

## Mean square BIBO stability of discrete-time stochastic control systems with delays and nonlinear perturbations

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**Abstract.** The problem of mean square bounded-input bounded-output(BIBO) stability is investigated for a class of discrete-time stochastic control systems with time delays and non-linear perturbations. In this paper, a special point  $\delta$  in the time delay's variation interval is introduced, and the variation interval is divided into two subintervals. Then, by defining a special Lyapunov–Krasovskii functional and checking its variation in the two subintervals, respectively, some novel delay-dependent stability criteria for the discrete-time stochastic control systems are derived. These conditions are expressed in the forms of linear matrix inequalities(LMIs), whose feasibility can be easily checked by using Matlab LMI Toolbox. Meanwhile, this paper provides a new method for studying discrete-time stochastic mean square BIBO stability. Finally, a numerical example is given to illustrate the validity of the main results.

### 1. Introduction

Because of the finite switching speed, memory effects and so on, time delay is unavoidable in technology and nature, and commonly exists in various mechanical, chemical processes, nuclear reactors, engineering, physical, biological, and economic systems. It is often an important source of instability and oscillation. That makes the design and hardware implementation of the control system become difficult. Thus, the stability of time-delay systems has been widely investigated. Please refer to [1]-[8], and there are some references there. In recent years, Bounded-Input Bounded-Output(BIBO) stability has been investigated by many researchers hoping to track out the reference input signal in real world, see [9]-[23] and some references therein. In [17,18], the sufficient condition for BIBO stability of the control system with no delays was proposed by the Bihari-type inequality. In [11,12], employing the parameters technique and the Gronwall inequality, the authors investigate the BIBO stability of the system without distributed time delays. In [19]-[21], based on Riccati-equations, by constructing appropriate Lyapunov functions, some BIBO stability criteria for a class of delayed control systems with nonlinear perturbations were established. In [22], the BIBO stability problem of a class of piecewise switched linear system was further investigated.

However, up to now, those previous results have been assumed to be in deterministic systems, but rarely in stochastic systems [10,23]. In [10], Zhou and Zhong discussed the BIBO stability in mean square of the

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stochastic delay system with nonlinear perturbation by auxiliary algebraic Riccati matrix equations. In [23], Yu and Liao got several BIBO stability in mean square in term of Razumikhin technique and comparison principle. In practice, when modeling real control systems, stochastic disturbances are probably part of the main sources leading to unwilling behaviors of concerned systems. The behavior of the stochastic process is a non-deterministic factor, namely, the system's later states are determined both by the process predictable actions and by random elements. Stochastic control systems are more applicable to solving problems that are environmental noise in nature or related to biological realities. Thus, the mean square BIBO stability for stochastic control systems are necessary.

It should be pointed out that, to date, almost all results concerning BIBO stability analysis problems for control systems have been on continuous-time models. In implementations and applications of control systems, however, discrete-time control systems play a more important role than their continuous-time counterparts in today's digital world. If one wants to simulate or compute the continuous-time systems, it is essential to formulate the discrete-time analogue so as to investigate the dynamical characteristics. Unfortunately, so far, the problem of BIBO stability of discrete-time control systems has not been fully investigated. It remains challenging. The relevant literature is also rare, Please refer to [9,10], and some references there. Yet, as far as we know, the mean square BIBO stabilization for the discrete-time stochastic variable delay systems has not been studied. The contribution of this paper is the initial attempt to study the BIBO stability analysis problem for such kind of control systems.

Most of the aforementioned efforts have given stability conditions on how to construct a suitable Lyapunov functional. Generally speaking, the more effective information about the time delay the constructed Lyapunov functional includes, the less conservatism the induced criterion may provide. In [24], a delay midpoint method was proposed to study the stability problem for a class of continuous-time linear systems. By employing the midpoint, the time delays variation interval was divided into two subintervals with equal length and the midpoint was involved in constructing the Lyapunov functional. This method then found many successful applications in [7,25]. In fact, at any instant, the value of the time delay is located in one subinterval since the point  $\delta$  of the time delay's variation interval is introduced, and motivated by which we may exploit new analysis method to achieve novel BIBO stability criteria for discrete-time stochastic systems.

Motivated by the above discussions, the main aim of this paper is to study the BIBO stabilization in mean square for the discrete-time stochastic control systems with time delays and nonlinear perturbations. By introducing a point in the time delay's variation interval, the time delays variation interval is divided into two subintervals. Then we constructed special Lyapunov–Krasovskii functional and checked its variation in the two subintervals respectively. That is, the proposed Lyapunov–Krasovskii functional is different when the time delay belongs to different subintervals. So, some novel delay-dependent stability criteria for the discrete-time stochastic control system are derived. At the end, a numerical example is provided to demonstrate the effectiveness of the derived results.

**Notations.** The notations are quite standard. Throughout this paper,  $R^n$  and  $R^{n \times m}$  denote, respectively, the  $n$ -dimensioned Euclidean space and the set of all  $n \times m$  real matrices. The superscript “ $T$ ” denotes the transpose and the notation  $X \geq Y$  (respective  $X > Y$ ) means that  $X$  and  $Y$  are symmetric matrices, and that  $X - Y$  is positive semi-definitive (respective positive definite).  $\|\cdot\|$  is the Euclidean norm in  $R^n$ .  $N^+$  is the positive integer set.  $I$  is the identity matrix with compatible dimension. If  $A$  is a matrix, denote  $\|A\|$  as its operator norm, i.e.,  $\|A\| = \sup\{\|Ax\| : \|x\| = 1\} = \sqrt{\lambda_{\max}(A^T A)}$ , where  $\lambda_{\max}(A)$  (respectively,  $\lambda_{\min}(A)$ ) means the largest (respectively, smallest) of  $A$ . Moreover, let  $(\Omega, F, \{F_t\}_{t \geq 0}, P)$  be a complete probability space with a filtration  $\{F_t\}_{t \geq 0}$  satisfying the usual conditions (i.e, the filtration contains all  $P$ -null sets and is right continuous).  $\mathbb{E}\{\cdot\}$  stands for the mathematical expectation operator with respect to the given probability measure  $P$ . A asterisk  $*$  in a matrix is used to denote a term that is induced by symmetry. Matrices, if not explicit and specified, are assumed to be of compatible dimensions.  $N[a, b] = \{a, a + 1, \dots, b\}$ . Sometimes, the arguments of function will be omitted in the analysis when no confusion would arise.

