and *almost D_δ-supercontinuous functions*. They note, in particular, that these two variants of continuity are "independent of continuity". One of the main purposes of this paper is to advocate exactly the opposite of this point of view. We argue that the distinction made by Kohli, Singh and Kumar [8] between the concepts of almost z-supercontinuity and continuity must be treated very carefully. It is imperative that this distinction be given a very strict interpretation. We claim that almost z-supercontinuity is a disguised form of continuity. In fact, we show that if the domain and codomain spaces of an almost z-supercontinuous function f are retopologized appropriately (see Theorem 4.1), then f is simply a continuous function. What this means is that the alleged new concept is in fact a classical notion in a modified form. To use the language of category theory, we contend that an almost z-supercontinuous function f exists because the wrong source and target have been chosen for the morphism f in the category *Top* whose objects are topological spaces and whose morphisms are continuous functions.

Section 2 considers the basic properties of semi-regular topologies and topologies generated by the collection of all cozero subsets of a topological space. Section 3 provides the relevant definitions of the

classes of functions that are considered in this paper. The change of topology approach is used in Section 4 to reconsider the class of almost z-supercontinuous functions introduced by Kohli, Singh and Kumar [8]. In Section 5 this same perspective of change of topology is focussed on the class of almost D_δ-supercontinuous functions defined by Kohli, Singh and Kumar [8].

The notation and terminology used in this paper are standard, see for example Dugundji [2]. In particular, no separation properties are assumed for topological spaces unless explicitly stated. We denote the interior of a subset A of a topological space (X, τ) by $\tau intA$ (or by $intA$, if there is no possibility of confusion), and the closure of A by τclA (or by clA).

2. Semi-regularization and complete regularization

In a topological space (X, τ) a set A is called τ regular open if $A = \tau int(\tau clA)$ and τ regular closed if $A = \tau cl(\tau intA)$. The collection $RO(X, \tau)$ of all τ regular open sets forms a base for a smaller topology τ_s on X called the semi-regularization of τ. The space (X, τ) is said to be semi-regular if $\tau_s = \tau$. Semi-regularization topologies are considered in some detail by Mršević, Reilly and Vamanamurthy [10], especially from the change of topology perspective. A result that we find useful is that (X, τ) is Hausdorff if and only if (X, τ_s) is Hausdorff [10, Proposition 1].

A subset B of a topological space (X, τ) is called a cozero set if there is a continuous real-valued function g on X such that $B = \{x \in X : g(x) \neq 0\}$. The complement of a cozero set is called a zero set. Since the intersection of two cozero sets is a cozero set, the collection of all cozero subsets of (X, τ) is a base for a topology τ_z on X, called the complete regularization of τ. It is clear that $\tau_z \subset \tau$ in general. Furthermore, the space (X, τ) is completely regular if and only if $\tau_z = \tau$. In general for any topological space (X, τ), we note that (X, τ_z) is completely regular. Thus (X, τ_z) is regular, and hence it is semi-regular. Therefore $(\tau_z)_s = \tau_z$. Now the inclusion $\tau_z \subset \tau$ implies that $(\tau_z)_s \subset \tau_s$. That is, we have $\tau_z \subset \tau_s$, for any topological space (X, τ).

3. Definitions

This section provides a list of definitions of variations of continuity that are relevant to this paper.

A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ between topological spaces is defined to be

(1) almost continuous [16] if for each $x \in X$ and for each regular open set V containing $f(x)$ there is an open set U containing x such that $f(U) \subset V$,

(2) δ-continuous [12] if for each $x \in X$ and for each regular open set V containing $f(x)$ there is a regular open set U containing x such that $f(U) \subset V$,

(3) supercontinuous [11] if for each $x \in X$ and for each open set V containing $f(x)$ there is a regular open set U containing x such that $f(U) \subset V$,

(4) z-continuous [15] if for each $x \in X$ and for each cozero subset V of Y containing $f(x)$ there is an open set U containing x such that $f(U) \subset V$,

(5) z-supercontinuous [5] if for each $x \in X$ and for each open set V of Y containing $f(x)$ there is a cozero subset U of X containing x such that $f(U) \subset V$,

(6) almost z-supercontinuous [8] if for each $x \in X$ and for each open set V of Y containing $f(x)$ there is a cozero subset U of X containing x such that $f(U) \subset int(clV)$.

