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Automorphisms of Tabačjn graphs

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Abstract. A *bicirculant* is a graph admitting an automorphism whose cyclic decomposition consists of two cycles of equal length. In this paper we consider automorphisms of the so-called *Tabačjn graphs*, a family of pentavalent bicirculants which are obtained from the generalized Petersen graphs by adding two additional perfect matchings between the two orbits of the above mentioned automorphism. As a corollary, we determine which Tabačjn graphs are vertex-transitive.

1. Introductory remarks

Tabačjn graphs were introduced recently in [1], as a natural generalization of generalized Petersen graphs [3] and rose window graphs [7]. In [1], the initial motivation was concerned with determining which of these graphs are arc-transitive. In particular, given natural numbers $n \geq 3$ and $1 \leq a, b, r \leq n-1$ with $r \neq n/2$ and $a \neq b$, the *Tabačjn graph* $T(n; a, b; r)$ is a pentavalent graph with vertex set $\{x_i \mid i \in \mathbb{Z}_n\} \cup \{y_i \mid i \in \mathbb{Z}_n\}$ and edge set $\{x_i x_{i+1} \mid i \in \mathbb{Z}_n\} \cup \{y_i y_{i+r} \mid i \in \mathbb{Z}_n\} \cup \{x_i y_i \mid i \in \mathbb{Z}_n\} \cup \{x_i y_{i+a} \mid i \in \mathbb{Z}_n\} \cup \{x_i y_{i+b} \mid i \in \mathbb{Z}_n\}$. A Tabačjn graph $T(n; a, b; r)$ clearly admits a $(2, n)$-semiregular automorphism $(x_0\ x_1\ \ldots\ x_{n-1})(y_0\ y_1\ \ldots\ y_{n-1})$ (see Section 2 for formal definitions), and our goal is to obtain conditions on the quadruple $(n; a, b; r)$ giving rise to a Tabačjn graph admitting additional automorphisms. In particular, we describe certain families of Tabačjn graphs which admits these additional automorphisms. As a consequence a complete classification of vertex-transitive Tabačjn graphs is obtained (see Theorem 5.4). However, our results do not determine the full automorphism groups of Tabačjn graphs, and thus they motivate us to propose the following problem.

Problem 1.1. *Determine the full automorphism groups of Tabajčn graphs.*

The paper is organized as follows. In Section 2 notions concerning this paper are introduced together with the notation and some auxiliary results that are needed in the subsequent sections. The rest of the paper is devoted to obtain conditions on the parameters $(n; a, b; r)$ giving rise to Tabačjn graphs admitting additional automorphisms next to the obvious $(2, n)$-semiregular automorphism, ending with Section 5 where vertex-transitive Tabačjn graphs are completely classified.

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2. Preliminaries

For a finite simple graph X let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ denote its vertex set, its edge set, its arc set and its automorphism group, respectively. For a vertex $v \in V(X)$ let $N(v)$ be the set of its neighbors. If for $x, y \in V(X)$ we have $\{x, y\} \in E(X)$, we will abbreviate this as xy.

A subgroup $G \leq \mathrm{Aut}(X)$ is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* provided it acts transitively on the sets of vertices, edges, and arcs of X, respectively. The graph X is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* if $\mathrm{Aut}(X)$ is vertex-transitive, edge-transitive, and arc-transitive, respectively. An arc-transitive graph is also called *symmetric*. A vertex-transitive and edge-transitive graph of odd valency is arc-transitive (see [8]). However, this is not true in general. There exist vertex-transitive and edge-transitive graphs of even valency which are not arc-transitive.

Given a transitive permutation group G on a set V, we say that a partition $\mathcal{B}$ of V is *G-invariant* if the elements of G permute the parts, that is, *blocks* of $\mathcal{B}$, setwise. If the *trivial* partitions $\{V\}$ and $\{\{v\} \mid v \in V\}$ are the only G-invariant partitions of V, then the action of G on V is said to be *primitive*, and is said to be *imprimitive* otherwise.

Let G be a transitive permutation group on a finite set V containing an abelian semiregular subgroup H. We say that $g \in G$ is a *mixer* relative to H (in short, a mixer when the subgroup H is clear from the context), if the orbits of H are not blocks of imprimitivity for $\langle g \rangle$.

A non-identity automorphism of a graph X is called *semiregular* (in particular (m, n)*–semiregular*), if it has m cycles of equal length n in its cycle decomposition. A graph X is called *n-bicirculant* (*bicirculant*, for short) if it admits a $(2, n)$-semiregular automorphism ρ.