## 2. Problem formulation and Preliminaries

Consider the following discrete-time stochastic control system with time-varying delays and nonlinear perturbations described by

$$\begin{cases} x(k+1) = Ax(k) + Bg(x(k)) + Cg(x(k-\tau(k))) + Du(k) + f(k, x(k), x(k-\tau(k))) \\ \quad + \sigma(k, x(k), x(k-\tau(k)))\omega(k+1), k \in N^+ \\ y(k) = Mx(k), \end{cases} \quad (1)$$

where  $x(k) = [x_1(k), x_2(k), \dots, x_n(k)]^T \in R^n$  denotes the state vector,  $u(k) = [u_1(k), u_2(k), \dots, u_m(k)]^T \in R^m$  is the control input vector,  $y(k) = [y_1(k), y_2(k), \dots, y_n(k)]^T \in R^n$  is the control output vector,  $g(x(k)) = [g_1(x(k)), g_2(x(k)), \dots, g_n(x(k))]^T \in R^n$ ,  $A, B, C, M \in R^{n \times n}$  are constant matrices, and  $D \in R^{n \times m}$  is a constant matrix, the positive integer  $\tau(k)$  is time varying delay satisfying

$$\tau_1 \leq \tau(k) \leq \tau_2, k \in N^+$$

with  $\tau_1$  and  $\tau_2$  are known positive integers. The initial condition associated with model (1) is given by

$$x(k) = \phi(k), k \in [-\tau_2, 0]$$

and  $f(k, x(k), x(k-\tau(k))) \in C(R^+ \times R^n \times R^n, R^n)$  is the nonlinear vector-valued perturbation bounded in magnitude as

$$\|f(k, x(k), x(k-\tau(k)))\|^2 \leq \alpha_1 \|x(k)\|^2 + \alpha_2 \|x(k-\tau(k))\|^2, \quad (2)$$

here  $\alpha_1, \alpha_2$  are known positive constants.  $\omega(k)$  is a scalar Wiener process (Brownian motion) on  $(\Omega, F, \{F_t\}_{t \geq 0}, P)$  with

$$E(\omega(k)) = 0, \quad E(\omega(k)^2) = 1, \quad E(\omega(i)\omega(j)) = 0, i \neq j.$$

To obtain the control law described by (1) of tracking out the reference input of the system, we let the controller be in the form of

$$u(k) = Kx(k) + r(k), \quad (3)$$

where  $K$  is the feedback gain matrix, and  $r(k)$  is the reference input.

**Assumption1.** For any  $\xi_1, \xi_2 \in R, \xi_1 \neq \xi_2$ ,

$$\gamma_i^- \leq \frac{g_i(\xi_1) - g_i(\xi_2)}{\xi_1 - \xi_2} \leq \gamma_i^+, \quad (4)$$

where  $\gamma_i^-$  and  $\gamma_i^+$  are known constant scalars.

**Remark1.** The constants  $\gamma_i^-, \gamma_i^+$  in Assumption 1 are allowed to be positive negative or zero. Hence, the function  $g(x(k))$  could be non-monotonic and is more general than the usual sigmoid functions or the recently commonly used Lipschitz conditions.

**Assumption2.**  $\sigma(k, x(k), x(k-\tau(k))) : R \times R^n \times R^n \rightarrow R^n$  is the continuous function, which is assumed to satisfy

$$\sigma^T(k, x(k), x(k-\tau(k)))\sigma^T(k, x(k), x(k-\tau(k))) \leq \begin{pmatrix} x(k) \\ x(k-\tau(k)) \end{pmatrix}^T \begin{pmatrix} G_1 & G_2 \\ * & G_3 \end{pmatrix} \begin{pmatrix} x(k) \\ x(k-\tau(k)) \end{pmatrix}, \quad (5)$$

**Remark2.** Choose  $G_1 = \rho_1 I, G_2 = 0, G_3 = \rho_2 I$ , we can find that (5) reduces to

$$\sigma^T(k, x(k), x(k-\tau(k)))\sigma^T(k, x(k), x(k-\tau(k))) \leq \rho_1 \|x(k)\|^2 + \rho_2 \|x(k-\tau(k))\|^2, \quad (6)$$

where  $\rho_1 > 0, \rho_2 > 0$  are known constant scalars. Thus the assumption condition (6), which was discussed in many references, is a special case of the assumption condition (5). It should be pointed out that the

delay-depended BIBO stability conditions of discrete stochastic control systems with varying time delays and nonlinear perturbations by (5) are generally less conservative than that by (6).

At the end of this section, let us introduce some important definitions and lemmas which will be used in the sequel.

**Definition1**([23,9]). A vector function  $r(k) = (r_1(k), r_2(k), \dots, r_n(k))^T$  is said to be an element of  $L_\infty^n$ , if  $\|r\|_\infty = \sup_{k \in \mathbb{N}[0, \infty)} \|r(k)\| < +\infty$ , where  $\|\cdot\|$  denotes the Euclid norm in  $R^n$  or the norm of a matrix.

**Definition2**([23,9]). The nonlinear stochastic control system (1) is said to be BIBO stabilized in mean square, if one can constructs a controller (3) such that the output  $y(k)$  satisfies

$$\mathbb{E}(\|y(k)\|^2) \leq N_1 + N_2 \|r\|_\infty^2, \tag{7}$$

where  $N_1, N_2$  are positive constants.

The following Lemmas has been referred to in many references.

**Lemma1.** For any given vectors  $v_i \in R^n, i = 1, 2, \dots, n$ , the following inequality holds:

$$\left[ \sum_{i=1}^n v_i \right]^T \left[ \sum_{i=1}^n v_i \right] \leq n \sum_{i=1}^n v_i^T v_i.$$

**Lemma2.** Let  $x, y \in R^n$  and any  $n \times n$  positive-definite matrix  $Q > 0$ . Then, we have

$$2x^T y \leq x^T Q^{-1} x + y^T Q y.$$

**Lemma3.** (Schur complement) Given the constant matrices  $\Omega_1, \Omega_2$  and  $\Omega_3$  with appropriate dimensions, where  $\Omega_1 = \Omega_1^T$  and  $\Omega_2 = \Omega_2^T > 0$ . Then  $\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$  if and only if

$$\begin{pmatrix} \Omega_1 & \Omega_3^T \\ * & -\Omega_2 \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -\Omega_2 & \Omega_3 \\ * & \Omega_1 \end{pmatrix} < 0.$$

### 3. Mean square BIBO stability of nonlinear stochastic control systems

In this section, we shall establish our main criterion based on the LMI approach. For the convenience of presentation, in the following, we denote

$$\begin{aligned} \Gamma_1 &= \text{diag}\{\gamma_1^-, \gamma_2^-, \dots, \gamma_n^-\}, \\ \Gamma_2 &= \text{diag}\{\gamma_1^+, \gamma_2^+, \dots, \gamma_n^+\}, \\ \Gamma_3 &= \text{diag}\{\gamma_1^- \gamma_1^+, \gamma_2^- \gamma_2^+, \dots, \gamma_n^- \gamma_n^+\}, \\ \Gamma_4 &= \text{diag}\left\{ \frac{\gamma_1^- + \gamma_1^+}{2}, \frac{\gamma_2^- + \gamma_2^+}{2}, \dots, \frac{\gamma_n^- + \gamma_n^+}{2} \right\}, \\ \delta &= \frac{\tau_1 + \tau_2}{2} - \frac{\min\{(-1)^{\tau_1 + \tau_2}, 0\}}{2}, \\ a &= \tau_2 - \tau_1 + 1, \quad b = \tau_2 - \delta, \\ c &= \begin{cases} \delta - \tau_1, & \tau_1 \leq \tau(k) \leq \delta \\ \tau_2 - \delta, & \delta < \tau(k) \leq \tau_2 \end{cases}, \\ \theta(k) &= \begin{cases} x(k - \tau_1), & \tau_1 \leq \tau(k) \leq \delta \\ x(k - \tau_2), & \delta < \tau(k) \leq \tau_2 \end{cases}, \\ \beta &= \begin{cases} \frac{\tau(k) - \tau_1}{\delta - \tau_1}, & \tau_1 \leq \tau(k) \leq \delta \\ \frac{\tau_2 - \tau(k)}{\tau_2 - \delta}, & \delta < \tau(k) \leq \tau_2 \end{cases}. \end{aligned}$$