In 1970 Mack [9] made the following definition. A subset B of a space (X, τ) is called a regular G_δ-set if B is the intersection of a sequence of closed sets in (X, τ) whose interiors contain B, so $B = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} intF_n$, where each F_n is closed in (X, τ). The complement of a regular G_δ-set is called a regular F_σ-set.

Then a function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is defined to be

(7) D_δ-continuous [7] if for each $x \in X$ and for each regular F_σ-subset V of Y containing $f(x)$ there is an open set U containing x such that $f(U) \subset V$.

(8) D_δ-supercontinuous [6] if for each $x \in X$ and for each open set V of Y containing $f(x)$ there is a regular F_σ-set U containing x such that $f(U) \subset V$.

4. Change of topology

The fundamental result, Theorem 4.1 below, shows that a change of topology on each of the domain and co-domain spaces of an almost z-supercontinuous function reduces it to a continuous function. Thus almost z-supercontinuity is not a new concept. It is the classical notion of continuity in disguise. The proof of Theorem 4.1 follows immediately from Proposition 3.1 of Kohli, Singh and Kumar [8].

Theorem 4.1. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a function. Then the following are equivalent:*

(1) $f : (X, \tau) \longrightarrow (Y, \sigma)$ is almost z-supercontinuous,

(2) $f : (X, \tau_z) \longrightarrow (Y, \sigma)$ is almost continuous,

(3) $f : (X, \tau) \longrightarrow (Y, \sigma_s)$ is z-supercontinuous,

(4) $f : (X, \tau_z) \longrightarrow (Y, \sigma_s)$ is continuous.

The equivalence of (1) and (4) in Theorem 4.1 is the fundamental defining characteristic of the class of almost z-supercontinuous functions. It shows that almost z-supercontinuity is a μ-continuity property in the sense of Gauld, Mršević, Reilly and Vamanamurthy [4]. The inclusion $\sigma_s \subset \sigma$, for any topology σ, and the equivalence of (1) and (3) in Theorem 4.1 show that, in general, z-supercontinuity is stronger than almost z-supercontinuity, and that for semi-regular codomains these notions are equivalent. On the other hand, the inclusion $\tau_z \subset \tau$, for any topology τ, and the equivalence of (1) and (2) of Theorem 4.1 indicate that almost continuity is, in general, weaker than almost z-supercontinuity, and that for completely regular domains they are equivalent.

One immediate conclusion from Theorem 4.1 is that each almost z-supercontinuous function is a morphism in the category *Top* where the objects are topological spaces and the morphisms are continuous functions. If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is almost z-supercontinuous, then f is a morphism in *Top* from (X, τ_z) to (Y, σ_s). It is not the case that f lies outside of *Top*. It is the case that the wrong objects in *Top* have been chosen for the source and target of the morphism f in the category *Top*.

The equivalence of (1) and (4) in Theorem 4.1 underlies much of the work in section 3 of Kohli, Singh and Kumar [8], but that is never explicit in their presentation. For example, the results of Theorem 3.2 and Theorem 3.5 of [8] are standard results for continuous functions restated in the setting of almost z-supercontinuous functions.

The equivalence of (1) and (4) in Theorem 4.1 can be used to provide elegant alternative proofs of existing results. Recall that Frolìk [3] defined a topological space (X, τ) to be quasi-compact if every cover of X by cozero sets in (X, τ) has a finite subcover. We observe that (X, τ) is quasi-compact if and only if (X, τ_z) is compact. The class of nearly compact spaces was introduced by Singal and Mathur [14]. A space (X, τ) is called nearly compact if every open cover of X has a finite subfamily such that the interiors of the closures of its members cover X. Carnahan [1, Theorem 4.1] proved that (X, τ) is nearly compact if and only if (X, τ_s) is compact.

