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On strongly conjugable extensions of hypergroups of type U with scalar identity

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Abstract. Let $\mathfrak{S}_n$ denote the class of hypergroups of type U on the right of size n with bilateral scalar identity. In this paper we consider the hypergroups $(H, \circ) \in \mathfrak{S}_7$ which own a proper and non-trivial subhypergroup h. For these hypergroups we prove that h is closed if and only if $(H - h) \circ (H - h) = h$. Moreover we consider the set $\mathfrak{E}_7$ of hypergroups in $\mathfrak{S}_7$ that own the above property. On this set, we introduce a partial ordering induced by the inclusion of hyperproducts. This partial ordering allows us to give a complete characterization of hypergroups in $\mathfrak{E}_7$ on the basis of a small set of minimal hypergroups, up to isomorphisms. This analysis gives a partial (negative) answer to a problem raised in [5] concerning the existence in $\mathfrak{S}_n$ of proper hypergroups having singletons as special hyperproducts.

1. Introduction

Hypergroups of type U on the right were introduced in [12] to analyze properties of quotient hypergroups H/h of a hypergroup H with respect to a subhypergroup $h \subseteq H$ ultraclosed on the right. The class of hypergroups of type U on the right is rather wide and rich in results [2, 4–6, 12, 14, 16], since it includes that of hypergroups of type C on the right [15, 16, 21] and, in particular, that of cogroups [3, 9, 17, 18] and that of quotient hypergroups G/g of a group G with respect to a non-normal subgroup $g \subseteq G$ (D-hypergroups) [9, 17, 18].

Also recently, several authors have studied diverse problems concerning existence and classification of hyperstructures, see e.g., [1, 2, 4–7, 10, 11, 13, 19, 20]. For example, hypergroups of $\mathfrak{S}_5$ have been classified in [5, 6], where $\mathfrak{S}_n$ denotes the class of hypergroups of type U on the right of size n with bilateral scalar identity. In particular, in [5] the authors proved that if $(H, \circ)$ belongs to $\mathfrak{S}_5$ and owns the identity ε then the following two properties are verified:

1. $(H, \circ)$ *is a group if and only if there exist* $x, y \in H - \{\varepsilon\}$ *such that the hyperproduct* $x \circ y$ *is a singleton.*
2. *If* $(H, \circ)$ *is not a group then, for every* $x, y \in H - \{\varepsilon\}$*, we have* $|x \circ y| \geq 3$ *and* $\varepsilon \in x \circ y$*;*

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Moreover, for every integer $k \geq 1$, they provided an example of hypergroup $(H, \circ)$ of class $\mathfrak{S}_{2k}$ that satisfies the following conditions:

$$(H, \circ) \text{ is not a group;} \tag{1}$$

$$\exists\, x, y \in H - \{\varepsilon\} \text{ such that } |x \circ y| = 1. \tag{2}$$

Now, since any hypergroup in $\mathfrak{S}_3$ is isomorphic to the group $\mathbb{Z}_3$, the problem to determine the minimum odd integer n such that there exists a hypergroup $(H, \circ) \in \mathfrak{S}_n$, satisfying (1) and (2) arises. This minimum integer n is 7 or 9. In fact, in this paper we describe a hypergroup in $\mathfrak{S}_9$ that satisfies the required conditions. From experimental attempts performed by means of a computer program described in [6], which generates tables of finite hypergroups, we have seen that the problem cannot be solved by a brute force approach in $\mathfrak{S}_7$, due to the huge computational cost. For this reason, we are motivated to study theoretical properties of hypergroups in $\mathfrak{S}_7$. In particular, we prove that such hypergroups own at most two proper and non-trivial subhypergroups. Moreover, when the hypergroup $(H, \circ)$ owns two proper and non-trivial subhypergroups, then $|x \circ y| > 1$, for every $x, y \in H - \{\varepsilon\}$, hence the condition (2) cannot be fulfilled. This result suggests to distinguish the hypergroups in $\mathfrak{S}_7$ according to the following three conditions:

1. $(H, \circ)$ owns only one closed, proper and non-trivial subhypergroup;
2. $(H, \circ)$ owns only one proper and non-trivial subhypergroup which is not closed;
3. $(H, \circ)$ does not own any proper and non-trivial subhypergroup.

In this paper we perform a complete analysis of the first case. Next section introduces some basic definitions and notations to be used throughout the paper. In Section 3, we recall some properties of closed subhypergroups and introduce the notion of strongly conjugable extension, whereupon we deduce necessary and sufficient conditions so that a hypergroup is a strongly conjugable extension. Section 4 contains the main results about subhypergroups of hypergroups in $\mathfrak{S}_7$. In particular we prove that every hypergroup $(H, \circ) \in \mathfrak{S}_7$ can own at most two proper and non-trivial subhypergroups. The closure on the left (resp., on the right) of a subhypergroup h in $(H, \circ)$ is a necessary and sufficient condition so that $(H, \circ)$ is a strongly conjugable extension of h. Moreover, if $(H, \circ)$ is a strongly conjugable extension, then $(H, \circ)$ owns only one closed subhypergroup. In Section 5 we consider, unless isomorphisms, the set $\mathfrak{E}_7$ of hypergroups in $\mathfrak{S}_7$ which are strongly conjugable extensions. We prove some properties about the elements and the size of hyperproducts of hypergroups in $\mathfrak{E}_7$. In addition, we define a partial ordering that allows us to give a complete characterization of hypergroups in $\mathfrak{E}_7$ using a set of minimal hypergroups. Finally, with the help of symbolic computation software, in last section we show that there are 182 minimal tables in $\mathfrak{E}_7$, up to isomorphisms.

In conclusion, we observe that the problem at the basis of our present work, namely, to establish if there exists a hypergroup $(H, \circ) \in \mathfrak{S}_7$ fulfilling conditions (1) and (2), remains still unsolved. By the way, that problem is now circumscribed to the analysis of the last two subclasses of $\mathfrak{S}_7$, that is, either $(H, \circ)$ owns only one proper and non-trivial subhypergroup which is not closed, or $(H, \circ)$ does not own any proper and non-trivial subhypergroup.

Remark 1.1. Throughout the paper, we will often show hyperproduct tables of hypergroups. These tables are usually obtained after long arguments that are aimed at proving the existence of hypergroups having certain properties. We inform the reader that, after these tables are obtained, we always check their associativity, possibly by means of computer routines as those described in [6]. Hence, the corresponding hypergroups are correctly defined, even if this is not always explicitly stated.

2. Basic definitions and results

A *hypergroupoid* is a nonempty set H endowed by a *hyperproduct*, that is, a mapping $\circ : H \times H \mapsto \wp^*(H)$, where $\wp^*(H)$ denotes the family of nonempty subsets of H. A *hypergroup* is a hypergroupoid $(H, \circ)$ whose hyperproduct is associative and fulfills the *reproducibility axiom*

$$\forall x \in H, \quad x \circ H = H \circ x = H. \tag{3}$$

A non-empty subset h of a hypergroup $(H, \circ)$ is called a *subhypergroup* of H if $x \circ h = h \circ x = h$, for all $x \in h$.
A subhypergroup h of a hypergroup $(H, \circ)$ is said

- *proper* if $H \neq h$;
- *closed on the right* (resp., *on the left*) if $h \circ (H - h) = H - h$ (resp., $(H - h) \circ h = H - h$);
- *closed* if $h \circ (H - h) = (H - h) \circ h = H - h$;
- *invertible on the right* (resp., *on the left*) if for every $x, y \in H$, $x \in y \circ h \Rightarrow y \in x \circ h$ (resp., $x \in h \circ y \Rightarrow y \in h \circ x$);
- *invertible* if it is invertible on the right and on the left;
- *invariant or normal* if $x \circ h = h \circ x$, for every $x \in H$;
- *conjugable* if it is closed and for every $x \in H$ there exists an element $y \in H$ such that $x \circ y \subseteq h$.

We recall that if h is a conjugable subhypergroup, then h is invertible. Moreover, if h is invertible on the right, then the family $\{x \circ h\}_{x \in H}$ is a partition of H and the quotient H/h is hypergroup under the hyperproduct

$$(x \circ h) \otimes (y \circ h) = \{z \circ h \mid z \in x \circ h \circ y \circ h\}.$$

If a hypergroup $(H, \circ)$ contains an element ε with the property that, for all $x \in H$, one has $x \in x \circ \varepsilon$ (resp., $x \in \varepsilon \circ x$), then we say that ε is a *right identity* (resp., *left identity*) of H. If $x \circ \varepsilon = \{x\}$ (resp., $\varepsilon \circ x = \{x\}$), for all $x \in H$, then ε is a *right scalar identity* (resp., *left scalar identity*). The element ε is said to be an *identity* (resp., *scalar identity or bilateral scalar identity*), if it is both right and left identity (resp., right and left scalar identity).
A hypergroup $(H, \circ)$ is said to be of *type U on the right* [2, 4, 10–12] if there exists an element ε which fulfills the following axioms:

$$\forall x \in H, \qquad x \circ \varepsilon = \{x\} \tag{4}$$

$$\forall x, y \in H, \qquad x \in x \circ y \Longrightarrow y = \varepsilon. \tag{5}$$

In this paper we denote by $\mathfrak{U}^1$ the class of hypergroups of type U on the right in which the right scalar identity is also left (not necessarily scalar) identity. Moreover, by $\mathfrak{S}$ we mark the subclass of hypergroups of type U on the right with bilateral scalar identity. For the sake of brevity these hypergroups are said to be *hypergroups with a scalar identity*. In the finite case, $\mathfrak{S}_n$ denotes the subclass of all hypergroups of size n with scalar identity.