**Theorem 1.** For given positive integers  $\tau_1 > 0, \tau_2 > 0$ , under Assumption 1 and Assumption 2, the nonlinear discrete-time stochastic control system (1) with the controller (3) is BIBO stabilizable in mean square, if there exist symmetric positive-definite matrices  $P, Q_1, Q_2, Q_3, Q_4, Z_1, Z_2$  with appropriate

dimensional, positive-definite diagonal matrices  $H, R, \Lambda_1, \Lambda_2$ , constant  $\lambda^* > 0$  such that the following two LMIs hold:

$$P + 2(\delta^2 Z_1 + c^2 Z_2) \leq \lambda^* I, \tag{8}$$

$$\Xi = \begin{pmatrix} \Xi_{11} & \lambda^* G_2 & \Xi_{13} & 0 & Z_1 & 0 & 0 & \sqrt{2}(A + DK) \\ * & \Xi_{22} & 0 & \Xi_{24} & \Xi_{25} & \Xi_{26} & 0 & 0 \\ * & * & \Xi_{33} & \Xi_{34} & 0 & 0 & \Xi_{37} & \frac{\sqrt{2}}{2} B \\ * & * & * & \Xi_{44} & 0 & 0 & \Xi_{47} & \frac{\sqrt{2}}{2} C \\ * & * & * & * & \Xi_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \Xi_{66} & 0 & 0 \\ * & * & * & * & * & * & \Xi_{77} & \frac{\sqrt{2}}{2} I \\ * & * & * & * & * & * & * & -\frac{1}{\lambda^*} I \end{pmatrix} < 0, \tag{9}$$

where

$$\Xi_{11} = Q_1 - 2a\Gamma_1 H + 2a\Gamma_2 R - Z_1 + Q_4 - \Gamma_3 \Lambda_1 - P + \left(\frac{a(\tau_2 - \tau_1)}{2} + \tau_2\right) Q_2 + 2\lambda^* \alpha_1 I + \lambda^* G_1 + 2(\delta^2 Z_1 + C^2 Z_2),$$

$$\Xi_{13} = aH - aR + \Gamma_4 \Lambda_1,$$

$$\Xi_{22} = 2\Gamma_1 H - 2\Gamma_2 R - \Gamma_3 \Lambda_2 - Q_2 - 3Z_2 + 2\lambda^* \alpha_2 I - \lambda^* G_3$$

$$\Xi_{24} = -H + R + \Gamma_4 \Lambda_2,$$

$$\Xi_{25} = (1 + \beta)Z_2,$$

$$\Xi_{26} = Z_2 + (1 - \beta)Z_2,$$

$$\Xi_{33} = -\Lambda_1 + aQ_3 + \frac{3}{2} B^T \lambda^* I B,$$

$$\Xi_{34} = \frac{1}{2} B^T \lambda^* I C,$$

$$\Xi_{37} = \frac{1}{2} B^T \lambda^* I,$$

$$\Xi_{44} = -\Lambda_2 - Q_3 + \frac{3}{2} C^T \lambda^* I C,$$

$$\Xi_{47} = \frac{1}{2} C^T \lambda^* I C,$$

$$\Xi_{55} = -Q_1 - Z_2 - \beta Z_2 - Z_1,$$

$$\Xi_{66} = -Q_4 - 2Z_2,$$

$$\Xi_{77} = -\frac{1}{2} \lambda^* I,$$

*Proof.* In order to establish the stability conditions, we introduce the following Lyapunov-Krasovskii functional candidate for system (1):

$$V(k) = \sum_{i=1}^6 V_i(k), \tag{10}$$

where

$$V_1(k) = x^T(k) P x(k),$$

$$V_2(k) = \sum_{i=k-\delta}^{k-1} x^T(i) Q_1 x(i) + \sum_{j=-\tau(k)}^{-1} \sum_{i=k+j}^{k-1} x^T(i) Q_2 x(i) + \sum_{j=-\tau_1-1}^{-1} \sum_{i=j+1}^{-1} \sum_{l=k+i}^{k-1} x^T(l) Q_2 x(l),$$

$$V_3(k) = 2 \sum_{j=-\tau_2+1}^{-\tau_1+1} \sum_{i=k-1+j}^{k-1} [(g(x(i)) - \Gamma_1 x(i))^T H + (\Gamma_2 x(i) - g(x(i)))^T R] x(i),$$

$$V_4(k) = \delta \sum_{j=-\delta}^{-1} \sum_{i=k-1+j}^{k-1} \eta^T(i) Z_1 \eta(i), \eta(k) = x(k+1) - x(k),$$

$$V_5(k) = \sum_{j=-\tau_2+1}^{-\tau_1+1} \sum_{i=k-1+j}^{k-1} g^T(x(i)) Q_3 g(x(i)),$$

$$V_6(k) = \begin{cases} (\delta - \tau_1) \sum_{j=-\delta}^{-\tau_1-1} \sum_{i=k+j}^{k-1} \eta^T(i) Z_2 \eta(i) \\ \quad + \sum_{i=k-\tau_1}^{k-1} x^T(i) Q_4 x(i), \tau_1 \leq \tau(k) \leq \delta \\ (\tau_2 - \delta) \sum_{j=-\tau_2}^{-\delta-1} \sum_{i=k+j}^{k-1} \eta^T(i) Z_2 \eta(i) \\ \quad + \sum_{i=k-\tau_2}^{k-1} x^T(i) Q_4 x(i), \delta < \tau(k) \leq \tau_2 \end{cases}.$$

Define  $\Delta V(k) = V(k+1) - V(k)$  then along the solution of (1), by lemma 1, we have