Theorem 4.2. ([8, Theorem 4.1]) *If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is an almost z-supercontinuous surjection, and (X, τ) is quasi-compact, then (Y, σ) is nearly compact.*

Proof. By Theorem 4.1, $f : (X, \tau_z) \longrightarrow (Y, \sigma_s)$ is continuous and onto. So (Y, σ_s) is the continuous image of the compact space (X, τ_z), and so is compact. Therefore (Y, σ) is nearly compact. □

Note that Corollary 4.2 of [8] follows immediately, since (Y, σ) is semiregular if and only if $\sigma = \sigma_s$.

We now present two propositions as examples of new results suggested by the equivalence of (1) and (4) of Theorem 4.1.

Proposition 4.3. *Let $f, g : (X, \tau) \longrightarrow (Y, \sigma)$ be almost z-supercontinuous functions and (Y, σ) be Hausdorff. Then $E = \{x \in X : f(x) = g(x)\}$ is closed in (X, τ_z).*

Proposition 4.4. *If* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *is almost z-supercontinuous and* (Y, σ) *is Hausdorff, then* $G(f)$, *the graph of* f, *is closed in* $(X \times Y, \tau_z \times \sigma_s)$.

The equivalence of (1) and (4) of Theorem 4.1 is an especially powerful tool in the discussion of purely mapping properties, for example the composition of functions. First we need to note the following equivalences.

(1) $f : (X, \tau) \longrightarrow (Y, \sigma)$ is δ-continuous if and only if $f : (X, \tau_s) \longrightarrow (Y, \sigma_s)$ is continuous [12, Theorem 2.5].

(2) $f : (X, \tau) \longrightarrow (Y, \sigma)$ is almost continuous if and only if $f : (X, \tau) \longrightarrow (Y, \sigma_s)$ is continuous [10, Proposition 12].

(3) $f : (X, \tau) \longrightarrow (Y, \sigma)$ is supercontinuous if and only if $f : (X, \tau_s) \longrightarrow (Y, \sigma)$ is continuous [11, Theorem 2.1].

(4) $f : (X, \tau) \longrightarrow (Y, \sigma)$ is z-supercontinuous if and only if $f : (X, \tau_z) \longrightarrow (Y, \sigma)$ is continuous [5, Theorem 6.3].

(5) $f : (X, \tau) \longrightarrow (Y, \sigma)$ is z-continuous if and only if $f : (X, \tau) \longrightarrow (Y, \sigma_z)$ is continuous [5, Theorem 6.5(a)].

The change of topology approach allows us to prove the next theorem simply by observing that the composition of two continuous functions is continuous. Proofs going back to first principles are not necessary. We note that (1) and (5) of Theorem 4.5 are Theorems 3.10 and 3.17 of Kohli and Kumar [5] respectively, and that (3) is part of Remark 3.6 of [8].

Theorem 4.5. *Let* $(X, \tau), (Y, \sigma)$ *and* (Z, ψ) *be topological spaces, and* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *and* $g : (Y, \sigma) \longrightarrow (Z, \psi)$ *be functions.*

(1) If f is z-supercontinuous and g is continuous then $g \circ f$ *is z-supercontinuous.*

(2) If f is z-supercontinuous and g is almost continuous then $g \circ f$ *is almost z-supercontinuous.*

(3) If f is almost z-supercontinuous and g is δ-*continuous then* $g \circ f$ *is almost z-supercontinuous.*

(4) If f is almost z-supercontinuous and g is supercontinuous then $g \circ f$ *is z-supercontinuous.*

(5) If f is z-continuous and g is z-supercontinuous then $g \circ f$ *is continuous.*

(6) If f is z-continuous and g is almost z-supercontinuous then $g \circ f$ *is almost continuous.*

5. Almost D_δ-supercontinuous functions

Let (X, τ) be a topological space, and let β denote the collection of all regular F_σ-subsets of (X, τ). The intersection of two regular F_σ-subsets of (X, τ) is a regular F_σ-subset of (X, τ), and hence β is the base of a topology τ^* on X. It is clear that $\tau^* \subset \tau$, (see Kohli and Singh [6]).

