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Multipliers

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Abstract. We describe the multiplier spaces $(H^{p,q,\alpha}, H^\infty)$, and $(H^{p,q,\alpha}, H^{\infty,v,\beta})$, where $H^{p,q,\alpha}$ are mixed norm spaces of analytic functions in the unit disk $\mathbb{D}$ and H^∞ is the space of bounded analytic functions in $\mathbb{D}$. We extend some results from [7] and [3], particularly Theorem 4.3 in [3].

1. Introduction

For $0 < p \leq \infty$, a function f analytic in the unit disk $\mathbb{D}$, $f \in H(\mathbb{D})$, is said to belong to the *Hardy space* H^p if

$$\|f\|_p = \sup_{0<r<1} M_p(r, f) < \infty, \quad 0 < p \leq \infty,$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt\right)^{1/p} < \infty\,, \quad 0 < p < \infty,$$

and

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)| < \infty\,.$$

It belongs to the *mixed norm space* $H^{p,q,\alpha}$, $0 < p, q \leq \infty$, $0 < \alpha < \infty$, if

$$\|f\|_{p,q,\alpha}^q = \int_0^1 (1-r)^{q\alpha-1} M_p(r, f)^q dr < \infty, \quad 0 < q < \infty,$$

and

$$\|f\|_{p,\infty,\alpha} = \sup_{0\leq r<1} (1-r)^\alpha M_p(r, f) < \infty.$$

$H_0^{p,\infty,\alpha}$ will be the subspace of $H^{p,\infty,\alpha}$ of functions f for which

$$\lim_{r\to 1}(1-r)^\alpha M_p(r, f) = 0.$$

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Obviously, H^∞ is the space of all bounded analytic functions in $\mathbb{D}$. A closed subspace of H^∞ consisting of functions analytic in $\mathbb{D}$, continuous on $\bar{\mathbb{D}}$, will be denoted by $\mathcal{A} = \mathcal{A}(\mathbb{D})$.

Let $g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n$ be analytic in $\mathbb{D}$. We define the multiplier transformation $D^s g$ of g, where s is any real number, by

$$D^s g(z) = \sum_{n=0}^{\infty} (n+1)^s \hat{g}(n) z^n.$$

If $0 < p \le \infty$, $0 < q \le \infty$ and $0 < \alpha < \infty$, the space of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{p,q,\alpha;s} := \|D^s f\|_{p,q,\alpha} < \infty$$

is denoted by $D^{-s}H^{p,q,\alpha}$. Similarly, are defined the spaces $D^{-s}H_0^{p,\infty,\alpha}$. If $s \ne 0$ we also write $H_s^{p,q,\alpha}$ instead of $D^{-s}H^{p,q,\alpha}$.

Let A and B be two quasi-normed spaces of functions analytic in $\mathbb{D}$. A function $g(z) = \sum_{k=0}^{\infty} \hat{g}(k)z^k$ is said to be multiplier from A to B if, whenever $f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k$ belongs to A, then

$$(f \star g)(z) = \sum_{k=0}^{\infty} \hat{f}(k)\hat{g}(k)z^k$$

belongs to B. The space C of all multipliers g from A to B with a quasi-norm

$$\|g\|_C = \sup\{\|f \star g\|_B : f \in A, \quad \|f\|_A \le 1\}$$

will be denoted by (A, B).

We denote the space of all Abel summable sequences by AS. The AS-dual of a space E of analytic functions in $\mathbb{D}$, i.e. the space (E, AS), is known as the Abel dual of E and will be denoted by E^a.

Our main goal of this paper is to describe the multiplier spaces $(H^{p,q,\alpha}, H^\infty)$, and $(H^{p,q,\alpha}, H^{\infty,v,\beta})$. We extend some results from [7] and [3], especially Theorem 4.3 in [3].