$$E\{\Delta V_1(k)\} = E\{x^T(k+1) P x(k+1) - x^T(k) P x(k)\} \tag{11}$$

$$\begin{aligned} E\{\Delta V_2(k)\} &= E\left\{ \left( \sum_{i=k-\delta+1}^k - \sum_{i=k-\delta}^{k-1} \right) x^T(i) Q_1 x(i) + \left( \sum_{j=-\tau(k+1)}^{-1} \sum_{i=k+1+j}^k - \sum_{j=-\tau(k)}^{-1} \sum_{i=k+j}^{k-1} \right) x^T(i) Q_2 x(i) \right. \\ &\quad \left. + \sum_{j=-\tau_2}^{-\tau_1-1} \sum_{i=j+1}^{-1} \left( \sum_{l=k+i+1}^k - \sum_{l=k+i}^{k-1} \right) x^T(l) Q_2 x(l) \right\} \\ &\leq E\{x^T(k) Q_1 x(k) - x^T(k-\delta) Q_1 x(k-\delta)\} + \left( \sum_{j=-\tau_2}^{-1} \sum_{i=k+1+j}^{k-1} - \sum_{j=-\tau(k)}^{-1} \sum_{i=k+j}^{k-1} \right) x^T(i) Q_2 x(i) \\ &\quad + \sum_{j=-\tau_2}^{-1} x^T(k) Q_2 x(k) + \frac{a(\tau_2-\tau_1)}{2} x^T(k) Q_2 x(k) - \sum_{j=-\tau_2}^{-\tau_1-1} \sum_{i=k+j+1}^{k-1} x^T(i) Q_2 x(i) \\ &\leq E\{x^T(k) Q_1 x(k) - x^T(k-\delta) Q_1 x(k-\delta)\} + \left( \sum_{j=-\tau_2}^{-1} \sum_{i=k+1+j}^{k-1} - \sum_{j=-\tau_1}^{-1} \sum_{i=k+j+1}^{k-1} \right) x^T(i) Q_2 x(i) \\ &\quad + \tau_2 x^T(k) Q_2 x(k) - x^T(k-\tau(k)) Q_2 x(k-\tau(k)) + \frac{a(\tau_2-\tau_1)}{2} x^T(k) Q_2 x(k) - \sum_{j=-\tau_2}^{-\tau_1-1} \sum_{i=k+j+1}^{k-1} x^T(i) Q_2 x(i) \\ &\leq E\{x^T(k) Q_1 x(k) - x^T(k-\delta) Q_1 x(k-\delta) - x^T(k-\tau(k)) Q_2 x(k-\tau(k))\} \\ &\quad + \left( \frac{a(\tau_2-\tau_1)}{2} + \tau_2 \right) x^T(k) Q_2 x(k), \end{aligned} \tag{12}$$

$$\begin{aligned} E\{\Delta V_3(k)\} &= 2E\left\{ \sum_{j=-\tau_2+1}^{-\tau_1+1} \left( \sum_{i=k+j}^k - \sum_{i=k-1+j}^{k-1} \right) [(g(x(i)) - \Gamma_1 x(i))^T H + (\Gamma_2 x(i) - g(x(i)))^T R] x(i) \right\} \\ &\leq 2E\{a[g(x(k)) - \Gamma_1 x(k)]^T H x(k) + a[\Gamma_2 x(k) - g(x(k))]^T R x(k) \\ &\quad - [g(x(k-\tau(k))) - \Gamma_1 x(k-\tau(k))]^T H x(k-\tau(k)) \\ &\quad - [\Gamma_2 x(k-\tau(k)) - g(x(k-\tau(k)))]^T R x(k-\tau(k))\}, \end{aligned} \tag{13}$$

$$\begin{aligned} E\{\Delta V_4(k)\} &= E\left\{ \delta \sum_{j=-\delta}^{-1} \left( \sum_{i=k+j}^k - \sum_{i=k-1+j}^{k-1} \right) \eta^T(i) Z_1 \eta(i) \right\} \\ &= E\left\{ \delta^2 \eta^T(k) Z_1 \eta(k) - \delta \sum_{j=k-\delta}^{k-1} \eta^T(j) Z_1 \eta(j) \right\} \end{aligned}$$

$$\leq E\{\delta^2 \eta^T(k) Z_1 \eta(k) - \sum_{j=k-\delta}^{k-1} \eta^T(j) Z_1 \sum_{j=k-\delta}^{k-1} \eta(j)\}, \tag{14}$$

$$\tag{15}$$

Note that

$$- \sum_{i=k-\delta}^{k-1} \eta^T(i) Z_1 \sum_{i=k-\delta}^{k-1} \eta(i) = \begin{pmatrix} x(k) \\ x(k-\delta) \end{pmatrix}^T \begin{pmatrix} -Z_1 & Z_1 \\ * & -Z_1 \end{pmatrix} \begin{pmatrix} x(k) \\ x(k-\delta) \end{pmatrix},$$

$$\begin{aligned} E\{\Delta V_5(k)\} &= E\left\{ \sum_{j=-\tau_2+1}^{-\tau_1+1} \left( \sum_{i=k+j}^k - \sum_{i=k-1+j}^{k-1} \right) g^T(x(i)) Q_3 g(x(i)) \right\} \\ &\leq E\{a g^T(x(k)) Q_3 g(x(k)) \sum_{j=k-\tau_2}^{k-\tau_1} g^T(x(j)) Q_3 g(x(j))\} \\ &\leq E\{a g^T(x(k)) Q_3 g(x(k)) - g^T(x(k-\tau(k))) Q_3 g(x(k-\tau(k)))\}, \end{aligned} \tag{16}$$

$$E\{\Delta V_6(k)\} = E\{x^T(k) Q_4 x(k) - \theta^T(k) Q_4 \theta(k) + c^2 \eta^T(k) Z_2 \eta(k) - c \psi(k)\}, \tag{17}$$

where

$$\psi(k) = \begin{cases} \sum_{i=k-\delta}^{k-\tau_1-1} \eta^T(i) Z_2 \eta(i), & \tau_1 \leq \tau(k) \leq \delta \\ \sum_{i=k-\tau_2}^{k-\delta-1} \eta^T(i) Z_2 \eta(i), & \delta < \tau(k) \leq \tau_2 \end{cases},$$

When  $\tau_1 \leq \tau(k) \leq \delta$ , it is easy to compute that

$$\begin{aligned} -c \psi(k) &\leq -(1-\beta) \sum_{i=k-\tau(k)}^{k-\tau_1-1} \eta^T(i) Z_2 \sum_{i=k-\tau(k)}^{k-\tau_1-1} \eta(i) - \sum_{i=k-\tau(k)}^{k-\tau_1-1} \eta^T(i) Z_2 \sum_{i=k-\tau(k)}^{k-\tau_1-1} \eta(i) \\ &\quad - \sum_{i=k-\delta}^{k-\tau(k)-1} \eta^T(i) Z_2 \sum_{i=k-\delta}^{k-\tau(k)-1} \eta(i) - \beta(1-\beta) \sum_{i=k-\delta}^{k-\tau(k)-1} \eta^T(i) Z_2 \sum_{i=k-\delta}^{k-\tau(k)-1} \eta(i). \end{aligned} \tag{18}$$

When  $\delta < \tau(k) \leq \tau_2$ , similarly we can have

$$\begin{aligned} -c \psi(k) &= -[(\tau_2 - \tau(k)) + (\tau(k) - \delta)] \sum_{i=k-\tau(k)}^{k-\delta-1} \eta^T(i) Z_2 \eta(i) - [(\tau_2 - \tau(k)) + (\tau(k) - \delta)] \sum_{i=k-\tau_2}^{k-\tau(k)-1} \eta^T(i) Z_2 \eta(i) \\ &\leq -\beta \sum_{i=k-\tau(k)}^{k-\delta-1} \eta^T(i) Z_2 \sum_{i=k-\tau(k)}^{k-\delta-1} \eta(i) - \sum_{i=k-\tau(k)}^{k-\delta-1} \eta^T(i) Z_2 \sum_{i=k-\tau(k)}^{k-\delta-1} \eta(i) - \sum_{i=k-\tau_2}^{k-\tau(k)-1} \eta^T(i) Z_2 \sum_{i=k-\tau_2}^{k-\tau(k)-1} \eta(i) \\ &\quad - (1-\beta) \sum_{i=k-\tau_2}^{k-\tau(k)-1} \eta^T(i) Z_2 \sum_{i=k-\tau_2}^{k-\tau(k)-1} \eta(i), \end{aligned} \tag{19}$$