For reader's convenience, we collect in the following lemma some preliminary results from [4, 10]:

Lemma 2.1. *Let $(H, \circ)$ be a hypergroup of type U on the right with right scalar identity ε. Then*

1. *if h is a subhypergroup of $(H, \circ)$, then we have $\varepsilon \in h$. Moreover, if $(H, \circ) \in \mathfrak{U}^1$ (resp., $(H, \circ) \in \mathfrak{S}$), then also $(h, \circ) \in \mathfrak{U}^1$ (resp., $(h, \circ) \in \mathfrak{S}$);*
2. *if $(H, \circ) \in \mathfrak{U}^1$, for all $x, y \in H$, we have $\varepsilon \in x \circ y \iff \varepsilon \in y \circ x$;*
3. *if $(H, \circ)$ is finite and G is a subgroup of $(H, \circ)$, then $|G|$ divides $|H|$.*

3. Strongly conjugable extensions

In this section we recall some properties of subhypergroups of a hypergroup that are closed on the left or on the right and introduce the notion of strongly conjugable extension. Moreover we find some necessary and sufficient conditions so that a hypergroup is a strongly conjugable extension and show a construction of hypergroups in the class $\mathfrak{U}^1$; under certain additional conditions, these hypergroups belong to the class $\mathfrak{S}_n$ and are strongly conjugable extensions.

We begin to observe that if h is a subhypergroup of a hypergroup $(H, \circ)$, it can occur that h is not a subset of $(H - h) \circ (H - h)$. An easy example is obtained by considering the set $H = \{a, b, c\}$ endowed with the hyperproduct represented by the table

$\circ$	a	b	c
a	$\{a\}$	H	H
b	H	$\{b,c\}$	$\{b,c\}$
c	H	$\{b,c\}$	$\{b,c\}$

.

In fact, $(H,\circ)$ is a hypergroup and $h = \{a\} \not\subset (H-h)\circ(H-h) = \{b,c\}$. The inclusion $h \subseteq (H-h)\circ(H-h)$ is verified when, for example, h is closed on the left or on the right. In fact, for reproducibility, for every $a \in h$ there exists an element $y \in H$ such that $a \in (H-h)\circ y$. Obviously $y \in H-h$, or else we have the contradiction $a \in (H-h)\circ h = H-h$. Therefore $a \in (H-h)\circ(H-h)$ and $h \subseteq (H-h)\circ(H-h)$.

Moreover we observe that the inclusion $h \subseteq (H-h)\circ(H-h)$ is a necessary but not sufficient condition so that h is closed on the left or on the right. In fact, the set $H = \{a,b,c\}$ endowed with the hyperproduct

$\circ$	a	b	c
a	$\{a\}$	H	H
b	H	H	$\{b,c\}$
c	H	$\{b,c\}$	H

is a hypergroup. The subset $h = \{a\}$ is a subhypergroup of H such that $h \subset (H-h)\circ(H-h) = H$ but h is neither closed on the left nor on the right.

Now we study the main properties of subhypergroups h which satisfy the equality $(H-h)\circ(H-h) = h$. We give the following

Definition 3.1. Let $(H,\circ)$ be a hypergroup and h a subhypergroup of H. We say that $(H,\circ)$ is a *strongly conjugable extension of* h if h is proper and $(H-h)\circ(H-h) = h$.

Lemma 3.1. *Let h be a subhypergroup of a hypergroup $(H,\circ)$. If $(H,\circ)$ is a strongly conjugable extension of h, then h is closed on the left in H if and only if h is closed on the right in H.*

Proof. The claim is an obvious consequence of the fact that

$$(H-h)\circ h = (H-h)\circ(H-h)\circ(H-h) = h\circ(H-h).$$

□

Proposition 3.1. *Let $(H,\circ)$ be a strongly conjugable extension of h. Then*

1. *h is closed in H;*
2. *$(H-h)\circ y = y\circ(H-h) = h$, for every $y \in H-h$;*
3. *$h\circ y = y\circ h = H-h$, for every $y \in H-h$.*

Proof. 1. If $h \cap (H-h)\circ h \neq \emptyset$, there exist $a, b \in h$ and $x \in H-h$ such that $b \in x\circ a$. Therefore $(H-h)\circ b \subseteq (H-h)\circ x\circ a \subseteq (H-h)\circ(H-h)\circ a = h\circ a = h$, whence we obtain the contradiction

$$H = H\circ b = [(H-h)\cup h]\circ b = [(H-h)\circ b]\cup h\circ b \subseteq h\cup h = h.$$

Hence $h \cap (H-h)\circ h = \emptyset$. Moreover, since

$$H = H\circ h = [(H-h)\cup h]\circ h = [(H-h)\circ h]\cup h\circ h = [(H-h)\circ h]\cup h$$

and $H = (H-h)\cup h$, we obtain $(H-h)\circ h = H-h$. Thus h is closed on the left and, for Lemma 3.1, it is closed in H.

2. For every $a \in h$ and $y \in H-h$, there exists $b \in H$ such that $a \in b\circ y$. From item 1., if $b \in h$ then we have the contradiction $a \in h\circ(H-h) = H-h$. Hence, $b \in H-h$ and $a \in (H-h)\circ y$. Consequently, $h \subseteq (H-h)\circ y \subseteq (H-h)\circ(H-h) = h$ and $(H-h)\circ y = h$. Analogously we can prove that $y\circ(H-h) = h$.

3. For every $x, y \in H-h$, there exists $a \in H$ such that $x \in a\circ y$. Since H is a strongly conjugable extension of h, if $a \in H-h$ then we have the contradiction $x \in a\circ y \subseteq (H-h)\circ(H-h) = h$. Therefore $a \in h$ and $x \in h\circ y$. So we have the inclusion $H-h \subseteq h\circ y$. Finally, since h is closed in H, we obtain that $H-h \subseteq h\circ y \subseteq h\circ(H-h) = H-h$, that is $h\circ y = H-h$. In the same way we can prove that $y\circ h = H-h$. □

As an immediate consequence of Proposition 3.1 we have the following

Corollary 3.1. *Let h be a subhypergroup of a hypergroup $(H, \circ)$. If $(H, \circ)$ is a strongly conjugable extension of h, then h is conjugable and invariant.*

The next proposition characterizes the strongly conjugable extensions by means of a theorem of Dresher-Ore. We recall the statement from [8]: *If h is a subhypergroup of a hypergroup $(H, \circ)$, the quotient $(H/h, \otimes)$ is a group if and only if h is a conjugable and invariant subhypergroup.*

Proposition 3.2. *Let h be a subhypergroup of a hypergroup $(H, \circ)$; the following conditions are equivalent:*

1. *$(H, \circ)$ is a strongly conjugable extension of h;*
2. *The quotient H/h is isomorphic to the group $\mathbb{Z}_2$.*

Proof. We begin to prove that 1. $\Rightarrow$ 2. From Corollary 3.1, h is a conjugable and invariant subhypergroup, and so $(H/h, \otimes)$ is a group. By item 3. of Proposition 3.1, h and $H - h$ are the classes of $(H/h, \otimes)$. Therefore the quotient H/h is isomorphic to the group $\mathbb{Z}_2$.

Now we prove that 2. $\Rightarrow$ 1. The subhypergroup h is conjugable because H/h is a group. Thus h is closed in H and $x \circ h \subseteq H - h$, for every $x \in H - h$. As H/h is isomorphic to $\mathbb{Z}_2$, we obtain $x \circ h = H - h$. Therefore it results $x \in x \circ h$ for every $x \in H$. Hence, if there exist $x, y \in H - h$ such that $x \circ y \cap (H - h) \neq \emptyset$ then we have $x \circ h \circ y \circ h \cap (H - h) \neq \emptyset$ and $(x \circ h) \otimes (y \circ h) \neq h$, a contradiction. So $[(H - h) \circ (H - h)] \cap (H - h) = \emptyset$ and obviously $(H - h) \circ (H - h) = h$. $\square$

The next example shows a construction of hypergroups in the class $\mathfrak{U}^1$. In some cases these hypergroups are strongly conjugable extensions:

Example 3.1. Let $(G, \cdot)$ be a group with identity 1 and let $\{A_g\}_{g \in G}$ be a set family such that the following conditions are verified:

(*I*) $|A_g| \geq 3$, for every $g \in G$;

(*II*) $(A_1, \circ)$ is a hypergroup of class $\mathfrak{U}^1$ with identity ε such that for every $a, b \in A_1 - \{\varepsilon\}$ we have (*i*) $\varepsilon \in a \circ b$ and (*ii*) $|a \circ b| \geq 2$.

Hence, we can define on $H = \bigcup_{g \in G} A_g$ the following hyperproduct:

$$x \bullet y = \begin{cases} x \circ y & \text{if } x, y \in A_1; \\ \{x\} & \text{if } x \in H - A_1,\ y = \varepsilon; \\ \{y\} & \text{if } x = \varepsilon,\ y \in H - A_1; \\ A_g - \{x\} & \text{if } x \in A_g,\ y \in A_1 - \{\varepsilon\},\ g \neq 1; \\ A_g & \text{if } x \in A_1 - \{\varepsilon\},\ y \in A_g,\ g \neq 1; \\ A_{gg'} & \text{if } x \in A_g,\ y \in A_{g'},\ g, g' \in G - \{1\}. \end{cases} \tag{6}$$

The set H, equipped with the hyperoperation $\bullet$, is a hypergroup of class $\mathfrak{U}^1$ with right scalar identity ε. We omit to verify reproducibility and associativity, with the exception of the case $(z \bullet x) \bullet y = z \bullet (x \bullet y)$, with $\{x, y\} \subseteq A_1 - \{\epsilon\}$ and $z \in A_g \neq A_1$, because it involves all the hypotheses given in (*I*), (*i*) and (*ii*). In fact as $|A_g| \geq 3$ we obtain

$$(z \bullet x) \bullet y = (A_g - \{z\}) \bullet y = \bigcup_{w \in A_g - \{z\}} w \bullet y = \bigcup_{w \in A_g - \{z\}} (A_g - \{w\}) = A_g.$$

Moreover, by (*i*) and (*ii*), we have that

$$z \bullet (x \bullet y) = z \bullet (x \circ y) = (z \bullet \varepsilon) \cup [z \bullet (x \circ y - \{\varepsilon\})] = \{z\} \cup (A_g - \{z\}) = A_g.$$

Furthermore, we note that the hypergroup $(H, \bullet)$ satisfies the following properties of which we omit the proofs:

1. If $(A_1, \circ) \in \mathfrak{S}$, then also $(H, \bullet) \in \mathfrak{S}$;
2. If G' is a subgroup of G, then the set $h = \bigcup_{g \in G'} A_g$ is a conjugable subhypergroup of $(H, \bullet)$;
3. If G' is a normal subgroup of G, then the subgroup $h = \bigcup_{g \in G'} A_g$ is invariant in $(H, \bullet)$;
4. If $|G| > 2$ and G' is a proper subgroup of G, then $(H, \bullet)$ is a strongly conjugable extension of $h = \bigcup_{g \in G'} A_g$ if and only if $[G : G'] = 2$. In particular, if $|G| = 2$, then $(H, \bullet)$ is a strongly conjugable extension of A_1.

