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A note on an error bound of Gauss-Turán quadrature with the Chebyshev weight

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Abstract. In two BIT papers error expansions in the Gauss and Gauss-Turán quadrature formulas with the Chebyshev weight function of the first kind, in the case when integrand is an analytic function in some region of the complex plane containing the interval of integration in its interior, have been obtained. On the basis of that, using a representation of the remainder term in the form of contour integral over confocal ellipses, the upper bound of the modulus of the remainder term, in the cases when certain parameter s ($s \in \mathbb{N}_0$) takes the specific values $s = 0, 1, 2$, has been obtained. Its form for a general s ($s \in \mathbb{N}_0$) has been supposed in one of the mentioned papers. Here, we prove that formula.

1. Introduction

Let Γ be an arbitrary simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and f a function analytic in its interior $\mathcal{D} = \operatorname{int}\Gamma$ and continuous in $\overline{\mathcal{D}}$.

Here, we consider the remainder term $R_{n,s}(f)$ of the well know Gauss-Turán quadrature formula with multiple nodes

$$\int_{-1}^{1} f(t)\omega(t)\mathrm{d}t = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R_{n,s}(f) \quad (n \in \mathbb{N}, s \in \mathbb{N}_0). \tag{1}$$

The weight function ω is a nonnegative and integrable function on the interval $[-1, 1]$. The Gauss-Turán quadrature (1) has the algebraic degree of precision $2n(s + 1) - 1$. In the case $s = 0$ it reduces to the standard Gaussian quadrature formula.

The nodes τ_ν in (1) are zeros of the orthogonal polynomials $\pi_n(t) \equiv \pi_{n,s}(t)$, known as "$s$−orthogonal" polynomials with respect to the weight function ω, which satisfy

$$\int_{-1}^{1} [\pi_n(t)]^{2s+1} t^k \omega(t)\mathrm{d}t = 0 \quad (k = 0, 1, \ldots, n - 1).$$

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We use the same notation as in [3]. In the sequel, Γ is an elliptical contour $\mathcal{E}_\rho$ with foci at ∓ 1 and the sum of semi-axes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} \mid z = \frac{1}{2}\left(\xi + \xi^{-1}\right),\ 0 \le \theta \le 2\pi \right\}, \quad \xi = \rho\, e^{i\theta}. \tag{2}$$

Using [3, Eq. (4.4)] (see also [1]), from [3, Eq. (3.8)] we get the error bound

$$\left|R_{n,s}(f)\right| \le \sum_{k=0}^{+\infty} |\alpha_{2n(s+1)+k}||\varepsilon_{n,k}^{(s)}| \le \frac{2}{\rho^{2n(s+1)}}\left(\max_{z\in\mathcal{E}_\rho}|f(z)|\right)\sum_{k=0}^{+\infty}\frac{|\varepsilon_{n,k}^{(s)}|}{\rho^k}. \tag{3}$$

When ω represents the Chebyshev weight function of the first kind, i. e.,

$$\omega(t) = (1-t^2)^{-1/2}, \tag{4}$$

the explicit expressions of the last sum in (3) have been obtained for the cases $s = 1$ and $s = 2$ in [3, Eqs. (4.6), (4.5)]. Previously, Hunter derived the corresponding explicit expression for the case $s = 0$ in [2, Eq. (4.4)] (see also [3, Eq. (4.7)]). Subject to those explicit expressions in [3, Remark 4.2] there was suggested the conjecture about the explicit expression for this sum in the general case $s \in \mathbb{N}_0$ (see also (6) below). In the following section we prove that conjecture.

2. An upper bound for $|R_{n,s}(f)|$ of the Gauss-Turán quadrature formula with the Chebyshev weight function of the first kind (4)

Lemma 2.1. *For each number $t \in \mathbb{N}_0$, it holds*

$$\sum_{i=0}^{t}(-1)^i\binom{m+s+i}{2s}\binom{2s+1}{i} = (-1)^t\frac{2s+1}{m+s+1}\binom{m+s+t+1}{2s+1}\binom{2s}{t}. \tag{5}$$

Proof. For $t = 0$ we have to show that

$$\binom{m+s}{2s}\binom{2s+1}{0} = \frac{2s+1}{m+s+1}\binom{m+s+1}{2s+1}\binom{2s}{0},$$

which is obvious.