Combining (16), (17) with (18), we have

$$\begin{aligned} E\{\Delta V_6(k)\} &= E\left\{ x^T(k) Q_4 x(k) - \theta^T(k) Q_4 \theta(k) + c^2 \eta^T(k) Z_2 \eta(k) \right. \\ &\quad + \begin{pmatrix} x(k-\tau(k)) \\ x(k-\delta) \\ \theta(k) \end{pmatrix}^T \begin{pmatrix} -2Z_2 & Z_2 & Z_2 \\ * & -Z_2 & 0 \\ * & * & -Z_2 \end{pmatrix} \begin{pmatrix} x(k-\tau(k)) \\ x(k-\delta) \\ \theta(k) \end{pmatrix} \\ &\quad + \beta \begin{pmatrix} x(k-\tau(k)) \\ x(k-\delta) \end{pmatrix}^T \begin{pmatrix} -Z_2 & Z_2 \\ * & -Z_2 \end{pmatrix} \begin{pmatrix} x(k-\tau(k)) \\ x(k-\delta) \end{pmatrix} \\ &\quad \left. + (1-\beta) \begin{pmatrix} x(k-\tau(k)) \\ \theta(k) \end{pmatrix}^T \begin{pmatrix} -Z_2 & Z_2 \\ * & -Z_2 \end{pmatrix} \begin{pmatrix} x(k-\tau(k)) \\ \theta(k) \end{pmatrix} \right\} \end{aligned}$$

From (4), it follows that

$$(g_i(x(k)) - \gamma_i^+ x_i(k))(g_i(x(k)) - \gamma_i^- x_i(k)) \leq 0, i = 1, 2, \dots, n.$$

which are equivalent to

$$\begin{pmatrix} x(k) \\ g(x(k)) \end{pmatrix}^T \begin{pmatrix} \gamma_i^- \gamma_i^+ e_i e_i^T & -\frac{\gamma_i^- + \gamma_i^+}{2} e_i e_i^T \\ * & e_i e_i^T \end{pmatrix} \begin{pmatrix} x(k) \\ g(x(k)) \end{pmatrix} \leq 0, \tag{20}$$

$$\begin{pmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{pmatrix}^T \begin{pmatrix} \gamma_i^- \gamma_i^+ e_i e_i^T & -\frac{\gamma_i^- + \gamma_i^+}{2} e_i e_i^T \\ * & e_i e_i^T \end{pmatrix} \begin{pmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{pmatrix} \leq 0, \tag{21}$$

where  $e_i$  denotes the unit column vector having one element on its  $i$ -th row and zeros elsewhere. Then from (19) and (20), for any matrices  $\Lambda_i = \text{diag}\{\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{in}\} > 0, i = 1, 2$ , it follows

$$\begin{pmatrix} x(k) \\ g(x(k)) \end{pmatrix}^T \begin{pmatrix} -\Gamma_3 \Lambda_1 & \Gamma_4 \Lambda_1 \\ * & -\Lambda_1 \end{pmatrix} \begin{pmatrix} x(k) \\ g(x(k)) \end{pmatrix} \geq 0, \tag{22}$$

$$\begin{pmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{pmatrix}^T \begin{pmatrix} -\Gamma_3 \Lambda_2 & \Gamma_4 \Lambda_2 \\ * & -\Lambda_2 \end{pmatrix} \begin{pmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{pmatrix} \geq 0, \tag{23}$$

then from (11) to (22), we have

$$E[\Delta V(k)] \leq E\{\xi^T(k)\Omega\xi(k) + \delta^2\eta^T(k)Z_1\eta(k) + c^2\eta^T(k)Z_2\eta(k) + x^T(k+1)Px(k+1)\} \tag{24}$$

where

$$\xi^T(k) = [x^T(k), x^T(k - \tau(k)), g^T(x(k)), g^T(x(k - \tau(k))), x^T(k - \delta), \theta^T(k), f^T].$$

$$\tilde{\Omega} = \begin{pmatrix} \tilde{\Omega}_{11} & 0 & \tilde{\Omega}_{13} & 0 & Z_1 & 0 & 0 \\ * & \tilde{\Omega}_{22} & 0 & \tilde{\Omega}_{24} & \tilde{\Omega}_{25} & \tilde{\Omega}_{26} & 0 \\ * & * & \tilde{\Omega}_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \tilde{\Omega}_{44} & 0 & 0 & 0 \\ * & * & * & * & \tilde{\Omega}_{55} & 0 & 0 \\ * & * & * & * & * & \tilde{\Omega}_{66} & 0 \\ * & * & * & * & * & * & 0 \end{pmatrix}$$

with

$$\tilde{\Omega}_{11} = Q_1 - 2a\Gamma_1 H + 2a\Gamma_2 R - Z_1 + Q_4 - \Gamma_3 \Lambda_1 - P + \left(\frac{a(\tau_2 - \tau_1)}{2} + \tau_2\right)Q_2,$$

$$\tilde{\Omega}_{13} = aH - aR + \Gamma_4 \Lambda_1,$$

$$\tilde{\Omega}_{22} = 2\Gamma_1 H - 2\Gamma_2 R - \Gamma_3 \Lambda_2 - Q_2 - 3Z_2$$

$$\tilde{\Omega}_{24} = -H + R + \Gamma_4 \Lambda_2,$$

$$\tilde{\Omega}_{25} = (1 + \beta)Z_2,$$

$$\tilde{\Omega}_{26} = Z_2 + (1 - \beta)Z_2,$$

$$\tilde{\Omega}_{33} = -\Lambda_1 + aQ_3,$$

$$\tilde{\Omega}_{44} = -\Lambda_2 - Q_3,$$

$$\tilde{\Omega}_{55} = -Q_1 - Z_2 - \beta Z_2 - Z_1,$$

$$\tilde{\Omega}_{66} = -Q_4 - 2Z_2.$$

Note that, from (8), we have

$$\begin{aligned}
 & E\{\delta^2\eta^T(k)Z_1\eta(k) + c^2\eta^T(k)Z_2\eta(k) + x^T(k+1)Px(k+1)\} \\
 & \leq E\{x^T(k+1)[p + 2(\delta^2Z_1 + c^2Z_2)]x(k+1) + x^T(k)2(\delta^2Z_1 + c^2Z_2)x(k)\} \\
 & \leq E\{x^T(k+1)\lambda^*Ix(k+1) + x^T(k)2(\delta^2Z_1 + c^2Z_2)x(k)\}
 \end{aligned}
 \tag{25}$$

from (1) and lemma 2,

$$\begin{aligned}
 E\{x^T(k+1)\lambda^*Ix(k+1)\} & \leq E\{\xi^T(k)\tilde{\Omega}^T\lambda^*I\tilde{\Omega}\xi(k) + \sigma^T\lambda^*I\sigma + \frac{3}{2}g^T(x(k))B^T\lambda^*IBg(x(k)) \\
 & \quad + \frac{3}{2}g^T(x(k-\tau(k)))C^T\lambda^*ICg(x(k-\tau(k))) + g^T(x(k))B^T\lambda^*ICg(x(k-\tau(k))) \\
 & \quad + g^T(x(k))B^T\lambda^*If(k, x(k), x(k-\tau(k))) + 5r^T(k)D^T\lambda^*IDr(k) \\
 & \quad + g^T(x(k-\tau(k)))C^T\lambda^*If(k, x(k), x(k-\tau(k))) \\
 & \quad + \frac{3}{2}f^T(k, x(k), x(k-\tau(k)))\lambda^*If(k, x(k), x(k-\tau(k)))\}
 \end{aligned}
 \tag{26}$$

where

$$\tilde{\Omega} = (\sqrt{2}(A + DK), 0, \frac{\sqrt{2}}{2}B, \frac{\sqrt{2}}{2}C, 0, 0, \frac{\sqrt{2}}{2}I)$$