The existence of a $(2, n)$-semiregular automorphism in a bicirculant enables us to label its vertex set and edge set in the following way. Let X be a connected n-bicirculant and let $\rho \in \mathrm{Aut}(X)$ be its $(2, n)$-semiregular automorphism. The vertices of X can be labeled by x_i and y_i with $i \in \mathbb{Z}_n$, such that

$$\rho = (x_0\, x_1\, \ldots\, x_{n-1})(y_0\, y_1\, \ldots\, y_{n-1}). \tag{1}$$

Observe that a mixer of X (relative to $\langle \rho \rangle$) is an automorphism α of X, for which partition

$$\{\{x_0, x_1, \ldots, x_{n-1}\}, \{y_0, y_1, \ldots, y_{n-1}\}\}$$

is not $\langle \alpha \rangle$-invariant.

To label edges of X, define the following three sets: $L := \{i \in \mathbb{Z}_n \mid x_0x_i\}$, $M := \{i \in \mathbb{Z}_n \mid x_0y_i\}$, $R := \{i \in \mathbb{Z}_n \mid y_0y_i\}$. Note that $L = -L$, $R = -R$, $M \neq \emptyset$ and $0 \notin L \cup R$. Now the edge set $E(X)$ can be partitioned into three subsets:

$$\begin{aligned}
\mathcal{L} &= \bigcup_{i \in \mathbb{Z}_n} \{x_i x_{i+l} \mid l \in L\} \quad \text{(left edges)},\\
\mathcal{M} &= \bigcup_{i \in \mathbb{Z}_n} \{x_i y_{i+m} \mid m \in M\} \quad \text{(middle (or spoke) edges)},\\
\mathcal{R} &= \bigcup_{i \in \mathbb{Z}_n} \{y_i y_{i+r} \mid r \in R\} \quad \text{(right edges)}.
\end{aligned}$$

We shall denote graph X by $BC_n[L, M, R]$ (this notation has been introduced in [6]). The vertices x_i, $i \in \mathbb{Z}_n$, will be referred to as *left vertices* and vertices y_i, $i \in \mathbb{Z}_n$, will be referred to as *right vertices* of $BC_n[L, M, R]$. Observe that the generalized Petersen graph $\mathrm{GP}(n, r)$ and a rose window graph $R_n(a, r)$ are bicirculants isomorphic to $BC_n[\{\pm 1\}, \{0\}, \{\pm r\}]$ and $BC_n[\{\pm 1\}, \{0, -a\}, \{\pm r\}]$, respectively.

Pick integers $n \geq 3$, $1 \leq a, b, r \leq n - 1$ with $r \neq n/2$ and $a \neq b$. A *Tabačjn graph* $T(n; a, b; r)$ is a pentavalent bicirculant isomorphic to $BC_n[\{\pm 1\}, \{0, a, b\}, \{\pm r\}]$ (an example is given in Figure 1).

Of course, a Tabačjn graph does not determine the quadruple $(n; a, b; r)$ uniquely. In the next two propositions some isomorphisms are given.

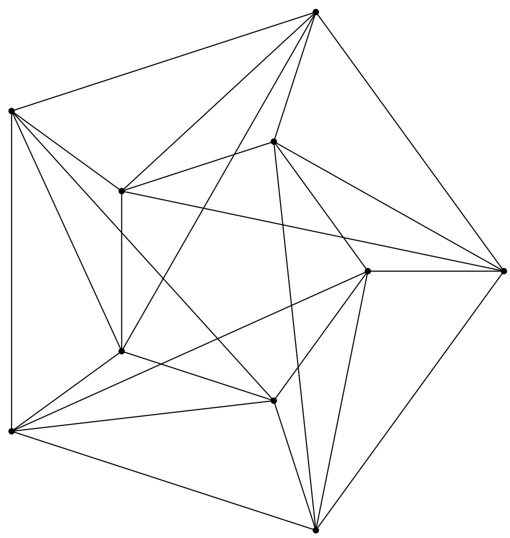

Figure 1: The Tabačjn graph $T(10; 3, 4; 1)$.