If we now suppose that (5) holds for some $t \in \mathbb{N}_0$, for proving that it holds for $t+1$ we have to show the identity

$$(-1)^t\frac{2s+1}{m+s+1}\binom{m+s+t+1}{2s+1}\binom{2s}{t} + (-1)^{t+1}\binom{m+s+t+1}{2s}\binom{2s+1}{t+1}$$
$$= (-1)^{t+1}\frac{2s+1}{m+s+1}\binom{m+s+t+2}{2s+1}\binom{2s}{t+1},$$

i. e.,

$$-\frac{2s+1}{m+s+1}\,\frac{m+t+1-s}{2s+1}\binom{m+s+t+1}{2s}\binom{2s}{t} + \binom{m+s+t+1}{2s}\frac{2s+1}{t+1}\binom{2s}{t}$$
$$= \frac{2s+1}{m+s+1}\,\frac{m+t+s+2}{2s+1}\binom{m+s+t+1}{2s}\frac{2s-t}{t+1}\binom{2s}{t},$$

which is equivalent to the following equality

$$-\frac{m+t+1-s}{m+s+1}+\frac{2s+1}{t+1}=\frac{(m+t+s+2)(2s-t)}{(m+s+1)(t+1)},$$

i. e.,

$$-(m+t+1-s)(t+1)+(2s+1)(m+s+1)=(m+t+s+2)(2s-t),$$

where the last one is obviously an identity. □

Theorem 2.2. *For the Gauss-Turán quadrature formula* (1) *with the Chebyshev weight function of the first kind* (4)*, if the function f is analytic in the interior $\mathcal{D}$ of the ellipse $\mathcal{E}_\rho$, given by* (2)*, and continuous in $\overline{\mathcal{D}}$, then there holds the following error bound*

$$\left|R_{n,s}(f)\right| \le 2\pi\left(\max_{z\in\mathcal{E}_\rho}|f(z)|\right)\frac{\displaystyle\sum_{k=0}^{s}(-1)^k\binom{2s+1}{s-k}\rho^{2n(s-k)}}{(\rho^{2n}-1)^{2s+1}}. \tag{6}$$

Proof. As first, we are expressing the numbers $\varepsilon_{n,k}^{(s)}$ defined by (3.9) (and (3.7), (4.2), (4.3)) in [3]. According to (3.7), (4.2) and (4.3) from [3], we have that for all $n\in\mathbb{N}$, $k\in\mathbb{N}_0$, holds

$$\omega_{n,k}^{(s)}=\sum_{j=0}^{[k/(2n)]}\overline{\beta}_{n,2nj}^{(s)}\overline{\gamma}_{n,k-2nj}^{(s)},$$

i. e.,

$$\omega_{n,k}^{(s)}=\sum_{j=0}^{+\infty}\overline{\beta}_{n,2nj}^{(s)}\overline{\gamma}_{n,k-2nj}^{(s)}.$$

If we define $\overline{\gamma}_{n,l}^{(s)}=0$ for $l<0$ and $\omega_{n,k}^{(s)}=0$ for $k<0$, we have

$$\begin{aligned}\epsilon_{n,k}^{(s)} &= \frac{1}{4}(\omega_{n,k}^{(s)}-\omega_{n,k-2}^{(s)})=\frac{1}{4}\sum_{j=0}^{+\infty}\overline{\beta}_{n,2nj}^{(s)}\left[\overline{\gamma}_{n,k-2nj}^{(s)}-\overline{\gamma}_{n,k-2-2nj}^{(s)}\right]\\ &= \frac{1}{4}\sum_{j=0}^{+\infty}\overline{\beta}_{n,2nj}^{(s)}\left[\overline{\gamma}_{n,k-2nj}^{(s)}-\overline{\gamma}_{n,k-2-2nj}^{(s)}\right]\end{aligned}$$

for all $n\in\mathbb{N}_0$, $k\in\mathbb{Z}$. It is easy to see from (4.3) in [3] that $\overline{\gamma}_{n,k-2nj}^{(s)}\ne\overline{\gamma}_{n,k-2-2nj}^{(s)}$ if and only if $k\equiv 0 \pmod{2n}$ and $0\le k-2nj\le 2sn$, i. e., $k=2nm$ for some $m\in\mathbb{N}_0$, $j\le m\le j+s$.