Then from (11) to (25), by (2), we have

$$E\{\Delta V(k)\} \leq E\{\xi^T(k)[\hat{\Omega} + \tilde{\Omega}^T\lambda^*I\tilde{\Omega}]\xi(k)\} + \rho\|r\|_\infty^2
 \tag{27}$$

where

$$\begin{aligned}
 \rho & = 5\lambda^*\|D\|^2 \\
 \hat{\Omega} & = \begin{pmatrix} \hat{\Omega}_{11} & \lambda^*G_2 & \hat{\Omega}_{13} & 0 & Z_1 & 0 & 0 \\ * & \hat{\Omega}_{22} & 0 & \hat{\Omega}_{24} & \hat{\Omega}_{25} & \hat{\Omega}_{26} & 0 \\ * & * & \hat{\Omega}_{33} & \hat{\Omega}_{34} & 0 & 0 & \hat{\Omega}_{37} \\ * & * & * & \hat{\Omega}_{44} & 0 & 0 & \hat{\Omega}_{47} \\ * & * & * & * & \hat{\Omega}_{55} & 0 & 0 \\ * & * & * & * & * & \hat{\Omega}_{66} & 0 \\ * & * & * & * & * & * & \hat{\Omega}_{77} \end{pmatrix}
 \end{aligned}$$

with

$$\begin{aligned}
 \hat{\Omega}_{11} & = Q_1 - 2a\Gamma_1H + 2a\Gamma_2R - Z_1 + Q_4 - \Gamma_3\Lambda_1 - P + \left(\frac{a(\tau_2 - \tau_1)}{2} + \tau_2\right)Q_2 + 2\lambda^*\alpha_1I + \lambda^*G_1 + 2(\delta^2Z_1 + C^2Z_2), \\
 \hat{\Omega}_{13} & = aH - aR + \Gamma_4\Lambda_1, \\
 \hat{\Omega}_{22} & = 2\Gamma_1H - 2\Gamma_2R - \Gamma_3\Lambda_2 - Q_2 - 3Z_2 + 2\lambda^*\alpha_2I - \lambda^*G_3 \\
 \hat{\Omega}_{24} & = -H + R + \Gamma_4\Lambda_2, \\
 \hat{\Omega}_{25} & = (1 + \beta)Z_2, \\
 \hat{\Omega}_{26} & = Z_2 + (1 - \beta)Z_2, \\
 \hat{\Omega}_{33} & = -\Lambda_1 + aQ_3 + \frac{3}{2}B^T\lambda^*IB,
 \end{aligned}$$

$$\begin{aligned} \hat{Q}_{34} &= \frac{1}{2}B^T \lambda^* IC, \\ \hat{Q}_{37} &= \frac{1}{2}B^T \lambda^* I, \\ \hat{Q}_{44} &= -\Lambda_2 - Q_3 + \frac{3}{2}C^T \lambda^* IC, \\ \hat{Q}_{47} &= \frac{1}{2}C^T \lambda^* IC, \\ \hat{Q}_{55} &= -Q_1 - Z_2 - \beta Z_2 - Z_1, \\ \hat{Q}_{66} &= -Q_4 - 2Z_2, \\ \hat{Q}_{77} &= -\frac{1}{2}\lambda^* I, \end{aligned}$$

If the LMI (9) holds, by using lemma 3 and (26), it follows that there exists a sufficient small positive  $\epsilon > 0$ , such that

$$E\{\Delta V(k)\} \leq -\epsilon E\{\|x(k)\|^2\} + \rho \|r\|_\infty^2. \tag{28}$$

It is easy to derive that

$$V(k) \leq \mu_1 \|x(k)\|^2 + \mu_2 \sum_{i=k-\tau_2}^{k-1} \|x(i)\|^2, \tag{29}$$

with

$$\begin{aligned} \mu_1 &= \lambda_{\max}(P), \\ \mu_2 &= \lambda_{\max}[Q_1] + a(\|H\|^2 + \lambda_{\max}(\Gamma_1 H) + \lambda_{\max}(R^T R)) + 4\delta^2 \lambda_{\max}(Z_1) + \lambda_{\max}(Q_4) + 4a^2 \lambda_{\max}(Z_2) \\ &\quad + a[2 + \lambda_{\max}(Q_3)]\Gamma_2^T \Gamma_2 + [\tau_2 + (\tau_2 - \tau_1)(\tau_2 - 1)]\lambda_{\max}(Q_2). \end{aligned}$$

For any  $\theta > 1$ , from (27) and (28), it follows that

$$\begin{aligned} \theta^{j+1}V(j+1) - \theta^jV(j) &= \theta^{j+1}\Delta V(j) - \theta^j(\theta - 1)V(j) \\ &\leq \theta^j[-\epsilon\theta + (\theta - 1)\mu_1]\|x(j)\|^2 + \rho\theta\|r\|_\infty^2 + (\theta - 1)\mu_2 \sum_{i=j-\tau_2}^{j-1} \|x(i)\|^2. \end{aligned} \tag{30}$$

Summing up both sides of (29) from 0 to  $k - 1$ , we can obtain

$$\theta^k V(k) - V(0) \leq (\mu_1(\theta - 1) - \epsilon\theta) \sum_{j=0}^{k-1} \theta^j \|x(j)\|^2 + \rho \sum_{j=0}^{k-1} \theta^{j+1} \|r\|_\infty^2 + \mu_2(\theta - 1) \sum_{j=0}^{k-1} \sum_{i=j-\tau_2}^{j-1} \theta^j \|x(i)\|^2. \tag{31}$$

Also it is easy to compute that

$$\begin{aligned} \sum_{j=0}^{k-1} \sum_{i=j-\tau_2}^{j-1} \theta^j \|x(i)\|^2 &\leq \left( \sum_{i=-\tau_2}^{-1} \sum_{j=0}^{i+\tau_2} + \sum_{i=0}^{k-1-\tau_2} \sum_{j=i+1}^{i+\tau_2} + \sum_{i=k-\tau_2}^{k-1} \sum_{i+1}^{k-1} \right) \mu^j \|x(i)\|^2 \\ &\leq \tau_2 \theta^{\tau_2} \sup_{s \in [-\tau_2, 0]} \|x(s)\|^2 + \tau_2 \theta^{\tau_2} \sum_{i=0}^{k-1} \theta^i \|x(i)\|^2. \end{aligned} \tag{32}$$