Proposition 2.1. [6] *Let L, M and R be subsets of $\mathbb{Z}_n$ such that $L = -L$, $R = -R$, $M \neq \emptyset$ and $0 \notin L \cup R$. Then*

$$BC_n[L, M, R] \cong BC_n[\lambda L, \lambda M + \mu, \lambda R] \qquad (\lambda \in \mathbb{Z}_n^*,\ \mu \in \mathbb{Z}_n),$$

with the isomorphism $\phi_{\lambda,\mu}$ given by $\phi_{\lambda,\mu}(x_i) = x_{\lambda i+\mu}$ and $\phi_{\lambda,\mu}(y_i) = y_{\lambda i}$.

Proposition 2.2. [1] *Let $n \geq 3$ and let $1 \leq a, b, r \leq n-1$ be such that $a \neq b$ and $r \neq n/2$. Then*

$$T(n; a, b; r) \cong T(n; a, b; -r) \cong T(n; -a, -b; r) \cong T(n; -a, b-a; r) \cong T(n; -b, a-b; r).$$

Moreover, if $gcd(n, r) = 1$, then also $T(n; a, b; r) \cong T(n; -ar^{-1}, -br^{-1}; r^{-1})$ holds.

Symmetric Tabačjn graphs are classified in [1]. In particular, it is proved in [1] that there are only three such graphs:

Proposition 2.3. [1] *A Tabačjn graph is symmetric if and only if it is isomorphic to one of the following graphs: $T(3; 1, 2; 1) \cong K_6$, $T(6; 2, 4; 1) \cong K_{6,6} - 6K_2$, and $T(6; 1, 5; 2)$, which is isomorphic to the icosahedron graph.*

However, we will show that there are infinitely many vertex-transitive Tabačjn graphs (see Theorem 5.4). In the following sections all arithmetic operations are to be taken modulo n if at least one argument is from $\mathbb{Z}_n$.

3. Automorphisms of Tabačjn graphs

In this section we describe certain families of Tabačjn graphs, which admits automorphisms different from the $(2, n)$-semiregular automorphism ρ defined in (1). Let $X = T(n; a, b; r)$, let $V(X) = \underline{L} \cup \underline{R}$, where $\underline{L} = \{x_i \mid i \in \mathbb{Z}_n\}$ and $\underline{R} = \{y_i \mid i \in \mathbb{Z}_n\}$. Let $A(\underline{L}, \underline{R}) = \{\alpha \in \mathrm{Aut}(X) \mid \alpha(\underline{L}) = \underline{L}, \alpha(\underline{R}) = \underline{R}\} \leq \mathrm{Aut}(X)$ be the subgroup of the automorphism group $\mathrm{Aut}(X)$ fixing the sets $\underline{L}$ and $\underline{R}$ set-wise. Note that ρ given in (1) belongs to $A(\underline{L}, \underline{R})$, and that $A(\underline{L}, \underline{R})^{\underline{L}} \lesssim D_{2n}$. Let $B(\underline{L}, \underline{R}) \leq \mathrm{Aut}(X)$ be the largest subgroup of the automorphism group $\mathrm{Aut}(X)$ such that $\{\underline{L}, \underline{R}\}$ is a $B(\underline{L}, \underline{R})$-invariant partition. Observe that $A(\underline{L}, \underline{R}) \leq B(\underline{L}, \underline{R})$, and if there exists $\sigma \in B(\underline{L}, \underline{R})$ such that $\sigma(\underline{L}) = \underline{R}$ (that is, $A(\underline{L}, \underline{R}) \neq B(\underline{L}, \underline{R})$) we can conclude that X is vertex-transitive.

Proposition 3.1. *Let X be a Tabačjn graph $T(n;a,b;r)$. Then the following hold:*

(i) There exists an automorphism $\gamma \in A(\underline{L},\underline{R})$ such that $\gamma^{\underline{L}} = 1$ and $\gamma^{\underline{R}} \neq 1$ if and only if $X \cong T(3m;m,2m;r)$ for some positive integer m.

(ii) $A(\underline{L},\underline{R})^{\underline{L}} \cong D_{2n}$ if and only if $X \cong T(n;a,-a;r)$.