Let us note that [3, Eq. (4.3)] can be modified in the following way:

$$\overline{\gamma}_{n,k}^{(s)}=\begin{cases}\dfrac{\pi}{2^{2s-1}}\displaystyle\sum_{\nu=0}^{j}\binom{2s+1}{s-\nu}, & k=2nj,2nj+2,\ldots,2n(j+1)-2,\\ 0, & \text{otherwise},\end{cases} \tag{7}$$

where $j\in\mathbb{N}_0$, since

$$\frac{\pi}{2^{2s-1}}\sum_{\nu=0}^{s}\binom{2s+1}{s-\nu}=\frac{1}{2}\left(\frac{\pi}{2^{2s-1}}\sum_{\nu=0}^{2s+1}\binom{2s+1}{s-\nu}\right)=\frac{1}{2}\frac{\pi}{2^{2s-1}}2^{2s+1}=2\pi.$$

Here, we used that $\binom{n}{k} = 0$ for $n \in \mathbb{N}_0,\ k < 0$.

Hence for $j \in \mathbb{N}_0$, according to (7),

$$\overline{\gamma}^{(s)}_{n,k-2nj} - \overline{\gamma}^{(s)}_{n,k-2-2nj} = \begin{cases} \dfrac{\pi}{2^{2s-1}}\dbinom{2s+1}{s-(m-j)}, & k = 2nm,\ m \geq j, \\ 0, & \text{otherwise}, \end{cases}$$

where $m \in \mathbb{N}_0$, i. e.,

$$\overline{\gamma}^{(s)}_{n,k-2nj} - \overline{\gamma}^{(s)}_{n,k-2-2nj} = \begin{cases} \dfrac{\pi}{2^{2s-1}}\dbinom{2s+1}{s-(m-j)}, & k = 2nm,\ m-s \leq j \leq m, \\ 0, & \text{otherwise}. \end{cases} \tag{8}$$

Finally, according to [3, Eq. (4.2)] and (8), we have

$$\varepsilon^{(s)}_{n,k} = \begin{cases} \pi \displaystyle\sum_{j=m-s}^{m} (-1)^j \binom{j+2s}{2s}\binom{2s+1}{s-(m-j)}, & k = 2nm,\ m \in \mathbb{N}_0, \\ 0, & \text{otherwise}. \end{cases} \tag{9}$$

The sum from (9) can be rewritten in the form

$$\pi \sum_{i=0}^{s} (-1)^{i+m-s} \binom{m+s+i}{2s}\binom{2s+1}{i} \tag{10}$$

and calculated by using Eq. (5) from the previous lemma.

Hence, if we put $t = s$ into (5), we get that (10) is equal to

$$(-1)^s \frac{2s+1}{m+s+1}\binom{m+2s+1}{2s+1}\binom{2s}{s},$$

and then (9) becomes

$$\varepsilon^{(s)}_{n,k} = \begin{cases} \pi(-1)^m \dfrac{2s+1}{m+s+1}\dbinom{m+2s+1}{2s+1}\dbinom{2s}{s}, & k = 2nm,\ m \in \mathbb{N}_0, \\ 0, & \text{otherwise}. \end{cases}$$

According to this, (3) obtains the form

$$\left|R_{n,s}(f)\right| \leq \frac{2\pi\|f\|_\rho}{\rho^{2n(s+1)}} \sum_{m=0}^{+\infty} \frac{2s+1}{m+s+1}\frac{1}{\rho^{2mn}}\binom{m+2s+1}{2s+1}\binom{2s}{s} = 2\pi\|f\|_\rho F(x),$$

where $\|f\|_\rho = \max_{z\in\mathcal{E}_\rho} |f(z)|$ and $x = \rho^{-2n}$ (therefore, $x \in (0,1)$) and

$$F(x) = (2s+1)\binom{2s}{s}\sum_{m=0}^{+\infty}\binom{m+2s+1}{2s+1}\frac{x^{m+s+1}}{m+s+1}.$$