4. The class $\mathfrak{S}_7$

In [5] the authors proved that a hypergroup $(H, \circ) \in \mathfrak{S}_5$ is a group if and only if there exists at least a pair (x, y) of elements not equal to the identity such that $|x \circ y| = 1$. Thus they have raised the problem to determine the minimum odd integer n such that there exists a hypergroup $(H, \circ) \in \mathfrak{S}_n$ fulfilling the conditions (1) and (2). For that minimum integer n it holds $5 < n \leq 15$. In fact, if $(H, \circ) \in \mathfrak{S}_5$ is not a group, then the direct product $H \times \mathbb{Z}_3$ is a proper hypergroup that satisfies the required conditions. Actually, we can restrict the possibilities only to $n = 7$ or $n = 9$. The following example shows a hypergroup in $\mathfrak{S}_9$ fulfilling (1) and (2).

Example 4.1. Let $(G, \cdot)$ be a group with identity 1 and let g be a normal, proper and non-trivial subgroup of $(G, \cdot)$. We define on G the following hyperproduct:

$$x \circ y = \begin{cases} \{xy\} & \text{if } x \in g \text{ or } y \in g; \\ xyg & \text{otherwise.} \end{cases} \tag{7}$$

Then, $(G, \circ)$ is a hypergroup of $\mathfrak{S}$ and g is a subgroup of $(G, \circ)$. We remark that the normality of g in $(G, \cdot)$ is exploited to prove $(x \circ y) \circ z = x \circ (y \circ z)$ when $x, y, z \in G - g$.

Next table shows a hypergroup $(H, \circ) \in \mathfrak{S}_9$ which is isomorphic to the hypergroup arising from the previous construction when $G = \mathbb{Z}_9$ and g is its subgroup having order 3. Using the notations $H = \{1, 2, \dots 9\}$, $h = \{1, 2, 3\}$, $A = \{4, 5, 6\}$ and $B = \{7, 8, 9\}$ we have:

$\circ$	1	2	3	4	5	6	7	8	9
1	{1}	{2}	{3}	{4}	{5}	{6}	{7}	{8}	{9}
2	{2}	{3}	{1}	{5}	{6}	{4}	{8}	{9}	{7}
3	{3}	{1}	{2}	{6}	{4}	{5}	{9}	{7}	{8}
4	{4}	{5}	{6}	B	B	B	h	h	h
5	{5}	{6}	{4}	B	B	B	h	h	h
6	{6}	{4}	{5}	B	B	B	h	h	h
7	{7}	{8}	{9}	h	h	h	A	A	A
8	{8}	{9}	{7}	h	h	h	A	A	A
9	{9}	{7}	{8}	h	h	h	A	A	A

(8)

Obviously, the hypergroup $(H, \circ) \in \mathfrak{S}_9$ fulfills conditions (1) and (2).

Motivated by the initial problem, in this section we study the main properties of hypergroups in $\mathfrak{S}_7$. In particular, in this section we show that all proper and non-trivial subhypergroups h of a hypergroup $(H, \circ)$ in $\mathfrak{S}_7$ are isomorphic to a certain hypergroup, denoted by B_4 in what follows. Moreover we prove that a hypergroup in $\mathfrak{S}_7$ can own at most two proper and non-trivial subhypergroups. Finally we prove that the closure on the left or on the right of h is a necessary and sufficient condition so that H is strongly conjugable extension of h. In this case h is the only proper and non-trivial subhypergroup of H.

We begin to recall that the hypergroups of type U on the right of size 4 have been classified in [14], unless isomorphisms, and so we can affirm that the hypergroups in $\mathfrak{S}_n$, with $2 \leq n \leq 4$, are the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2$ and the following two hypergroups:

H_4:

$\circ$	ε	b	c	d
ε	$\{\varepsilon\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{b\}$	$\{\varepsilon\}$	$\{d\}$	$\{c\}$
c	$\{c\}$	$\{d\}$	$\{\varepsilon, b\}$	$\{\varepsilon, b\}$
d	$\{d\}$	$\{c\}$	$\{\varepsilon, b\}$	$\{\varepsilon, b\}$

B_4:

$\circ$	ε	b	c	d
ε	$\{\varepsilon\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{b\}$	$\{\varepsilon, c, d\}$	$\{\varepsilon, c, d\}$	$\{\varepsilon, c, d\}$
c	$\{c\}$	$\{\epsilon, b, d\}$	$\{\varepsilon, b, d\}$	$\{\varepsilon, b, d\}$
d	$\{d\}$	$\{\varepsilon, b, c\}$	$\{\varepsilon, b, c\}$	$\{\varepsilon, b, c\}$

By means of these hypergroups and Lemma 2.1, we can characterize the subhypergroups of the hypergroups in the class $\mathfrak{S}_7$. If $(H, \circ) \in \mathfrak{U}^1$, then the subhypergroup $h = \{\varepsilon\}$ is said to be *trivial*.

Theorem 4.1. *Let $(H, \circ) \in \mathfrak{S}_7$ and let h be a proper and non-trivial subhypergroup. Then, h is isomorphic to B_4.*

Proof. From Lemma 2.1, we have that $(h, \circ) \in \mathfrak{S}$ and, moreover, h is not isomorphic to $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ because $|h|$ does not divide $|H| = 7$. For the same reason h is not isomorphic to H_4 or else there is a subgroup of size 2 in $(H, \circ)$. So $|h| \in \{5, 6\}$ or $h \cong B_4$.

Let $|h| = 6$ and $H = h \cup \{x\}$. For every element $a \in h - \{\varepsilon\}$, we have $x \in H = H \circ a = (h \cup \{x\}) \circ a = h \circ a \cup x \circ a$. Therefore $x \in x \circ a$ and $a = \varepsilon$, that is absurd.

Let $|h| = 5$ and $H = h \cup \{x, y\}$. For every element $a \in h - \{\varepsilon\}$, we have $H = H \circ a = (h \cup \{x, y\}) \circ a = h \circ a \cup x \circ a \cup y \circ a = h \cup x \circ a \cup y \circ a$. Thus, since $x \notin x \circ a$ and $y \notin y \circ a$, we have

$$\forall a \in h - \{\varepsilon\}, \quad y \in x \circ a, \quad x \in y \circ a. \tag{9}$$

Now, taking an element $b \in h - \{\varepsilon\}$, we have $b \circ b \neq \{\varepsilon\}$ otherwise $S = \{\varepsilon, b\}$ is a subgroup of $(H, \circ)$. Therefore, since $b \notin b \circ b$, there exists an element $c \in h - \{\varepsilon, b\}$ such that $c \in b \circ b$. By (9), we have that $y \in x \circ b \cap x \circ c$. Moreover we can put $x \circ b = \{y\} \cup A$, with $A \subseteq h$. Therefore $y \in x \circ c \subseteq x \circ (b \circ b) = (x \circ b) \circ b = (\{y\} \cup A) \circ b = y \circ b \cup A \circ b$. So, as $A \circ b \subseteq h$, we have $y \in y \circ b$ and $b = \varepsilon$, that is impossible.

Thus $|h| \notin \{5, 6\}$ and $h \cong B_4$. □

Corollary 4.1. *Let h, k be proper and non-trivial subhypergroups of a hypergroup $(H, \circ) \in \mathfrak{S}_7$. Then either $h \cap k = \{\varepsilon\}$ or $h = k$.*

Proof. By item 1. of Lemma 2.1, we have $\varepsilon \in h \cap k$. For this reason $h \cap k = \{\varepsilon\}$ or there exists an element $x \in (h \cap k) - \{\varepsilon\}$. In the latter case, by Theorem 4.1, considering the table of B_4, we obtain $h = x \circ x \circ x = k$. □

Corollary 4.2. *Every hypergroup $(H, \circ) \in \mathfrak{S}_7$ owns at most two proper and non-trivial subhypergroups.*

Proof. If we suppose that $(H, \circ)$ owns three proper and non-trivial subhypergroups h_1, h_2, h_3, then, from Theorem 4.1 and Corollary 4.1, we obtain $|h_1| = |h_2| = |h_3| = 4$ and $h_1 \cap h_2 = h_1 \cap h_3 = h_2 \cap h_3 = \{\varepsilon\}$. This fact leds to $|H| \geq 10$, that is a contradiction. □

Corollary 4.3. *Let $(H, \circ) \in \mathfrak{S}_7$ be a hypergroup having two proper and non-trivial subhypergroups. Then $|x \circ y| > 1$ for all $x, y \in H - \{\varepsilon\}$.*