Proof. To prove part (i) suppose first that there exists an automorphism $\gamma \in A(\underline{L},\underline{R})$ such that $\gamma^{\underline{L}} = 1$ and $\gamma^{\underline{R}} \neq 1$. Then without loss of generality we may assume that $\gamma(y_0) = y_a$. Therefore $N(y_0) \cap \underline{L} = N(y_a) \cap \underline{L}$, that is $\{x_0, x_{-a}, x_{-b}\} = \{x_0, x_a, x_{a-b}\}$. This implies that $x_{-a} = x_{a-b}$ and $x_{-b} = x_a$, and consequently $2a = b$ and $a = -b$. It follows that $3a = 3b = 0$, and so n is of the form $n = 3m$ for some positive integer m. Since $a \neq b$ we can conclude that $a = m$ and $b = 2m$, and thus $X \cong T(3m;m,2m;r)$. Conversely, observe that the mapping γ defined by

$$\gamma(x_i) = x_i \text{ and } \gamma(y_i) = y_{i+m}\ ,\ i \in \mathbb{Z}_{3m},$$

is an automorphism of $T(3m;m,2m;r)$ such that $\gamma^{\underline{L}} = 1$ and $\gamma^{\underline{R}} \neq 1$. Namely,

$$\gamma(x_i x_{i+1}) = x_i x_{i+1},\ \gamma(x_i y_i) = x_i y_{i+m},\ \gamma(x_i y_{i+m}) = x_i y_{i+2m},$$

$$\gamma(x_i y_{i+2m}) = x_i y_i,\ \text{ and } \gamma(y_i y_{i+r}) = y_{i+m} y_{i+m+r}$$

are all edges in $T(3m;m,2m;r)$.

To prove part (ii) suppose that $A(\underline{L},\underline{R})^{\underline{L}} \cong D_{2n}$. Then there exists $\tau \in A(\underline{L},\underline{R})$ such that $\tau(x_i) = x_{-i}$ for every $i \in \mathbb{Z}_n$. Assume first that one of the neighbours of x_0 in $\underline{R}$ is fixed by τ. Without loss of generality we may assume that this neighbour is y_0: $\tau(y_0) = y_0$. Then $\{x_0, x_{-a}, x_{-b}\} = N(y_0) \cap \underline{L} = \tau(N(y_0) \cap \underline{L}) = \{x_0, x_a, x_b\}$. If $a = -a$, then $b = -b$ and so $a = b$, a contradiction. Therefore $b = -a$. Assume next that none of the neighbours of x_0 in $\underline{R}$ is fixed by τ. Without loss of generality we may assume that $\tau(y_a) = y_b$. Then we have $\{x_0, x_{-a}, x_{b-a}\} = \tau(N(y_a) \cap \underline{L}) = N(y_b) \cap \underline{L} = \{x_0, x_b, x_{b-a}\}$. This shows that $b = -a$.

Conversely, observe that the mapping τ defined by

$$\tau(x_i) = x_{-i} \text{ and } \tau(y_i) = y_{-i} \qquad (i \in \mathbb{Z}_n)$$

is an automorphism of $T(n;a,-a;r)$. Namely, for any n, a and r the mapping τ maps edges of $T(n;a,-a;r)$ to its edges:

$$\tau(x_i x_{i+1}) = x_{-i} x_{-i-1},\ \tau(x_i y_i) = x_{-i} y_{-i},\ \tau(x_i y_{i+a}) = x_{-i} y_{-i-a},$$

$$\tau(x_i y_{i-a}) = x_{-i} y_{-i+a},\ \text{ and } \tau(y_i y_{i+r}) = y_{-i} y_{-i-r}.$$

Therefore $\tau, \rho \in A(\underline{L},\underline{R})$, and consequently we have $A(\underline{L},\underline{R})^{\underline{L}} \cong D_{2n}$ in $T(n;a,-a;r)$. □

Proposition 3.2. *Let X be a Tabačjn graph $T(n;a,b;r)$ admitting an automorphism $\sigma \in B(\underline{L},\underline{R})$ such that $\sigma(\underline{L}) = \underline{R}$. Then one of the following holds:*

(i) $X \cong T(n;a,b;1)$;

(ii) $X \cong T(n;a,b;r)$ where $r^2 \equiv 1 \pmod n$, $ar \equiv -a \pmod n$ and $br \equiv -b \pmod n$;

(iii) $X \cong T(n;a;ar;r)$ where $r^2 \equiv 1 \pmod n$.