Further, we have that

$$\begin{aligned} F'(x) &= (2s+1)\binom{2s}{s}\sum_{m=0}^{+\infty}\binom{m+2s+1}{2s+1}x^{m+s} \\ &= (s+1)\binom{2s+1}{s}\sum_{m=0}^{+\infty}\binom{2s+2+m-1}{m}x^{m+s} \\ &= (s+1)\binom{2s+1}{s}\frac{x^s}{(1-x)^{2s+2}}. \end{aligned}$$

The right-hand side of the inequality in (6), expressed as a function of x, is equal to

$$2\pi\|f\|_\rho \frac{\sum_{k=0}^{s}(-1)^k\binom{2s+1}{s-k}x^{s+k+1}}{(1-x)^{2s+1}} = 2\pi\|f\|_\rho \frac{\sum_{k=0}^{s}(-1)^k\binom{2s+1}{s+k+1}x^{s+k+1}}{(1-x)^{2s+1}}.$$

Therefore, it remains to check the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\sum_{k=0}^{s}(-1)^k\binom{2s+1}{s+k+1}x^{s+k+1}}{(1-x)^{2s+1}}\right\} = (s+1)\binom{2s+1}{s}\frac{x^s}{(1-x)^{2s+2}}. \tag{11}$$

We have

$$\begin{aligned}
&(1-x)^{2s+2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{k=0}^{s}(-1)^k\binom{2s+1}{s+k+1}\frac{x^{s+k+1}}{(1-x)^{2s+1}}\right\}\\
&\quad = (1-x)\sum_{k=0}^{s}(-1)^k(s+k+1)\binom{2s+1}{s+k+1}x^{s+k} + (2s+1)\sum_{k=0}^{s}(-1)^k\binom{2s+1}{s+k+1}x^{s+k+1}.
\end{aligned}$$

For $0 \le k \le s-1$ the coefficient which multiplies x^{s+k+1} on the right hand side in the previous equality is equal to

$$\begin{aligned}
(-1)^{k+1}(s+k+2)\binom{2s+1}{s+k+2} - (-1)^k(s+1+k)\binom{2s+1}{s+k+1} + (2s+1)(-1)^k\binom{2s+1}{s+k+1}&\\
= (-1)^k\left(-(2s+1)\binom{2s}{s+k+1} + (s-k)\binom{2s+1}{s+k+1}\right)&\\
= (-1)^k\left(-(2s+1)\binom{2s}{s+k+1} + (s-k)\binom{2s+1}{s-k}\right)&\\
= (-1)^k\left(-(2s+1)\binom{2s}{s+k+1} + (2s+1)\binom{2s}{s-k-1}\right)&\\
= (-1)^k\left(-(2s+1)\binom{2s}{s+k+1} + (2s+1)\binom{2s}{s+k+1}\right) = 0,&
\end{aligned}$$

while the coefficient which multiplies x^s in the corresponding expression is equal to

$$(s+1)\binom{2s+1}{s+1} = (s+1)\binom{2s+1}{s}.$$

This confirms the identity (11), which implies that $F(x)$ has the form

$$F(x) = \frac{\sum_{k=0}^{s}(-1)^k\binom{2s+1}{s+k+1}x^{s+k+1}}{(1-x)^{2s+1}} + C,$$

where C is a constant for which we can easily deduce that $C = 0$, if we put $x \to 0+$ in the equality

$$(2s+1)\binom{2s}{s}\sum_{m=0}^{+\infty}\binom{m+2s+1}{2s+1}\frac{x^{m+s+1}}{m+s+1} = \frac{\displaystyle\sum_{k=0}^{s}(-1)^k\binom{2s+1}{s+k+1}x^{s+k+1}}{(1-x)^{2s+1}} + C.$$

The proof of the theorem is completed. $\square$

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