Proof. Let h and k be two distinct proper and non-trivial subhypergroups. By absurd, we suppose that there exist $x, y \in H - \{\varepsilon\}$ such that $|x \circ y| = 1$. From Theorem 4.1 and Corollary 4.1, we have $h \cong B_4 \cong k$ and $h \cap k = \{\varepsilon\}$. Obviously, we must have $H = h \cup k = (h - \{\varepsilon\}) \cup k$. Since $|x \circ y| = 1$, we can assume that $x \in h - \{\varepsilon\}$, $y \in k - \{\varepsilon\}$ and $h - \{\varepsilon\} = \{x, z, w\}$. Consequently $(x \circ x) \circ y = \{\varepsilon, z, w\} \circ y = \{y\} \cup z \circ y \cup w \circ y$. Now, by reproducibility of k and H, we have $H = H \circ y = [(h - \{\varepsilon\}) \cup k] \circ y = [(h - \{\varepsilon\}) \circ y] \cup k \circ y = x \circ y \cup z \circ y \cup w \circ y \cup k$. Since $x \notin (x \circ y) \cup k$ we have $x \in z \circ y \cup w \circ y \subset (x \circ x) \circ y = x \circ (x \circ y)$. Finally, from axiom (5), we have $x \circ y = \{\varepsilon\}$ and the contradiction $y \in (x \circ x) \circ y = x \circ (x \circ y) = x \circ \varepsilon = \{x\}$. □

Next table shows an example of hypergroup in $\mathfrak{S}_7$ with two proper and non-trivial subhypergroups:

$\circ$	1	2	3	4	5	6	7
1	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$
2	$\{2\}$	$\{1,3,4\}$	$\{1,3,4\}$	$\{1,3,4\}$	$H-\{2\}$	$H-\{2\}$	$H-\{2\}$
3	$\{3\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	$H-\{3\}$	$H-\{3\}$	$H-\{3\}$
4	$\{4\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$H-\{4\}$	$H-\{4\}$	$H-\{4\}$
5	$\{5\}$	$H-\{5\}$	$H-\{5\}$	$H-\{5\}$	$\{1,6,7\}$	$\{1,6,7\}$	$\{1,6,7\}$
6	$\{6\}$	$H-\{6\}$	$H-\{6\}$	$H-\{6\}$	$\{1,5,7\}$	$\{1,5,7\}$	$\{1,5,7\}$
7	$\{7\}$	$H-\{7\}$	$H-\{7\}$	$H-\{7\}$	$\{1,5,6\}$	$\{1,5,6\}$	$\{1,5,6\}$

The aforesaid subhypergroups are $h = \{1,2,3,4\}$ and $k = \{1,5,6,7\}$. Obviously, in line with Corollary 4.1 and Corollary 4.3, it results $h \cap k = \{1\}$ and $|x \circ y| > 1$, for all $x, y \in H - \{1\}$.

4.1. Strongly conjugable extensions in $\mathfrak{S}_7$

In this section we study the case where a hypergroup $(H, \circ) \in \mathfrak{S}_7$ owns a closed, proper and non-trivial subhypergroup h. In this case, we will prove that $(H, \circ)$ is a strongly conjugable extension of h. We premise a proposition which is true for hypergroups in the class $\mathfrak{U}^1$.

Proposition 4.1. *If $(H, \circ) \in \mathfrak{U}^1$ and h is a subhypergroup of $(H, \circ)$, then the following conditions are equivalent:*

1. *h is closed in $(H, \circ)$;*
2. *h is closed on the left in $(H, \circ)$;*
3. *h is closed on the right in $(H, \circ)$.*

Proof. The implications 1. ⇒ 2. and 1. ⇒ 3. are obvious. So we prove that 2. ⇒ 3., whence 2. ⇒ 1. By hypothesis $(H - h) \circ h = H - h$. If $h \circ (H - h) \cap h \neq \emptyset$ then there exist a pair (a, b) of elements in h and an element $x \in H - h$ such that $a \in b \circ x$. Moreover, from the reproducibility of h, there exists $c \in h$ such that $\varepsilon \in c \circ a$. Thus we have $\varepsilon \in c \circ a \subseteq c \circ (b \circ x) = (c \circ b) \circ x$ and consequently there exists $d \in c \circ b \subseteq h$ such that $\varepsilon \in d \circ x$. Therefore, by Lemma 2.1, we obtain the contradiction $\varepsilon \in x \circ d \subseteq (H - h) \circ h = H - h$. So it must be $h \circ (H-h) \cap h = \emptyset = (H-h) \cap h$. Finally, given that $H = h \circ H = h \circ [(H - h) \cup h] = [h \circ (H-h)] \cup h \circ h = [h \circ (H-h)] \cup h$ and $H = (H - h) \cup h$, we obtain $h \circ (H - h) = H - h$.

The proof of the implication 3. ⇒ 2., whence 3. ⇒ 1., is analogous. □

Lemma 4.1. *Let $(H, \circ) \in \mathfrak{S}_7$ and let h be a closed, proper and non-trivial subhypergroup of $(H, \circ)$. Then $x \circ b = H - (h \cup \{x\})$, for all $x \in H - h$ and $b \in h - \{\varepsilon\}$.*

Proof. By Theorem 4.1, we have $h \cong B_4$. Moreover, for every $b \in h - \{\varepsilon\}$ and $x \in H - h$, we have $|x \circ b| \in \{1, 2\}$. In fact $x \circ b \subset (H - h) \circ h = H - h$, $x \notin x \circ b$ and $|H - h| = 3$. Now, since $h \cong B_4$, if $x \circ b = \{y\}$ we obtain

$$y \circ b = (x \circ b) \circ b = x \circ (b \circ b) = x \circ (h - \{b\}) = \{x\} \cup x \circ (h - \{\varepsilon, b\}).$$

Thus, setting $h - \{\varepsilon, b\} = \{c, d\}$ and $H - h = \{x, y, z\}$, we have $x \circ c = x \circ d = \{z\}$ because $y \notin y \circ b$ e $x \notin x \circ (h - \{\varepsilon, b\})$. Therefore we obtain $z \in x \circ d \subseteq x \circ (c \circ c) = (x \circ c) \circ c = z \circ c$ and $c = \varepsilon$, that is impossible. Finally $|x \circ b| = 2$ and $x \circ b = H - (h \cup \{x\})$. □

Corollary 4.4. *Let $(H, \circ) \in \mathfrak{S}_7$ and let h be a proper and non-trivial subhypergroup of $(H, \circ)$. Then the following conditions are equivalent:*

1. *h is closed in $(H, \circ)$;*
2. *h is closed on the right in $(H, \circ)$;*
3. *h is closed on the left in $(H, \circ)$;*
4. *$(H, \circ)$ is a strongly conjugable extension of h.*

Proof. The equivalence among the items 1., 2. and 3. follows from Proposition 4.1. The implication 4. $\Rightarrow$ 1. descends from Proposition 3.1 and the fact that all conjugable subhypergroups are closed. Now we prove that 1. $\Rightarrow$ 4. Since h is closed it suffices to prove that $(H-h)\circ(H-h)\subseteq h$. By axiom (5), if there exists a triple (x,y,z) of elements in $H-h$ such that $z\in x\circ y$, necessarily $z\neq x$. Therefore, by Lemma 4.1, taking an element $b\in h-\{\varepsilon\}$, we obtain $x\in z\circ b\subseteq(x\circ y)\circ b=x\circ(y\circ b)$. Consequently $\varepsilon\in y\circ b\subseteq(H-h)\circ h=H-h$, that is impossible. So, for every pair (x,y) of elements in $H-h$, we have $x\circ y\subseteq h$. □

Theorem 4.2. *If* $(H,\circ)\in\mathfrak{S}_7$ *is a strongly conjugable extension, then* $(H,\circ)$ *owns exactly one subhypergroup of size* 4.

Proof. Let $(H,\circ)$ be a strongly conjugable extension of h. By definition, h is a proper subhypergroup. Moreover $h\neq\{\varepsilon\}$. In fact, if $h=\{\varepsilon\}$, then for every $x\in H-\{\varepsilon\}$ we have $x\circ x=\{\varepsilon\}$ and so $k=\{\varepsilon,x\}$ is a subgroup of $(H,\circ)$, in contradiction with item 3. of Lemma 2.1. Therefore, for Theorem 4.1, $h\cong B_4$. Now, if k is another subhypergroup of size 4, we obtain $k\cong B_4$ and $h\cap k\neq\{\varepsilon\}$. In fact, if $h\cap k=\{\varepsilon\}$ then we get the inclusion $k=(k-\{\varepsilon\})\circ(k-\{\varepsilon\})\subseteq(H-h)\circ(H-h)=h$, whence the contradiction $k=h\cap k=\{\varepsilon\}$. Finally, from Corollary 4.1, we obtain that $h=k$. □

An example of hypergroup in $\mathfrak{S}_7$, which is a strongly conjugable extension, can be obtained by means of the construction in Example 3.1. In fact, if $|G|=2$ and $H=A_1\cup A_2$ with $A_1\cong B_4$ and $|A_2|=3$, then the hypergroup $(H,\bullet)$ belongs to the class $\mathfrak{S}_7$ and is a strongly conjugable extension of A_1. In particular, setting $A_1=\{1,2,3,4\}$ and $A_2=\{5,6,7\}$, the hyperproduct is given by the following multiplicative table:

$\bullet$	1	2	3	4	5	6	7
1	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$
2	$\{2\}$	$\{1,3,4\}$	$\{1,3,4\}$	$\{1,3,4\}$	A_2	A_2	A_2
3	$\{3\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	A_2	A_2	A_2
4	$\{4\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	A_2	A_2	A_2
5	$\{5\}$	$\{6,7\}$	$\{6,7\}$	$\{6,7\}$	A_1	A_1	A_1
6	$\{6\}$	$\{5,7\}$	$\{5,7\}$	$\{5,7\}$	A_1	A_1	A_1
7	$\{7\}$	$\{5,6\}$	$\{5,6\}$	$\{5,6\}$	A_1	A_1	A_1

(10)

Lastly, we observe that in the class $\mathfrak{S}_7$ there exist hypergroups $(H,\circ)$ with a subhypergroup of size 4 which is not closed. Obviously, such hypergroups are not strongly conjugable extensions. Here is an example:

$\circ$	1	2	3	4	5	6	7
1	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$
2	$\{2\}$	$\{1,3,4\}$	$\{1,3,4\}$	$\{1,3,4\}$	$H-\{2\}$	$H-\{2\}$	$H-\{2\}$
3	$\{3\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	$H-\{3\}$	$H-\{3\}$	$H-\{3\}$
4	$\{4\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$H-\{4\}$	$H-\{4\}$	$H-\{4\}$
5	$\{5\}$	$H-\{5\}$	$H-\{5\}$	$H-\{5\}$	$H-\{5\}$	$H-\{5\}$	$H-\{5\}$
6	$\{6\}$	$H-\{6\}$	$H-\{6\}$	$H-\{6\}$	$H-\{6\}$	$H-\{6\}$	$H-\{6\}$
7	$\{7\}$	$H-\{7\}$	$H-\{7\}$	$H-\{7\}$	$H-\{7\}$	$H-\{7\}$	$H-\{7\}$

The set $h=\{1,2,3,4\}$ is a non-closed subhypergroup of $(H,\circ)$.