Substituting (31) into (30) leads to

$$\theta^k V(k) - V(0) \leq \eta_1(\theta) \sup_{s \in [-\tau_2, 0]} \|x(s)\|^2 + \eta_2(\theta) \sum_{i=0}^{k-1} \theta^i \|x(i)\|^2 + \rho \sum_{j=0}^{k-1} \theta^{j+1} \|r\|_\infty^2, \tag{33}$$

where

$$\begin{aligned} \eta_1(\theta) &= \mu_2(\theta - 1)\tau_2\theta^{\tau_2}, \\ \eta_2(\theta) &= \mu_2(\theta - 1)\tau_2\theta^{\tau_2} + \mu_1(\theta - 1) - \epsilon\theta. \end{aligned}$$

Since  $\eta_2(1) < 0$ , there must exist a positive  $\theta_0 > 1$  such that  $\eta_2(\theta_0) < 0$ . Then we have

$$\begin{aligned} V(k) &\leq \eta_1(\theta_0)\left(\frac{1}{\theta_0}\right)^k \sup_{s \in [-\tau_2, 0]} \|x(s)\|^2 + \left(\frac{1}{\theta_0}\right)^k V(0) + \rho \sum_{j=0}^{k-1} \frac{1}{\theta^{k-j-1}} \|r\|_\infty^2 \\ &\leq (\eta_1(\theta_0) + \mu_1 + \mu_2 \tau_2) \sup_{s \in [-\tau_2, 0]} \|x(s)\|^2 + \frac{\rho}{\theta_0 - 1} \|r\|_\infty^2. \end{aligned} \tag{34}$$

On the other hand, by (10) we can get

$$V(k) \geq \lambda_{\min}(P) \|x(k)\|^2. \tag{35}$$

Combining (33) with (34), we have

$$E\{\|x(k)\|^2\} \leq \frac{\eta_1(\theta_0) + \mu_1 + \mu_2 \tau_2}{\lambda_{\min}(P)} \sup_{s \in [-\tau_2, 0]} E\|x(s)\|^2 + \frac{\rho}{\theta_0 - 1} \|r\|_\infty^2. \tag{36}$$

Thus

$$E\{\|y(k)\|^2\} \leq \|M\|^2 E\{\|x(k)\|^2\} \leq N_1 + N_2 \|r\|_\infty^2, \tag{37}$$

where

$$\begin{aligned} N_1 &= \|M\|^2 \frac{\eta_1(\theta_0) + \mu_1 + \mu_2 \tau_2}{\lambda_{\min}(P)} \sup_{s \in [-\tau_2, 0]} E\|x(s)\|^2, \\ N_2 &= \frac{\rho}{\theta_0 - 1} \|M\|^2. \end{aligned}$$

By Definition 2, the nonlinear discrete-time stochastic control system (1) is said to be BIBO stabilized in mean square. This completes the proof.  $\square$

If the stochastic term  $\omega(k)$  is removed in (1), the following results can be obtained.

**Corollary 1.** For given positive integers  $\tau_1 > 0, \tau_2 > 0$ , under Assumption 1 and Assumption 2, the nonlinear discrete-time stochastic control system (1) with the controller (3) is BIBO stabilizable in mean square, if there exist symmetric positive-definite matrices  $P, Q_1, Q_2, Q_3, Q_4, Z_1, Z_2$ , with appropriate dimensional, positive-definite diagonal matrices  $H, R, \Lambda_1, \Lambda_2$ , constant  $\lambda^* > 0$  such that the following two LMIs hold:

$$P + 2(\delta^2 Z_1 + c^2 Z_2) \leq \lambda^* I, \tag{38}$$

$$\Xi = \begin{pmatrix} \Xi_{11} & 0 & \Xi_{13} & 0 & Z_1 & 0 & 0 & \sqrt{2}(A + DK) \\ * & \Xi_{22} & 0 & \Xi_{24} & \Xi_{25} & \Xi_{26} & 0 & 0 \\ * & * & \Xi_{33} & \Xi_{34} & 0 & 0 & \Xi_{37} & \frac{\sqrt{2}}{2} B \\ * & * & * & \Xi_{44} & 0 & 0 & \Xi_{47} & \frac{\sqrt{2}}{2} C \\ * & * & * & * & \Xi_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \Xi_{66} & 0 & 0 \\ * & * & * & * & * & * & \Xi_{77} & \frac{\sqrt{2}}{2} I \\ * & * & * & * & * & * & * & -\frac{1}{\lambda^*} I \end{pmatrix} < 0 \tag{39}$$

where

$$\begin{aligned} \Xi_{11} &= Q_1 - 2a\Gamma_1 H + 2a\Gamma_2 R - Z_1 + Q_4 - \Gamma_3 \Lambda_1 - P + \left(\frac{a(\tau_2 - \tau_1)}{2} + \tau_2\right) Q_2 + 2\lambda^* \alpha_1 I + 2(\delta^2 Z_1 + C^2 Z_2), \\ \Xi_{13} &= aH - aR + \Gamma_4 \Lambda_1, \\ \Xi_{22} &= 2\Gamma_1 H - 2\Gamma_2 R - \Gamma_3 \Lambda_2 - Q_2 - 3Z_2 + 2\lambda^* \alpha_2 I \end{aligned}$$

$$\begin{aligned}
E_{24} &= -H + R + \Gamma_4 \Lambda_2, \\
E_{25} &= (1 + \beta)Z_2, \\
E_{26} &= Z_2 + (1 - \beta)Z_2, \\
E_{33} &= -\Lambda_1 + aQ_3 + \frac{3}{2}B^T \lambda^* IB, \\
E_{34} &= \frac{1}{2}B^T \lambda^* IC, \\
E_{37} &= \frac{1}{2}B^T \lambda^* I, \\
E_{44} &= -\Lambda_2 - Q_3 + \frac{3}{2}C^T \lambda^* IC, \\
E_{47} &= \frac{1}{2}C^T \lambda^* IC, \\
E_{55} &= -Q_1 - Z_2 - \beta Z_2 - Z_1, \\
E_{66} &= -Q_4 - 2Z_2, \\
E_{77} &= -\frac{1}{2}\lambda^* I,
\end{aligned}$$

*Proof.* The proof is straightforward and hence omitted.  $\square$

**Corollary 2 .** System (1) is also stable in mean square when all the conditions in Theorem 1 and Corollary 1 are satisfied, if the bounded input  $r(t) = 0$  in (3).

**Remark 3.** It is obvious that  $\delta$  divides the time delay's variation interval into two subintervals,  $[\tau_1, \delta]$  and  $(\delta, \tau_2]$ . Then the Lyapunov-Krasovskii functional is constructed for each subinterval, that is, the proposed Lyapunov-Krasovskii functional is different when the time-delay  $\tau(k)$  belongs to different subintervals. It has two features. Firstly, it makes full use of the information on the considered time delay. Secondly, the new state  $x(k - \delta)$  is introduced by  $V(k)$ .

**Remark 4.** In this paper, novel BIBO stability conditions for the system (1) are derived from checking the variation of derivatives of the Lyapunov-Krasovskii functional in each subinterval. It is different from [9], which checked the variation of the Lyapunov functional in the whole variation interval of the delay.