2. The multiplier space $(H^{p,q,\alpha}, H^\infty)$

Let $\omega : R \to R$ be nonincreasing function of class C^∞ such that $\omega(t) = 1$, for $t \le 1$, and $\omega(t) = 0$, for $t \ge 2$. Let $\varphi(t) = \omega(t/2) - \omega(t)$, $t \in R$, and let

$$w_0(z) = 1 + z, \quad \text{and} \quad w_n(z) = \sum_{k=2^{n-1}}^{2^{n+1}} \varphi(\frac{k}{2^{n-1}})z^k, \quad n = 1, 2, \ldots \quad .$$

In [4] the authors showed that for any $f \in H(\mathbb{D})$ we have

$$f(z) = \sum_{n=0}^{\infty} (w_n \star f)(z), \quad z \in \mathbb{D}$$

and

$$\|w_n \star f\|_p \le C\|f\|_p, \quad 0 < p \le \infty, \quad n = 0, 1, 2, \ldots.$$

For an extension of these results see [6].

The following two lemmas will be needed in sequel. For a proof of the first one see [3]. The proof of the second is not too much different.

Lemma 1. *Let $0 < p, q \le \infty$, $0 < \alpha < \infty$ and $f \in H(\mathbb{D})$. Then the following statements are equivalent:*

(i) $f \in H^{p,q,\alpha}$;

(ii) *The sequence* $\{2^{-n\alpha}\|w_n \star f\|_p\}$ *belongs to* l^q;

(iii) *The sequence* $\{\|w_n \star D^{-\alpha} f\|_p\}$ *belongs to* l^q.

Here, as usual, l^q is the space of all sequences $\lambda = \{\lambda_n\}$ such that $\|\lambda\|_{l^q}^q = \|\lambda\|_q^q = \sum_{n=0}^{\infty} |\lambda_n|^q < \infty$, $0 < q < \infty$; l^∞ is the space of all bounded sequences and c_0 is its subspace consisting of zero sequences.

Lemma 2. *Let* $0 < p \le \infty$, $0 < \alpha < \infty$ *and* $f \in H(\mathbb{D})$. *Then the following statements are equivalent:*

(i) $f \in H_0^{p,\infty,\alpha}$;

(ii) *The sequence* $\{2^{-n\alpha}\|w_n \star f\|_p\}$ *belongs to* c_0;

(iii) *The sequence* $\{\|w_n \star D^{-\alpha} f\|_p\}$ *belongs to* c_0.

Recall that an analytic function f on the unit disk $\mathbb{D}$ is a Cauchy transform if it admits representation

$$f(z) = C[\mu](z) = \int_{\mathbb{T}} \frac{1}{1 - ze^{-i\theta}} d\mu(e^{i\theta}), \quad z \in \mathbb{D}, \tag{1}$$

where $\mu \in M(\mathbb{T})$. Recall that $M(\mathbb{T})$ is a Banach space of all complex Borel measures μ on the boundary $\mathbb{T}$ of $\mathbb{D}$ under the total variation norm $\|\mu\|$.

The space M_+ of all Cauchy transforms is a Banach space under the norm

$$\|f\|_{M_+} = \inf\{\|\mu\| : \mu \in M(\mathbb{T}) \text{ and (1) holds }\}.$$

First, we characterize the multipliers $(H^{p,q,\alpha}, H^\infty)$, for $0 < p \le \infty$.

Theorem 1. *Let* $0 < p, q \le \infty$, $0 < \alpha < \infty$, $p_0 = \min\{1, p\}$, $p_1 = \max\{1, p\}$, $q_1 = \max\{1, q\}$ *and* p_1' *and let* q_1' *be the conjugate exponents of* p_1 *and* q_1 *respectively. Then*

$$(H^{p,q,\alpha}, H^\infty) = H^{p_1', q_1', 1}_{\alpha + 1/p_0}. \tag{2}$$

Proof. We consider the case $p = \infty$, since the remaining cases have been considered in [7]. We will use the fact that if $g \in H(\mathbb{D})$, then

$$\|g\|_{(H^{p,q,\alpha}, H^\infty)} \approx \|\{2^{n\alpha}\|w_n \star g\|_{(H^p, H^\infty)}\}\|_{l^{q_1'}}, \quad \text{see [7]}. \tag{3}$$

Since, by Lemma 1, we have

$$\|D^{\alpha+1} g\|_{1, q_1', 1} \approx \|\{2^{-n}\|w_n \star D^{\alpha+1} g\|_1\}\|_{l^{q_1'}} \approx \|\{2^{n\alpha}\|w_n \star g\|_1\}\|_{l^{q_1'}},$$

to prove (2), for $p = \infty$, by (3) it is sufficient to prove that

$$\|w_n \star g\|_{(H^\infty, H^\infty)} \approx \|w_n \star g\|_{M_+} \approx \|w_n \star g\|_1.$$

By the equality $(H^\infty, H^\infty) = M_+$, (see [2]), we have that the first relation holds .