Remark 4.1. Previous results obtained in the present section and, in particular, Corollary 4.2, Corollary 4.3, Corollary 4.4 and Theorem 4.2, suggest to tackle the problem open in [5] and quoted at the beginning of this section by discriminating between three subcases:

1. $(H,\circ)$ owns only one closed, proper and non-trivial subhypergroup;
2. $(H,\circ)$ owns only one proper and non-trivial subhypergroup which is not closed;
3. $(H,\circ)$ does not own any proper and non-trivial subhypergroup.

In what follows we address the first subcase; to this aim, we denote by $\mathfrak{E}_7$ the subclass of hypergroups in $\mathfrak{S}_7$ that are strongly conjugable extensions.

5. Structure of the class $\mathfrak{E}_7$

In this section we deepen the knowledge of the class $\mathfrak{E}_7$. In particular, we introduce a partial ordering induced by the inclusion of hyperproducts, which allows us to give a complete characterization of hypergroups in $\mathfrak{E}_7$ on the basis of a small set of minimal hypergroups. A similar description is obtained in [7] for the hypergroups of type U on the right having order 6 whose right scalar identity is not also a left identity.

For Theorem 4.2 and Corollary 3.1, every hypergroup $(H, \circ) \in \mathfrak{E}_7$ owns exactly one conjugable and invariant subhypergroup $h \cong B_4$. Moreover, from Lemma 4.1, we know that $x \circ b = H - (h \cup \{x\})$, for every $x \in H - h$ and $b \in h - \{\varepsilon\}$. Thus, setting $H = \{1, 2, 3, 4, 5, 6, 7\}$ and $h = \{1, 2, 3, 4\}$, up to isomorphisms, we obtain the partial hyperproduct table

$\star$	1	2	3	4	5	6	7
1	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$
2	$\{2\}$	$\{1,3,4\}$	$\{1,3,4\}$	$\{1,3,4\}$	X_1	X_2	X_3
3	$\{3\}$	$\{1,2,4\}$	$\{1,2,4\}$	$\{1,2,4\}$	X_4	X_5	X_6
4	$\{4\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	X_7	X_8	X_9
5	$\{5\}$	$\{6,7\}$	$\{6,7\}$	$\{6,7\}$	Y_1	Y_2	Y_3
6	$\{6\}$	$\{5,7\}$	$\{5,7\}$	$\{5,7\}$	Y_4	Y_5	Y_6
7	$\{7\}$	$\{5,6\}$	$\{5,6\}$	$\{5,6\}$	Y_7	Y_8	Y_9

(11)

where the sets X_i and Y_i for $i = 1, 2, \ldots, 9$ are non-empty subsets of $H - h$ and h, respectively.

In what follows, we prove some properties pertaining to the sets X_i, Y_i. These results will be useful in the next section to determine the isomorphism classes of certain minimal hypergroups in $\mathfrak{E}_7$. In the forthcoming results we will tacitly refer to the table (11), in particular whenever we make use of the notations X_i, Y_i.

Lemma 5.1. *If $b \in h - \{1\}$ and $x, y \in H - h$, with $x \neq y$, then*

$$b \circ x = \{y\} \;\Rightarrow\; b \circ y = y \circ b = H - (h \cup \{y\}).$$

Proof. By hypothesis we have $x \in y \circ b = H - (h \cup \{y\})$ and $(h - \{b\}) \circ x = (b \circ b) \circ x = b \circ (b \circ x) = b \circ y$. Moreover, from Proposition 3.1, we obtain $H - h = h \circ x = (h - \{b\}) \circ x \cup b \circ x = b \circ y \cup \{y\}$, and so $|b \circ y| \geq 2$. Finally, since $|y \circ b| = 2$ and $b \circ y \subseteq b \circ (x \circ b) = (b \circ x) \circ b = y \circ b$, we have that $b \circ y = y \circ b = H - (h \cup \{y\})$. □

Proposition 5.1. *For every $i = 1, 2, \ldots, 9$, we have $|X_i| \geq 2$.*

Proof. Suppose that there exists $i \in \{1, 2, \ldots, 9\}$ such that $|X_i| = 1$. Obviously there exist $b \in h - \{1\}$ and $x, y \in H - h$ such that $X_i = b \circ x = \{y\}$. We distinguish two cases:

1. *Case $x = y$*: We have $(h - \{b\}) \circ x = (b \circ b) \circ x = b \circ (b \circ x) = b \circ x = \{x\}$, and so $h \circ x = \{x\}$. Furthermore, since h is invariant, we obtain $H - h = x \circ h = h \circ x = \{x\}$.
2. *Case $x \neq y$*: Assuming $H - h = \{x, y, z\}$, from Lemma 5.1, we have $b \circ y = \{x, z\}$ and $(h - \{1, b\}) \circ x \subseteq (b \circ b) \circ x = b \circ (b \circ x) = b \circ y = \{x, z\}$. Now, if there exists $c \in h - \{1, b\}$ such that $x \in c \circ x$, we have the contradiction $y \in b \circ x \subseteq b \circ (c \circ x) = (b \circ c) \circ x = (h - \{b\}) \circ x = \{x\} \cup (h - \{1, b\}) \circ x \subseteq \{x, z\}$. Thus

   $$(h - \{1, b\}) \circ x = \{z\}.$$

   At last, putting $h - \{1, b\} = \{c, d\}$, we have $c \circ x = d \circ x = \{z\}$. Then, from Lemma 5.1, we obtain that $c \circ z = \{x, y\}$, and moreover $z \in d \circ x \subseteq (c \circ c) \circ x = c \circ (c \circ x) = c \circ z = \{x, y\}$.

Hence, in both cases, we come to a contradiction. So $|X_i| \geq 2$. □

The forthcoming proposition concerns the sets Y_i. We premise the following

Lemma 5.2. *For every $i = 1, 2 \ldots, 9$, we have:*

1. $Y_i \circ 2 = Y_i \circ 3 = Y_i \circ 4$;
2. $Y_i \notin \{\{1\}, \{1,2\}, \{1,3\}, \{1,4\}\}$.

Proof. Let $x, y \in \{5,6,7\}$ and $Y_i = x \circ y$, for some $i \in \{1, 2, \ldots, 9\}$.

1. From (11), we have $y \circ 2 = y \circ 3 = y \circ 4$. Consequently we obtain

$$x \circ (y \circ 2) = x \circ (y \circ 3) = x \circ (y \circ 4),$$

and so $Y_i \circ 2 = Y_i \circ 3 = Y_i \circ 4$.

2. If $Y_i = x \circ y = \{1\}$, by item 1., we obtain

$$2 = 1 \circ 2 = Y_i \circ 2 = Y_i \circ 3 = 1 \circ 3 = 3,$$

that is a contradiction. Moreover, if $Y_i = x \circ y = \{1, 2\}$, then we have

$Y_i \circ 2 = \{1,2\} \circ 2 = 1 \circ 2 \cup 2 \circ 2 = \{2\} \cup (h - \{2\}) = h$;

$Y_i \circ 3 = \{1,2\} \circ 3 = 1 \circ 3 \cup 2 \circ 3 = \{3\} \cup (h - \{2\}) = (h - \{2\})$,

that is impossible by 1. Hence $Y_i \neq \{1, 2\}$. Analogously one can prove that $Y_i \neq \{1, 3\}$ and $Y_i \neq \{1, 4\}$.

□

Proposition 5.2. *For every $i = 1, 2, \ldots, 9$, we have $1 \in Y_i$ and $|Y_i| \geq 3$.*

Proof. Let $i \in \{1, 2, \ldots, 9\}$ and $x, y \in H - h$ such that $Y_i = x \circ y$. From Proposition 5.1, we have $|a \circ x| \geq 2$ for every $a \in h - \{1\}$. Hence, since $a \circ x \subseteq H - h = \{5, 6, 7\}$, we obtain that $a \circ x \in \{\{5,6\}, \{5,7\}, \{6,7\}, \{5,6,7\}\}$, for every $a \in h - \{1\}$. Therefore we can distinguish two cases:

Case 1: If $2 \circ x \cap 3 \circ x \cap 4 \circ x = \emptyset$ then, withous loss of generality, we can suppose $2 \circ x = \{5, 6\}$, $3 \circ x = \{5, 7\}$ and $4 \circ x = \{6, 7\}$. Thus, if we suppose by absurd that $1 \notin Y_i = x \circ y$, by axiom (5) we have

$2 \notin 2 \circ x \circ y = 5 \circ y \cup 6 \circ y$;
$3 \notin 3 \circ x \circ y = 5 \circ y \cup 7 \circ y$;
$4 \notin 4 \circ x \circ y = 6 \circ y \cup 7 \circ y$.