Proof. Let $G = B(\underline{L},\underline{R})$ and let $x = x_0$. Then X can be viewed as the coset graph with respect to the vertex stabilizer G_x. In particular the assumptions imply that there exists $\sigma \in G$ such that $L = \{\rho^i G_x \mid i \in \mathbb{Z}_n\}$ and $R = \{\rho^i \sigma G_x \mid i \in \mathbb{Z}_n\}$. Since L and R are orbits of $\langle\rho\rangle$ one can see that $\langle\rho\rangle$ is normal in G. Consequently, $\sigma\rho G_x = \rho^s \sigma G_x$ and $\sigma\rho\sigma G_x = \rho^s \sigma^2 G_x = \rho^{s+t} G_x$ for some $s \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$, implying that

$$\sigma\rho^i G_x = \rho^{si}\sigma G_x \text{ and } \sigma\rho^i\sigma G_x = \rho^{si+t} G_x,\ \ i \in \mathbb{Z}_n.$$

Moreover, applying the adjacency conditions we get $s = \pm r$ and $t \in \{0, -a, -b\}$.

All these combined together imply that in the vertex labeling with x_i and y_i we can, without loss of generality, assume that there exists $\sigma \in B(\underline{L}, \underline{R})$ such that $\sigma(x_i) = y_{ri}$, and

$$\text{either } \sigma(y_i) = x_{ri} \text{ or } \sigma(y_i) = x_{ri-a}, \quad i \in \mathbb{Z}_n.$$

It follows that two cases need to be considered. Observe also, that since $\sigma(y_i y_{i+r})$ must be an edge in $\mathcal{L}$, both cases give $r^2 \equiv \pm 1 \pmod{n}$.

Case 3.3.

$\sigma(y_i) = x_{ri-a}$, $i \in \mathbb{Z}_n$.

Then $\sigma(N(y_0)) = N(x_{-a})$, implying that either $-ar = -a + b$ and $-br = -a$, or $-ar = -a$ and $-br = -a + b$. In the first case $-br^2 = -br + b$, and thus for $r^2 \equiv -1 \pmod{n}$ we get $-br = 0$, a contradiction. If, however, $r^2 \equiv 1 \pmod{n}$ then since $br = a$ it follows that $b = ar$ and thus $X \cong T(n; a; ar; r)$. In the second case we have $br = a - b$, and thus Proposition 2.2 implies that $T(n; a, b; r) \cong T(n; a - b, -b; r) = T(n; br, -b; r) \cong T(n; (-b)(-r), -b; -r)$. If $r^2 \equiv -1 \pmod{n}$ then $ar = a$ implies that $-a = ar^2 = ar$, and so $a = -a$ and $2a = 0$. Since $\sigma(N(y_0)) = N(x_{-a})$, we also have $x_{r^2} = x_{-a\pm 1}$, implying that $-1 = r^2 = -a \pm 1$, and so $a = 2$. This combined together with $2a = 0$ imply that $n = 4$, a contradiction (namely there is no element in $\mathbb{Z}_4^*$ whose square is equal to -1 modulo 4). Therefore we must have $r^2 \equiv 1 \pmod{n}$.

Case 3.4.

$\sigma(y_i) = x_{ri}$, $i \in \mathbb{Z}_n$.

Then $\sigma(x_0 y_a) = y_0 x_{ar}$ and $\sigma(x_0 y_b) = y_0 x_{br}$, and thus $ar, br \in \{-a, -b\}$. If $r = -1$ then X is isomorphic to $T(n; a, b; 1)$. We may therefore assume that $r \neq \pm 1$.

Suppose first that $ar = -a$. Then $br = -b$ and $r^2 \equiv 1 \pmod{n}$, giving the graphs stated in (ii). Namely, for $r^2 \equiv -1 \pmod{n}$ one can easily see that $2a = 2b = 0$, which is impossible since $a \neq b$. Suppose now that $ar = -b$. Then $br = -a$, and thus $ar^2 = -br$ and $br^2 = -ar$. If $r^2 \equiv -1 \pmod{n}$ then $a = br$ and $b = ar$, implying that $a = ar^2 = -a$ and $b = br^2 = -b$, and thus $2a = 2b = 0$, a contradiction. Therefore, we have $r^2 \equiv 1 \pmod{n}$ and $T(n; a, b; r) = T(n; a, -ar; r) = T(n; a, a(-r); -r)$. □

4. Tabačjn graphs admitting mixers

For a Tabačjn graph $T(n; a, b; r)$ we will only consider mixers relative to $\langle \rho \rangle$. That is, we say that $T(n; a, b; r)$ admits a mixer in case it admits a mixer relative to $\langle \rho \rangle$. It is the aim of this section to characterize Tabačjn graphs admitting mixers (see Proposition 4.3). In this respect the so-called rose window graphs, first defined in [7], will be needed. A *rose window graph* $R_n(a; r)$ is a tetravalent bicirculant isomorphic to $BC_n[\{\pm 1\}, \{0, -a\}, \{\pm r\}]$. Observe that $R_n(a; r)$ is isomorphic to a spanning subgraph of $T(n; -a, b; r)$. Edge-transitive rose window graphs were classified in [4] and the automorphism groups of these graphs were determined in [5]. In particular the following proposition can be deduced from [4, Corollary 1.3] (see also [5, 7]).