**Remark 5.** The BIBO stability conditions for the discrete-time system had been investigated in the recently reported paper [9]. However, the stochastic disturbances and nonlinear perturbations in the control systems had not been taken into consideration. In [9], the time delay was a constant, which was a special case of this paper when  $\tau_1 = \tau_2$ .

#### 4. Example

In this section, a numerical example is presented to show the validity of the main results derived from Section 3.

Considering the stochastic control system (1) with control law (3), the parameters are given by

$$A = \begin{pmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{pmatrix}, \quad B = \begin{pmatrix} -0.1 & 0.1 \\ -0.1 & 0.5 \end{pmatrix}, \quad C = \begin{pmatrix} 0.5 & 0.1 \\ 0.5 & 0.5 \end{pmatrix}, \quad D = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0.2 \end{pmatrix},$$

$G_1 = 0.001I$ ,  $G_2 = 0.002I$ ,  $G_3 = 0.02I$ ,  $f = [0.1x(k), \sqrt{0.2}x(k - \tau(k))]^T$ ,  $g_1(s) = \sin(0.2s) - 0.6 \cos(s)$ ,  $g_2(s) = \tanh(-0.4s)$ ,  $\varepsilon_i = 1, i = 1, 2, 3, 4$ . So it can be very easy to be verified that

$$\Gamma_1 = \begin{pmatrix} -0.8 & 0 \\ 0 & -0.4 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} -0.8 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} -0.64 & \\ 0 & 0 \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} 0 & 0 \\ 0 & -0.2 \end{pmatrix}.$$

$\tau_1 = 1$ ,  $\tau_2 = 5$ , meantime, we have  $a = 5$ ,  $b = 2$ ,  $c = 2$ ,  $\delta = 3$ .

Meanwhile, the corresponding values of  $\beta$  for various  $\tau(k)$  are listed as follows.

When  $\tau(k) = 1;5$  then  $\beta = 0$ ; when  $\tau(k) = 2$  then  $\beta = 1$ ; when  $\tau(k) = 3;4$  then  $\beta = 0.5$ ;  
By using the Matlab LMI Toolbox, we solved LMI(8) and LMI(9) and obtained the feasible solutions as follows:

when  $\beta = 0$ ,

$$P = \begin{pmatrix} 173.3668 & 0.2021 \\ 0.2021 & 112.6277 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 0.8581 & 0.0070 \\ 0.0070 & 0.3097 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 3.3990 & -0.0027 \\ -0.0027 & 2.6835 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 11.3028 & 0.0310 \\ 0.0310 & 14.0202 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1.4484 & 0.0010 \\ 0.0010 & 0.7920 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 4.6486 & 0.2164 \\ 0.2164 & 4.7879 \end{pmatrix},$$

$$Q_4 = \begin{pmatrix} 15.1835 & -0.0018 \\ -0.0018 & 15.0268 \end{pmatrix}, \quad R = \begin{pmatrix} 3.3329 & 0 \\ 0 & 7.4569 \end{pmatrix}, \quad H = \begin{pmatrix} 3.3431 & 0 \\ 0 & 8.6943 \end{pmatrix},$$

$$\Lambda_1 = \begin{pmatrix} 35.8656 & 0 \\ 0 & 42.0701 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 17.4126 & 0 \\ 0 & 25.0698 \end{pmatrix}, \quad K = \begin{pmatrix} 24.5051 & -1.9133 \\ -1.9133 & 20.7397 \end{pmatrix}$$

and  $\lambda^* = 238.9518$ .

when  $\beta = 0.5$ ,

$$P = \begin{pmatrix} 171.7588 & 0.2540 \\ 0.2540 & 111.8846 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 0.9518 & 0.0031 \\ 0.0031 & 0.4241 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 3.3952 & -0.0025 \\ -0.0025 & 2.6953 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 11.2321 & 0.0155 \\ 0.0155 & 13.1525 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1.4611 & 0.0007 \\ 0.0007 & 0.8352 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 4.4460 & 0.2055 \\ 0.2055 & 4.5303 \end{pmatrix},$$

$$Q_4 = \begin{pmatrix} 14.0165 & -0.0019 \\ -0.0019 & 14.0519 \end{pmatrix}, \quad R = \begin{pmatrix} 3.2462 & 0 \\ 0 & 7.1631 \end{pmatrix}, \quad H = \begin{pmatrix} 3.2553 & 0 \\ 0 & 8.3139 \end{pmatrix},$$

$$\Lambda_1 = \begin{pmatrix} 34.3462 & 0 \\ 0 & 40.0259 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 17.1514 & 0 \\ 0 & 24.2894 \end{pmatrix}, \quad K = \begin{pmatrix} 23.4860 & -1.7442 \\ -1.7442 & 20.0778 \end{pmatrix},$$

and  $\lambda^* = 234.9575$ .

when  $\beta = 1$ ,

$$P = \begin{pmatrix} 171.3560 & 0.2578 \\ 0.2578 & 111.3159 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 0.9531 & 0.0031 \\ 0.0031 & 0.4286 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 3.3791 & -0.0029 \\ -0.0029 & 2.7095 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 11.5764 & 0.0182 \\ 0.0182 & 13.0461 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1.4514 & 0.0006 \\ 0.0006 & 0.8124 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 4.4404 & 0.2055 \\ 0.2055 & 4.5245 \end{pmatrix},$$

$$Q_4 = \begin{pmatrix} 13.6802 & -0.0015 \\ -0.0015 & 13.8779 \end{pmatrix}, \quad R = \begin{pmatrix} 3.2446 & 0 \\ 0 & 7.1255 \end{pmatrix}, \quad H = \begin{pmatrix} 3.2534 & 0 \\ 0 & 8.2736 \end{pmatrix},$$

$$\Lambda_1 = \begin{pmatrix} 34.2897 & 0 \\ 0 & 39.9642 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 17.0144 & 0 \\ 0 & 24.2527 \end{pmatrix}, \quad K = \begin{pmatrix} 23.4731 & -1.7453 \\ -1.7453 & 20.0593 \end{pmatrix},$$

and  $\lambda^* = 233.0185$ .

**Remark6.** From this example, we can see that the stability criterion in Theorem 1 in this paper also depends on actual time delays themselves, not just on the difference between the maximum and minimum time-delay bounds, that is, not just depends on the time-delay interval.

## 5. Conclusions

The problem of bounded-input bounded-output(BIBO) stabilization in mean square sense for a class of discrete-time stochastic control systems with time varying delays and nonlinear perturbations has been considered in this paper. A point of the time delay's variation interval has been introduced, and the variation interval has been divided into two subintervals. Then, by defining a special Lyapunov–Krasovskii functional and checking its variation in the two subintervals respectively, some novel delay-dependent stability criteria for the discrete-time stochastic control system has been obtained. These stability conditions in this paper depended on actual time delays themselves as well, not just on the difference between the maximum and minimum time-delay bounds, that is, not just depended on the time-delay interval. And they were expressed in the forms of linear matrix inequalities(LMIs), whose feasibility can be easily checked by using Matlab LMI Toolbox. Meanwhile, this paper provided a new method for studying discrete-time stochastic BIBO stabilization. At the end, a numerical example has been given to illustrate the validity of the main results.

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