Consequently we obtain $5 \circ y \subseteq \{1, 4\}$, $6 \circ y \subseteq \{1, 3\}$ and $7 \circ y \subseteq \{1, 5\}$. Whence, since $1 \notin h \circ y$ and $H \circ y = H$, there exists $a \in H - h$ such that $1 \in a \circ y$ and $|a \circ y| \leq 2$. This fact contradicts Lemma 5.2. Hence $1 \in Y_i$.

Case 2: If there exists an element $t \in 2 \circ x \cap 3 \circ x \cap 4 \circ x$ then, by (11) we have $t \circ y \subset h$. Now, for Lemma 5.2, we know that $t \circ y \neq \{1\}$ and so $t \circ y \cap (h - \{1\}) \neq \emptyset$. Moreover, taking an element $a \in t \circ y \cap (h - \{1\})$, we have $t \in a \circ x$ and thus $a \in t \circ y \subseteq (a \circ x) \circ y = a \circ (x \circ y)$. Consequently, also in this case, by axiom (5) we obtain that $1 \in x \circ y = Y_i$.

Finally, again for Lemma 5.2, we have that $|Y_i| \geq 3$, for every $i \in \{1, 2, \ldots 9\}$. □

As an immediate consequence of Proposition 5.1 and Proposition 5.2, due to the structure of the hypergroups in $\mathfrak{E}_7$ shown in (11), we obtain the following result:

Theorem 5.1. *If $(H, \circ) \in \mathfrak{E}_7$ then $|x \circ y| \geq 2$ whenever x, y are two elements different from the identity.*

5.1. A semiordering in $\mathfrak{E}_7$

Earlier in this section, we proved that all the hypergroups in $\mathfrak{E}_7$ can be described by the table (11), and fulfil the conditions stated in Propositions 5.1 and 5.2. Hence, we are motivated to consider $\mathfrak{E}_7$ as the set of all hypergroups whose hyperoperation is represented as in (11). We stress the fact that, with this convention, every hypergroup in $\mathfrak{S}_7$ that is a strongly conjugable extension, is isomorphic to (at least) one hypergroup in $\mathfrak{E}_7$.

For notational and descriptive simplicity, hereafter we will use a shorthand notation to represent the multiplicative table of hypergroups $(H, \circ) \in \mathfrak{E}_7$. More precisely, we denote with B, X_H, A and Y_H the arrays

$$B = \begin{array}{|c|c|c|c|}\hline \{1\} & \{2\} & \{3\} & \{4\} \\ \hline \{2\} & \{1,3,4\} & \{1,3,4\} & \{1,3,4\} \\ \hline \{3\} & \{1,2,4\} & \{1,2,4\} & \{1,2,4\} \\ \hline \{4\} & \{1,2,3\} & \{1,2,3\} & \{1,2,3\} \\ \hline \end{array} \qquad X_H = \begin{array}{|c|c|c|}\hline \{5\} & \{6\} & \{7\} \\ \hline X_1 & X_2 & X_3 \\ \hline X_4 & X_5 & X_6 \\ \hline X_7 & X_8 & X_9 \\ \hline \end{array}$$

$$A = \begin{array}{|c|c|c|c|}\hline \{5\} & \{6,7\} & \{6,7\} & \{6,7\} \\ \hline \{6\} & \{5,7\} & \{5,7\} & \{5,7\} \\ \hline \{7\} & \{5,6\} & \{5,6\} & \{5,6\} \\ \hline \end{array} \qquad Y_H = \begin{array}{|c|c|c|}\hline Y_1 & Y_2 & Y_3 \\ \hline Y_4 & Y_5 & Y_6 \\ \hline Y_7 & Y_8 & Y_9 \\ \hline \end{array}.$$

With these notations, we can represent the hyperproduct table (11) in a compact way by means of the following block table:

$$\begin{array}{|c|c|}\hline B & X_H \\ \hline A & Y_H \\ \hline \end{array}. \tag{12}$$

Furthermore, we can define two special subsets of $\mathfrak{E}_7$ as follows: Let

$$X = \begin{array}{|c|c|c|}\hline \{5\} & \{6\} & \{7\} \\ \hline \{5,6,7\} & \{5,6,7\} & \{5,6,7\} \\ \hline \{5,6,7\} & \{5,6,7\} & \{5,6,7\} \\ \hline \{5,6,7\} & \{5,6,7\} & \{5,6,7\} \\ \hline \end{array} \tag{13}$$

$$Y = \begin{array}{|c|c|c|}\hline \{1,2,3,4\} & \{1,2,3,4\} & \{1,2,3,4\} \\ \hline \{1,2,3,4\} & \{1,2,3,4\} & \{1,2,3,4\} \\ \hline \{1,2,3,4\} & \{1,2,3,4\} & \{1,2,3,4\} \\ \hline \end{array}. \tag{14}$$

We denote by $\mathfrak{E}_7(X)$ (resp., $\mathfrak{E}_7(Y)$) the set of hypergroups in $\mathfrak{E}_7$ with block tables (12) such that $X_H = X$ (resp., $Y_H = Y$). Obviously the sets $\mathfrak{E}_7(X)$ and $\mathfrak{E}_7(Y)$ share only the hypergroup $(H, \bullet)$ defined in (10), whose block table is

$$\begin{array}{|c|c|}\hline B & X \\ \hline A & Y \\ \hline \end{array}. \tag{15}$$

In what follows, we prove that $\mathfrak{E}_7$ owns a rather special structure with respect to the partial ordering given here below. This structure allows us to obtain all hypergroups in $\mathfrak{E}_7$ starting from the minimal elements in that set.

Definition 5.1. Given two hypergroupoids $(H, \circ)$ and $(H, *)$, we say that $(H, \circ)$ is a *hyperproduct restriction* of $(H, *)$ if $x \circ y \subseteq x * y$, for all $x, y \in H$. In this case we say also that $(H, *)$ a *hyperproduct extension* of $(H, \circ)$, and we write $(H, \circ) \preceq (H, *)$.

Definition 5.2. A hypergroup $(H, \circ) \in \mathfrak{E}_7$ is said to be *minimal* if there exists no hypergroup in $\mathfrak{E}_7$ different from $(H, \circ)$ which is a hyperproduct restriction of $(H, \circ)$.

We observe that, in the light of these definitions, the hypergroup represented from the block table (15) is maximum in the partially ordered set $(\mathfrak{E}_7\,,\preceq)$.

Henceforth, we denote by $\mathfrak{M}_7$ (resp., $\mathfrak{M}_7(X)$ and $\mathfrak{M}_7(Y)$) the set of hypergroups in $\mathfrak{E}_7$ (resp., in $\mathfrak{E}_7(X)$ and $\mathfrak{E}_7(Y)$) that are minimal according to Definition 5.2. Every hypergroup in $\mathfrak{E}_7 - \mathfrak{M}_7$ can be obtained by hyperproduct extension of some hypergroups in $\mathfrak{M}_7$. Moreover, every hypergroup in $\mathfrak{M}_7$ owns non-trivial hyperproduct extensions belonging to $\mathfrak{E}_7$. These results descend from Theorem 5.2; its proof is largely based on the following preliminary results:

Lemma 5.3. *Let $(H, \circ) \in \mathfrak{E}_7$. For every $a, b \in h - \{1\}$ and $u, v, x, y \in H - h$, we have*

1. *if $u \neq v$ then $(x \circ u) \cup (x \circ v) = h$;*
2. *if $x \neq y$ then $(x \circ u) \cup (y \circ u) = h$;*
3. *if $u \neq v$ then $(a \circ u) \cup (a \circ v) = H - h$;*
4. *if $a \neq b$ then $\{u\} \cup (a \circ u) \cup (b \circ u) = H - h$.*

Proof. 1. If $u \neq v$ then we can set $H - h = \{t, u, v\}$. Thus, if $x \in H - h$ then, by Proposition 5.2, there exist two distinct elements $a, b \in h - \{1\}$ such that $x \circ t \supseteq \{1, a, b\}$. So, by (11), we have $(x \circ u) \cup (x \circ v) = x \circ \{u, v\} = x \circ (t \circ 2) = (x \circ t) \circ 2 \supseteq \{1, a, b\} \circ 2 = h$, whence the claim follows.

2. Let $x \neq y$ and $H - h = \{x, y, z\}$. If $u \in H - h$ then, by Proposition 5.1, we know that $|2 \circ u| \geq 2$. Hence, from (11) and item 1. we have that $(x \circ u) \cup (y \circ u) = \{x, y\} \circ u = (z \circ 2) \circ u = z \circ (2 \circ u) = h$.

3. Let $\{t, u, v\} = H - h$ and $a \in h - \{1\}$. By Proposition 5.1 we have $|a \circ t| \geq 2$. Therefore, from (11), we obtain $(a \circ u) \cup (a \circ v) = a \circ \{u, v\} = a \circ (t \circ 2) = (a \circ t) \circ 2 = H - h$.