Proposition 4.1. [4, Corollary 1.3] *A rose window graph $R_n(a, r)$ is edge-transitive if and only if it is isomorphic to a graph belonging to one of the following four families:*

(i) $R_n(2, 1)$,

(ii) $R_{2m}(m + 2, m + 1)$,

(iii) $R_{12m}(3m + 2, 3m - 1)$ *and* $R_{12m}(3m - 2, 3m + 1)$,

(iv) $R_{2m}(2b, r)$, *for which b satisfies $b^2 \equiv \pm 1 \pmod{m}$, $2 \leq 2b \leq m$, and r satisfies $r = 1$, or $r = m - 1$ and m is even.*

Coming back to Tabačjn graphs let the set of the spoke edges of a Tabačjn graph $T(n; a, b; r)$ be partitioned into

$$\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_a \cup \mathcal{M}_b,$$

where $\mathcal{M}_0 = \{x_i y_i \mid i \in \mathbb{Z}_n\}$, $\mathcal{M}_a = \{x_i y_{i+a} \mid i \in \mathbb{Z}_n\}$, and $\mathcal{M}_b = \{x_i y_{i+b} \mid i \in \mathbb{Z}_n\}$.

With the help of computer package Magma [2] one can see that the following proposition holds.

Proposition 4.2. *Let X be a Tabačjn graph $T(n; a, b; r)$ such that the set $\mathcal{O} = \mathcal{M}_a \cup \mathcal{L} \cup \mathcal{R}$ is an edge orbit for $Aut(X)$. Then X is isomorphic to $T(8; 2, 4; 3)$, and $Aut(X)$ has two edge orbits.*

Proof. If a Tabačjn graph $X = T(n; a, b; r)$ is such that the set $\mathcal{O} = \mathcal{M}_a \cup \mathcal{L} \cup \mathcal{R}$ is an edge orbit for $\mathrm{Aut}(X)$ then the graph $(V(X), \mathcal{O})$ is isomorphic to a cubic vertex-transitive and edge-transitive bicirculant $BC_n[\{\pm 1\}, \{0\}, \{\pm r\}]$, that is, to a symmetric generalized Petersen graph $\mathrm{GP}(n, r)$. Recall that there are only seven symmetric generalized Petersen graphs. These are $\mathrm{GP}(4, 1)$, $\mathrm{GP}(5, 2)$, $\mathrm{GP}(8, 3)$, $\mathrm{GP}(10, 2)$, $\mathrm{GP}(10, 3)$, $\mathrm{GP}(12, 5)$, and $\mathrm{GP}(24, 5)$ (see [3]). With the use of program package Magma [2] one can then obtain the graph given in the statement of the proposition. □

Now we are ready to characterize Tabačjn graphs admitting mixers.

Proposition 4.3. *A Tabačjn graph $X = T(n; a, b; r)$ admits a mixer relative to $\langle \rho \rangle$ if and only if one of the following holds:*

(i) *X is vertex-transitive and edge-transitive, in which case $X \cong T(3; 1, 2; 1)$, or $X \cong T(6; 2, 4; 1)$, or $X \cong T(6; 1, 5; 2)$;*

(ii) *$X - \mathcal{M}_0$ is isomorphic to an edge-transitive rose window graph admitting a vertex-transitive and edge-transitive subgroup giving an invariant partition consisting of blocks of the form $\{x_i, y_j\}$;*

(iii) *$X \cong T(8; 2, 4; 3)$.*

Proof. The existence of a mixer implies that we may assume that there exists an automorphism of X mapping an edge from $\mathcal{L}$ to an edge from $\mathcal{M}_a$. The assumptions clearly imply that $\mathrm{Aut}(X)$ is vertex-transitive and that $\langle \rho \rangle$ is not normal in $\mathrm{Aut}(X)$. In particular, if $\langle \rho \rangle$ is normal in $\mathrm{Aut}(X)$ then $\underline{L}$ and $\underline{R}$ are blocks of imprimitivity for $\mathrm{Aut}(X)$, and thus edges in $\mathcal{M}_a$ cannot be in the same orbit as edges in $\mathcal{L}$. Also, the existence of a mixer implies that the action of $\mathrm{Aut}(X)$ on the edge set of X has at most three orbits. In particular, we may, without loss of generality, assume that $\mathrm{Aut}(X)$ on $E(X)$ has one of the following orbits:

(i) $\mathcal{O}_1 = \mathcal{M}_0 \cup \mathcal{M}_a \cup \mathcal{M}_b \cup \mathcal{L} \cup \mathcal{R} = E(X)$;

(ii) $\mathcal{O}_1 = \mathcal{M}_0$ and $\mathcal{O}_2 = \mathcal{M}_a \cup \mathcal{M}_b \cup \mathcal{L} \cup \mathcal{R}$;

(iii) $\mathcal{O}_1 = \mathcal{M}_0 \cup \mathcal{M}_b$ and $\mathcal{O}_2 = \mathcal{M}_a \cup \mathcal{L} \cup \mathcal{R}$;

(iv) $\mathcal{O}_1 = \mathcal{M}_0$, $\mathcal{O}_2 = \mathcal{M}_a \cup \mathcal{L} \cup \mathcal{R}$ and $\mathcal{O}_3 = \mathcal{M}_b$;

By Proposition 4.2, (iv) cannot occur. If (i) holds then X is edge-transitive, and thus Proposition 2.3 applies. If (ii) holds then $Y = (V(X), \mathcal{O}_2)$ is a vertex-transitive and edge-transitive spanning subgraph of X, which is isomorphic to a rose window graph given in Proposition 4.1, and $\mathrm{Aut}(X) \leq \mathrm{Aut}(Y)$. In particular, $\mathrm{Aut}(X)$ is a vertex-transitive and edge-transitive subgroup of $\mathrm{Aut}(Y)$, implying that there must be an $\mathrm{Aut}(X)$-invariant partition in Y consisting of blocks of the form $\{x_i, y_j\}$. If (iii) holds then, by Proposition 4.2, $X \cong T(8; 2, 4; 3)$.

That the graphs given in the statement of the theorem indeed admit a mixer relative to $\langle \rho \rangle$ follows from edge-transitivity of the graphs and edge-transitivity of the spanning subgraphs, respectively. □

Remark 4.4. By [5] the automorphism groups of the rose window graphs $R_n(2, 1)$ and $R_{2m}(m + 2, m + 1)$ both act imprimitively with the corresponding invariant partition $\{\{x_i, y_{i-1}\} \mid i \in \mathbb{Z}_n\}$ (see [5]). Therefore, adding edges between vertices in the blocks of this partition results in a Tabačjn graph isomorphic to $T(n; 1, n - 1; 1) \cong C_n[K_2]$ and $T(2m; m - 2, 2m - 1; m + 1)$, respectively.

5. Vertex-transitive Tabačjn graphs

In this section vertex-transitive Tabačjn graphs are characterized (see Theorem 5.4). In the following two propositions we first show that three particular families of Tabačjn graphs consist of vertex-transitive graphs.

Proposition 5.1. *Given natural numbers $n \geq 3$ and $1 \leq a, b \leq n-1$, where $a \neq b$, the Tabačjn graph $T(n; a, b; 1)$ is vertex-transitive.*

Proof. The permutation

$$\alpha = \prod_{i=0}^{n-1} (x_i \; y_{-i})$$

is an automorphism of $T(n; a, b; 1)$ which together with the automorphism ρ given in (1) gives a vertex-transitive subgroup of automorphisms $\langle \alpha, \rho \rangle$. $\square$

Proposition 5.2. *For $r \in \mathbb{Z}_n^*$ such that $r^2 \equiv 1 \pmod{n}$ the Tabačjn graphs*

(i) $T(n; a, ar; r)$, and

(ii) $T(n; a, b; r)$, where $ar \equiv -a \pmod{n}$ and $br \equiv -b \pmod{n}$,

are vertex-transitive graphs.