Kohli, Singh and Kumar [8, Definition 2.1] made the following definition. The function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is defined to be almost D_δ-supercontinuous if for each $x \in X$ and each σ-open set V containing $f(x)$, there is a regular F_σ-subset of U of (X, τ) containing x and such that $f(U) \subset \sigma int(\sigma cl V)$.

The fundamental characterization of almost D_δ-supercontinuous functions is given by the next theorem. It shows that almost D_δ-supercontinuity is not a new concept, but that it is the classical notion of continuity in disguise. Its proof follows immediately from Proposition 3.1 of Kohli, Singh and Kumar [8].

Theorem 5.1. *If* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *is a function between topological spaces, then the following are equivalent:*

(1) $f : (X, \tau) \longrightarrow (Y, \sigma)$ *is almost* D_δ-*supercontinuous,*

(2) $f : (X, \tau^*) \longrightarrow (Y, \sigma)$ *is almost continuous,*

(3) $f : (X, \tau) \longrightarrow (Y, \sigma_s)$ *is* D_δ-*supercontinuous,*

(4) $f : (X, \tau^*) \longrightarrow (Y, \sigma_s)$ *is continuous.*

It is now clear that we can give a discussion of the properties of the class of almost D_δ-supercontinuous functions exactly parallel to that in Section 4 of the properties of the class of almost z-supercontinuous functions. We shall state some results without providing the proofs, which are entirely analogous to the proofs of the corresponding results in Section 4.

Recall that a space (X, τ) is D_δ-compact [8, Definition 2.5] if every cover of X by regular F_σ-subsets of X has a finite subcover. Hence (X, τ) is D_δ-compact if and only if (X, τ^*) is compact.

Theorem 5.2. ([8, Theorem 4.1]) *If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is an almost D_δ-supercontinuous surjection, and (X, τ) is D_δ-compact, then (Y, σ) is nearly compact.*

Proposition 5.3. *Let $f, g : (X, \tau) \longrightarrow (Y, \sigma)$ be almost D_δ-supercontinuous functions and (Y, σ) be Hausdorff. Then $E = \{x \in X : f(x) = g(x)\}$ is closed in (X, τ^*).*

Proposition 5.4. *If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is almost D_δ-supercontinuous and (Y, σ) is Hausdorff, then $G(f)$, the graph of f, is closed in $(X \times Y, \tau^* \times \sigma_s)$.*

Before considering composition of functions we note the following equivalences.

(1) $f : (X, \tau) \longrightarrow (Y, \sigma)$ is D_δ-continuous if and only if $f : (X, \tau) \longrightarrow (Y, \sigma^*)$ is continuous [6, Theorem 6.3].

(2) $f : (X, \tau) \longrightarrow (Y, \sigma)$ is D_δ-supercontinuous if and only if $f : (X, \tau^*) \longrightarrow (Y, \sigma)$ is continuous [6, Theorem 6.1].

We now state a result entirely analogous to the corresponding result for almost z-supercontinuity, and which is proved by observing that the composition of two continuous functions is continuous. Theorem 5.5 (1) and (5) are Theorems 3.10 and 3.18 of Kohli and Singh [6] respectively.

Theorem 5.5. *Let $(X, \tau), (Y, \sigma)$ and (Z, ψ) be topological spaces, and $f : (X, \tau) \longrightarrow (Y, \sigma)$ and $g : (Y, \sigma) \longrightarrow (Z, \psi)$ be functions.*

(1) If f is D_δ-supercontinuous and g is continuous then $g \circ f$ is D_δ-supercontinuous.

(2) If f is D_δ-supercontinuous and g is almost continuous then $g \circ f$ is almost D_δ-supercontinuous.

(3) If f is almost D_δ-supercontinuous and g is δ-continuous then $g \circ f$ is almost D_δ-supercontinuous.

(4) If f is almost D_δ-supercontinuous and g is supercontinuous then $g \circ f$ is D_δ-supercontinuous.

(5) If f is D_δ-continuous and g is D_δ-supercontinuous then $g \circ f$ is continuous.

(6) If f is D_δ-continuous and g is almost D_δ-supercontinuous then $g \circ f$ is almost continuous.

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