4. Let $a \neq b$ and $\{a, b, c\} = h - \{1\}$. By (11), we have $(c \circ c) \circ u = \{1, a, b\} \circ u = \{u\} \cup a \circ u \cup b \circ u$. Moreover, by Proposition 5.1, we can suppose that there exist two distinct elements $x, y \in H - h$ such that $\{x, y\} \subseteq c \circ u$. Then, by the preceding item 3., we have $c \circ (c \circ u) \supseteq c \circ \{x, y\} = (c \circ x) \cup (c \circ y) = H - h$. So $\{u\} \cup a \circ u \cup b \circ u = H - h$ and the proof is complete. □

Lemma 5.4. *Let $(H, \circ) \in \mathfrak{E}_7$. For every $a, b, c \in h - \{1\}$ and $x, y, z \in H - h$, we have*

1. *$a \circ b \circ c = h$;*
2. *$a \circ b \circ x = a \circ x \circ b = x \circ a \circ b = H - h$;*
3. *$a \circ x \circ y = x \circ a \circ y = x \circ y \circ a = h$;*
4. *$x \circ y \circ z = H - h$.*

Proof. Firstly, we observe that $(H, \circ)$ is a strongly conjugable extension of the subhypergroup h and so, by Proposition 3.1, we have

$a \circ b \circ x \cup a \circ x \circ b \cup x \circ a \circ b \subseteq H - h$;
$a \circ x \circ y \cup x \circ a \circ y \cup x \circ y \circ a \subseteq h$;
$x \circ y \circ z \subseteq H - h$.

1. It is obvious since $h \cong B_4$.
2. By Proposition 5.1, there exist two distinct elements $u, v \in H - h$ such that $\{u, v\} \subseteq b \circ x$. Hence, from item 3. of Lemma 5.3, we have $a \circ b \circ x \supseteq a \circ \{u, v\} = (a \circ u) \cup (a \circ v) = H - h$.
   Analogously, if $\{u, v\} \subseteq a \circ x$ then, by (11) we deduce $a \circ x \circ b \supseteq \{u, v\} \circ b = (u \circ b) \cup (v \circ b) = H - h$.
   Moreover, we can set $H - h = \{x, y, z\}$ and $x \circ a = \{y, z\}$. Thus $x \circ a \circ b = \{y, z\} \circ b = y \circ b \cup z \circ b = H - h$.
3. For Proposition 5.2, there exists an element $b \in h - \{1\}$ such that $\{1, b\} \subset x \circ y$. Hence, by (11), we have $a \circ x \circ y \supseteq a \circ \{1, b\} = \{a\} \cup a \circ b = h$.
   If we set $H - h = \{x, z, w\}$ and $x \circ a = \{z, w\}$ then, by item 2. of Lemma 5.3, we have $x \circ a \circ y = \{z, w\} \circ y = z \circ y \cup w \circ y = h$.
   Moreover, for Proposition 5.2, we know that there exist two distinct elements $b, c \in h - \{1\}$ such that $\{1, b, c\} \subset x \circ y$. Therefore, by (11), we get that $x \circ y \circ a \supseteq \{1, b, c\} \circ a = h$.

4. Lastly, we can suppose that $\{1, b\} \subset y \circ z$. Hence, by (11), we have $x \circ y \circ z \supseteq x \circ \{1, b\} = \{x\} \cup x \circ b = H - h$. $\square$

We observe that the content of the preceding lemma leads to the following conclusion: Let $(H, \circ) \in \mathfrak{E}_7$ and let $\circ_3$ denote the ternary extension of the hyperoperation $\circ$. Then, the ternary hypergroup $(H, \circ_3)$ is Abelian.

Theorem 5.2. *Let* $(H, \bullet)$ *be the hypergroup described in* (15), *and let* $(H, \circ)$ *and* $(H, *)$ *be two hypergroupoids. If* $(H, \circ) \in \mathfrak{E}_7$ *and* $(H, \circ) \preceq (H, *) \preceq (H, \bullet)$ *then* $(H, *) \in \mathfrak{E}_7$.

Proof. If $(H, \circ) \in \mathfrak{E}_7$ and $x \circ y \subseteq x * y \subseteq x \bullet y$ for every $x, y \in H$ then the hyperoperation $*$ fulfills the axioms (3), (4) and (5). It remains to prove associativity. For every $x, y, z \in H - \{1\}$, we have

$$(x \circ y) \circ z \subseteq (x * y) * z \subseteq (x \bullet y) \bullet z, \qquad x \circ (y \circ z) \subseteq x * (y * z) \subseteq x \bullet (y \bullet z).$$

By virtue of Lemma 5.4, we have $x \circ y \circ z = x \bullet y \bullet z$, hence we get that $(x * y) * z = x * (y * z)$ and we have the claim. The remaining case, when one of x, y, z is the identity, associativity is trivial. $\square$

The subsequent theorem shows that every hypergroup in $\mathfrak{E}_7$ can be obtained from those belonging to $\mathfrak{E}_7(X)$ and $\mathfrak{E}_7(Y)$. This fact will allow us to obtain the tables of minimal hypergroups of $\mathfrak{E}_7$ starting from minimal hypergroups of $\mathfrak{E}_7(X)$ and $\mathfrak{E}_7(Y)$. The description of these tables will be given in the next section.

Theorem 5.3. *Using the notations* (13) *and* (14) *we have:*

1. $(H, \circ) = \begin{array}{|c|c|}\hline B & X_H \\ \hline A & Y_H \\ \hline\end{array} \in \mathfrak{E}_7 \quad \Longrightarrow \quad (H, \diamond) = \begin{array}{|c|c|}\hline B & X_H \\ \hline A & Y \\ \hline\end{array} \in \mathfrak{E}_7(Y)$;
2. $(H, \circ) = \begin{array}{|c|c|}\hline B & X_H \\ \hline A & Y_H \\ \hline\end{array} \in \mathfrak{E}_7 \quad \Longrightarrow \quad (H, *) = \begin{array}{|c|c|}\hline B & X \\ \hline A & Y_H \\ \hline\end{array} \in \mathfrak{E}_7(X)$;
3. $(H, \diamond) = \begin{array}{|c|c|}\hline B & X_H \\ \hline A & Y \\ \hline\end{array} \in \mathfrak{E}_7(Y)$ *and* $(H, *) = \begin{array}{|c|c|}\hline B & X \\ \hline A & Y_H \\ \hline\end{array} \in \mathfrak{E}_7(X) \quad \Longrightarrow$

   $(H, \circ) = \begin{array}{|c|c|}\hline B & X_H \\ \hline A & Y_H \\ \hline\end{array} \in \mathfrak{E}_7$.

Proof. The first two items follow at once from Theorem 5.2. In fact, if $(H, \bullet)$ is the hypergroup described in (15), then we get $(H, \circ) \preceq (H, \diamond) \preceq (H, \bullet)$ and $(H, \circ) \preceq (H, *) \preceq (H, \bullet)$. Therefore $(H, \star)$ and $(H, \diamond)$ are hypergroups respectively in $\mathfrak{E}_7(X)$ and $\mathfrak{E}_7(Y)$.

In order to prove the third item, it suffices to show that the hyperoperation $\circ$ is associative for every triple of elements in $H - \{1\}$. We proceed by cases, supposing that $a, b, c \in h - \{1\}$ and $x, y, z \in H - h$.

1. Associativity for the triple (a, b, c) is obvious.
2. (a, b, x): From hypotheses and Lemma 5.4, we get that

   $$(a \circ b) \circ x = (a \diamond b) \diamond x = H - h = a \diamond (b \diamond x) = a \circ (b \circ x).$$

   Essentially in the same way, we can prove associativity for triples (a, x, b) and (x, a, b).
3. (a, x, y): From hypotheses and Lemma 5.4, we have $a \circ (x \circ y) = a * (x * y) = h$ and $(a \circ x) \circ y \subseteq h$. Moreover, for Proposition 5.1, there exist two distinct elements $u, v \in H - h$ such that $\{u, v\} \subseteq a \diamond x$. Hence, for the second item of Lemma 5.3, we obtain that $(a \circ x) \circ y = (a \diamond x) \circ y = (a \diamond x) * y \supseteq \{u, v\} * y = u * y \cup v * y = h$. Hence $a \circ (x \circ y) = h = (a \circ x) \circ y$.
4. (x, a, y): The proof is similar to the previous item. In this case we suppose that $\{u, v\} \subseteq a \diamond y$ and use the first item of Lemma 5.3.
5. (x, y, a): From hypotheses and Lemma 5.4, we get

   $$(x \circ y) \circ a = (x * y) * a = h = x * (y * a) = x \circ (y \circ a).$$

6. (x, y, z): From hypotheses and Lemma 5.4, we have $x \circ (y \circ z) = x * (y * z) = H - h$ and $(x \circ y) \circ z \subseteq H - h$. Besides, for Proposition 5.2, there exist two distinct elements $a, b \in h - \{1\}$ such that $\{1, a, b\} \subseteq x * y$. Thus, from the fourth item of Lemma 5.3, we obtain that $(x \circ y) \circ z = (x * y) \circ z \supseteq \{1, a, b\} \circ z = \{z\} \cup a \circ z \cup b \circ z = \{z\} \cup a \diamond z \cup b \diamond z = H - h$. Hence $x \circ (y \circ z) = H - h = (x \circ y) \circ z$.

□

The following corollary is an immediate consequence of Theorem 5.3; it is exploited in the next section to characterize $\mathfrak{M}_7$ in terms of $\mathfrak{M}_7(X)$ and $\mathfrak{M}_7(Y)$.

Corollary 5.1. *If $\mathfrak{M}_7$, $\mathfrak{M}_7(X)$ and $\mathfrak{M}_7(Y)$ are respectively the sets of minimal hypergroups in $\mathfrak{E}_7$, $\mathfrak{E}_7(X)$ and $\mathfrak{E}_7(Y)$, then a hypergroup $(H, \circ) \in \mathfrak{M}_7$ if and only if there exist two hypergroups*

$$(H, \diamond) = \begin{array}{|c|c|} \hline B & X_H \\ \hline A & Y \\ \hline \end{array} \in \mathfrak{M}_7(Y) \quad \textit{and} \quad (H, *) = \begin{array}{|c|c|} \hline B & X \\ \hline A & Y_H \\ \hline \end{array} \in \mathfrak{M}_7(X)$$

such that $(H, \circ) = \begin{array}{|c|c|} \hline B & X_H \\ \hline A & Y_H \\ \hline \end{array}$.