Proof. Since $r^2 \equiv 1 \pmod{n}$, the mapping α defined by

$$\alpha(x_i) = y_{-ri} \text{ and } \alpha(y_i) = x_{-ri}$$

is an automorphism of $T(n; a, ar; r)$ as well as of $T(n; a, b; r)$, if $ar \equiv -a \pmod{n}$ and $br \equiv -b \pmod{n}$. In both graphs this automorphism α together with the automorphism ρ given in (1) gives a vertex-transitive subgroup of automorphisms $\langle \alpha, \rho \rangle$. $\square$

Remark 5.3. *By Proposition 2.2, $T(n; a, a + ar; r) \cong T(n; -a, (-a)(-r); -r)$, and thus, by Proposition 5.2, this graph is vertex-transitive.*

We are now ready to prove the main theorem of this paper.

Theorem 5.4. *Let X be a Tabačjn graph $T(n; a, b; r)$. Then the following hold:*

(i) X is vertex-transitive and edge-transitive if and only if it is isomorphic to one of the graphs $T(3; 1, 2; 1)$, $T(6; 2, 4; 1)$ and $T(6; 1, 5; 2)$.

(ii) X is vertex-transitive but not edge-transitive if and only if it is isomorphic to one of the following graphs:

(a) $X \cong T(n; a, b; 1)$, where $(n, a, b) \notin \{(3, 1, 2), (6, 2, 4)\}$;

(b) $X \cong T(n; a, b; r)$, where $r^2 \equiv 1 \pmod{n}$, $ar \equiv -a \pmod{n}$ and $br \equiv -b \pmod{n}$;

(c) $X \cong T(n; a; ar; r)$, where $r^2 \equiv 1 \pmod{n}$;

or $X - \mathcal{M}_0$ is isomorphic to an edge-transitive rose window graph admitting a vertex-transitive and edge-transitive subgroup giving an invariant partition consisting of blocks of the form $\{x_i, y_j\}$.

Proof. Let X be a vertex-transitive Tabačjn graph $T(n; a, b; r)$. If it is also edge-transitive then Proposition 2.3 implies that it is isomorphic to one of the graphs $T(3; 1, 2; 1)$, $T(6; 2, 4; 1)$ and $T(6; 1, 5; 2)$.

We may therefore assume that X is not edge-transitive, and thus that

$$(n, a, b, r) \notin \{(3, 1, 2, 1), (6, 2, 4, 1), (6, 1, 5, 2)\}.$$

If X does not admit a mixer then, by vertex-transitivity of X, we have $A(\underline{L}, \underline{R}) \neq B(\underline{L}, \underline{R}) = \mathrm{Aut}(X)$. Thus there exists $\sigma \in \mathrm{Aut}(X)$ such that $\sigma(\underline{L}) = \underline{R}$, and $\{\underline{L}, \underline{R}\}$ is an $\mathrm{Aut}(X)$-invariant partition. Therefore, by Proposition 3.2, either $X \cong T(n; a, b; 1)$, or $X \cong T(n; a, b; r)$, where $r^2 \equiv 1 \pmod{n}$, $ar \equiv -a \pmod{n}$ and $br \equiv -b \pmod{n}$, or $X \cong T(n; a; ar; r)$, where $r^2 \equiv 1 \pmod{n}$. That these graphs are indeed vertex-transitive follows from Propositions 5.1 and 5.2, respectively. If, however, X admits a mixer relative to $\langle \rho \rangle$ then Proposition 4.3 applies. $\square$

References

[1] A. Arroyo, I. Hubard, K. Kutnar, E. O'Reilly, P. Šparl, Classification of symmetric Tabačjn graphs, submitted to Graphs and Combinatorics.
[2] W. Bosma, J. Cannon, C. Playoust, The Magma Algebra System I: The User Language, Journal of Symbolic Computation 24 (1997) 235–265.
[3] R. Frucht, J. E. Graver, M. E. Watkins, The groups of the generalized Petersen graphs, Proceedings of the Cambridge Philosophical Society 70 (1971) 211–218.
[4] I. Kovács, K. Kutnar, D. Marušič, Classification of edge-transitive rose window graphs, Journal of Graph Theory 65 (2010) 216–231.
[5] I. Kovács, K. Kutnar, J. Ruff, Rose window graphs underlying rotary maps, Discrete Mathematics 310 (2010) 1802–1811.
[6] I. Kovács, B. Kuzman, A. Malnič, S. Wilson, Characterization of edge-transitive 4-valent bicirculants, Journal of Graph Theory 69 (2012) 441–463.
[7] S. Wilson, Rose window graphs, Ars Mathematica Contemporanea 1 (2008) 7–19.
[8] W. T. Tutte, Connectivity in graphs, University of Toronto Press, Toronto, 1966.