6. Computation of minimal hypergroups in $\mathfrak{E}_7$

In this section we present transversal sets for the classes $\mathfrak{M}_7(X)$ and $\mathfrak{M}_7(Y)$ defined in the preceding section. In the light of Corollary 5.1, these two lists are sufficient to completely describe the class $\mathfrak{M}_7$. We recall that, owing to Theorem 5.2, all hypergroups in $\mathfrak{E}_7$ can be obtained, modulo isomorphisms, as hyperproduct extensions of the hypergroups in $\mathfrak{M}_7$. To produce these lists, we employ two algorithms described in the Appendix of the paper [6]. The first routine, which is called `findHgroups`, determines all the hypergroups in a given class specified by an incomplete hyperproduct table. The second routine, called `sieveHgroups`, takes as input a hypergroup list and produces as output a list of representatives of its isomorphism classes. For brevity, we illustrate only the construction of the transversal set of $\mathfrak{M}_7(X)$; the construction of the corresponding list for $\mathfrak{M}_7(Y)$ is almost completely analogous.

Firstly, we produce a complete listing of the hypergroups in $\mathfrak{E}_7(X)$; to this aim we make use of the routine `findHgroups`, adapted to the hyperproduct structure described in (11) having fixed the hyperproducts $X_i = \{4, 5, 6\}$ for $i = 1, \ldots, 9$. The hyperproducts $Y_1, \ldots, Y_9$ must fulfil the conditions in Proposition 5.2. Hence, an high-level description of the routine `findHgroups` adapted to our pourposes is the following:

$\mathfrak{E}_7(X) = \emptyset$
$\mathcal{Y} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$
for all $Y_1 \in \mathcal{Y}$
  for all $Y_2 \in \mathcal{Y}$
    for all $Y_3 \in \mathcal{Y}$
      for all $Y_4 \in \mathcal{Y}$
        for all $Y_5 \in \mathcal{Y}$
          for all $Y_6 \in \mathcal{Y}$
            for all $Y_7 \in \mathcal{Y}$
              for all $Y_8 \in \mathcal{Y}$
                for all $Y_9 \in \mathcal{Y}$
                  define $(H, \circ)$ as in (11) with $X_1 = \ldots = X_9 = \{4, 5, 6\}$
                  if $(H, \circ)$ is associative and reproducible **then**
                    add $(H, \circ)$ to $\mathfrak{E}_7(X)$
End.

On termination, the list $\mathfrak{E}_7(X)$ contains 11776 hypergroups. Next, we compute $\mathfrak{M}_7(X)$. Starting from the list $\mathfrak{E}_7(X)$, we discard those hypergroups that are extensions of some other hypergroups in the same set. The selection is performed in the following way: Let $H_1, \ldots, H_k$ denote the hypergroups in $\mathfrak{E}_7(X)$. To each

Nr.	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9
1	123	124	134	124	123	1234	134	1234	123
2	123	124	134	124	123	1234	134	1234	124
3	123	124	134	124	123	1234	1234	134	123
4	123	124	134	124	123	1234	1234	134	124
5	123	124	134	124	134	123	134	123	124
6	123	124	134	124	1234	123	1234	134	124
7	123	124	134	134	123	124	124	134	123
8	123	124	134	134	123	1234	1234	134	124
9	123	124	134	134	1234	123	124	123	1234
10	123	124	134	134	1234	123	124	134	1234
11	123	124	134	134	1234	123	1234	123	124
12	123	124	134	134	1234	123	1234	134	124
13	123	124	1234	124	123	1234	1234	1234	134
14	123	124	1234	1234	1234	134	124	123	1234

Table 1: Hyperproducts $Y_1, \dots, Y_9$ defining the isomorphism classes of $\mathfrak{M}_7(X)$.

Nr.	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
1	56	57	67	57	67	56	67	56	57
2	56	57	67	57	67	56	67	57	56
3	56	57	67	57	567	56	567	67	56
4	56	57	67	67	57	56	67	57	56
5	56	57	567	57	56	67	67	57	56
6	56	57	567	57	56	67	67	567	57
7	56	57	567	57	67	56	567	56	67
8	56	57	567	57	567	67	67	56	57
9	57	56	67	56	67	57	67	57	56
10	57	56	567	56	57	67	67	57	56
11	57	56	567	56	57	67	567	67	57
12	57	56	567	56	57	567	567	57	67
13	57	56	567	56	67	57	567	57	67

Table 2: Hyperproducts $X_1, \dots, X_9$ defining the isomorphism classes of $\mathfrak{M}_7(Y)$.

hypergroup, associate a binary-valued variable *flag* whose value is set initially to one. Then, we perform the following algorithm:

$\mathfrak{M}_7(X) = \emptyset$
for $i = 1, \dots, k$
  for $j = 1, \dots, k$
    if $[i \neq j$ and $flag(j) = 1]$ **then**
      if $H_i \leq H_j$ **then**
        $flag(j) := 0$
        add H_i to $\mathfrak{M}_7(X)$
End.

Upon termination, those hypergroups whose corresponding flag is set to 1 are minimal. In particular, the number of flag variables whose value is one gives the cardinality of $\mathfrak{M}_7(X)$. This number is 390.

Finally, we employ the routine `sieveHgroups` to select a set of pairwise not isomorphic hypergroups from $\mathfrak{M}_7(X)$. To speed up the construction of that transversal set, it is opportune to identify the smallest possible set of candidate isomorphisms in $\mathfrak{E}_7$. By observing the structure of the hyperproduct table (11), we conclude that every permutation $\pi \in S_7$ that is also a hypergroup isomorphism in $\mathfrak{E}_7$ necessarily fulfills the equations $\pi(1) = 1$, $\pi(\{2,3,4\}) = \{2,3,4\}$ and $\pi(\{5,6,7\}) = \{5,6,7\}$. The possible permutations are 36 and consitute a group isomorphic to $S_3 \times S_3$. By applying `sieveHgroups` to the list $\mathfrak{M}_7(X)$ with these permutations we obtain 14 hypergroups pairwise not isomorphic. With reference to (11), we show in Table 1 the sets $Y_1, \dots, Y_9$ characterizing this transversal set.

The construction of a transversal of $\mathfrak{M}_7(Y)$ proceeds analogously, having fixed the hyperproducts $Y_i = \{1,2,3,4\}$ for $i = 1,\dots,9$. The condition to be satisfied by the sets $X_1,\dots,X_9$ is the one given in Proposition 5.1. Hence, we let $X_i \in \mathcal{X} = \{\{5,6\},\{5,7\},\{6,7\},\{5,6,7\}\}$ in the routine `findHgroups`. As a result, the cardinality of $\mathfrak{E}_7(Y)$ is 19268, and that of $\mathfrak{M}_7(Y)$ is 49. Finally, the selection of pairwise non-isomorphic hypergroups ends up with 14 representatives, that are listed in Table 2.

Therefore, from Corollary 5.1 we obtain the following result:

Theorem 6.1. *There exist* 182 *isomorphism classes in* $\mathfrak{M}_7$. *A transversal is given by the hyperproduct table* (11) *when the hyperproducts* Y_i *are chosen from the rows of Table 1 and the hyperproducts* X_i *from the rows of Table 2.*

A rigorous proof of this theorem should exploit the following fact, which is rather self-apparent: in the block-notations analogous to those in Corollary 5.1, let

$$(H,\diamond) = \begin{array}{|c|c|}\hline B & X_H \\ \hline A & Y_H \\ \hline\end{array} \in \mathfrak{M}_7(Y) \quad \text{and} \quad (H,*) = \begin{array}{|c|c|}\hline B & X'_H \\ \hline A & Y'_H \\ \hline\end{array} \in \mathfrak{M}_7(X).$$

Hence, if f is a isomorphism, $(H,\diamond) \xrightarrow{f} (H,*)$, then the same f determines also the following isomorphisms:

$$\begin{array}{|c|c|}\hline B & X \\ \hline A & Y_H \\ \hline\end{array} \xrightarrow{f} \begin{array}{|c|c|}\hline B & X \\ \hline A & Y'_H \\ \hline\end{array}, \qquad \begin{array}{|c|c|}\hline B & X_H \\ \hline A & Y \\ \hline\end{array} \xrightarrow{f} \begin{array}{|c|c|}\hline B & X'_H \\ \hline A & Y \\ \hline\end{array}.$$

7. Conclusions

In this paper we have analyzed hypergroups that are strongly conjugable extensions of one of their subhypergroups, see Definition 3.1. Successively, we considered the class $\mathfrak{S}_7$ of hypergroups of type U on the right having bilateral scalar identity and order 7. The interest in this class arises from various facts recently discovered in closely related hypergroup classes, see e.g., [2, 5–7]. In particular, we have shown that a nontrivial subhypergroup h of a hypergroup $(H,\circ) \in \mathfrak{S}_7$ is closed if and only if $(H,\circ) \in \mathfrak{S}_7$ is a strongly conjugable extension of h. We have developed the analysis of this kind of hypergroups and we have found that they can be characterized by means of a small set of hypergroups that are minimal with respect to a semiordering induced by hyperproduct inclusion. This approach to the representation of hypergroup classes was firstly proposed in [7] and is quite relevant to hyperstructure theory since the generality of hyperproduct operations hinders the study of the automorphism group and the isomorphism classes of this kind of hyperstructures.

Finally, we observe that the problem raised in [5], namely, to establish the minimal (odd) cardinality n of a hypergroup in $\mathfrak{S}_n$ fulfilling conditions (1) and (2) remains still unsolved. Nevertheless, the results presented herein considerably simplify its solution. Indeed, we know from [5] that $n > 5$, and the hypergroup presented in (8) shows that $n \le 9$. By Theorem 5.1, the problem is now circumscribed to two subclasses of $\mathfrak{S}_7$, that is, the hypergroups that own only one proper and non-trivial subhypergroup which is not closed, and those that do not own any proper and non-trivial subhypergroup.

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