Obviously, $\|w_n \star g\|_1 \ge \|w_n \star g\|_{M_+}$.

Let $f_r(z) = (1 - rz)^{-2}$, $0 < r < 1$. Then

$$\|w_n \star g \star f_r\|_1 \le \|w_n \star g\|_{M_+} \|f_r\|_{(M_+, H^1)}.$$

Recall that

$$\|f_r\|_{(M_+, H^1)} = \sup\{\|f_r \star h\|_1 : h \in M_+, \|h\|_{M_+} \le 1\}.$$

If $h \in M_+$ and $\|h\|_{M_+} \le 1$, then $h(z) = \int\limits_T \frac{d\mu(\xi)}{1 - z\bar{\xi}}$, where $\|\mu\| \le 1$. Hence

$$\|f_r \star h\|_1 = \|D^1 h\|_1 \approx \|h_r'\|_1 \le \frac{C}{1 - r}.$$

By taking $r = 1 - 2^{-n}$, we get

$$2^{-n}\|w_n \star D^1 g\|_1 \approx \|w_n \star g\|_1 \le C\|w_n \star g\|_{M_+}.$$

□

A similar argument, based on Lemma 2, shows that the following is true:

Theorem 2. *Let* $0 < p \le \infty$, $0 < \alpha < \infty$, $p_0 = \min\{1, p\}$, $p_1 = \max\{1, p\}$, *and* p_1' *is the conjugate of* p_1 . *Then*

$$(H_0^{p,\infty,\alpha}, H^\infty) = H^{p_1',1,1}_{\alpha+1/p_0}. \tag{4}$$

3. Multipliers $(H^{p,q,\alpha}, H^{\infty,v,\beta})$

We define $a \ominus b = \infty$ if $a \le b$, and

$$\frac{1}{a \ominus b} = \frac{1}{b} - \frac{1}{a}, \quad \text{for} \quad 0 < b < a.$$

If X is any quasi-normed space of analytic functions in $\mathbb{D}$ that contains polynomials, then for $0 < q \le \infty$, we define the space

$$X[q] = \left\{ f \in H(\mathbb{D}) : \|f\|_{X[q]} = \| \{\|w_n \star f\|_X\} \|_{l^q} \right\}.$$

We will also write $X[l^q]$ instead of $X[q]$. We define $X[c_0]$ to be the subspace of $X[l^\infty]$ consisting of functions $f \in H(\mathbb{D})$ such that $\{\|w_n \star f\|_X\} \in c_0$.

For a proof of the next theorem see [3]

Theorem 3. *Let* $0 < p, q, u, v \le \infty, 0 < \alpha, \beta < \infty$. *Then*

(i) $(H^p[q], H^u[v]) = (H^p, H^u)[q \ominus v]$.

(ii)$(H^{p,\infty,\alpha}[q], H^{u,\infty,\beta}[v]) = (H^{p,\infty,\alpha}, H^{u,\infty,\beta})[q \ominus v]$.

(iii) $(H^p[c_0], H^u[l^v] = (H^p, H^u)[(c_0, l^v)] = (H^p, H^u)[l^v]$;

(iv) $(H^p[c_0], H^u[c_0]) = (H^p, H^u)[(c_0, c_0)] = (H^p, H^u)[l^\infty]$;

(v) $(H^p[l^\infty], H^u[c_0]) = (H^p, H^u)[(l^\infty, c_0)] = (H^p, H^u)[c_0]$;

(vi) $(H^p[l^q], H^u[c_0]) = (H^p, H^u)[(l^q, c_0)] = (H^p, H^u)[l^\infty], \quad 0 < q < \infty$.

As a corollary we have

Theorem 4.

(i) $(H^{p,q,\alpha}, H^{u,v,\beta}) = (H^{p,\infty,\alpha}, H^{u,\infty,\beta})[q \ominus v], \quad 0 < p, q, u, v \le \infty, 0 < \alpha, \beta < \infty.$

(ii) $(H^{p,q,\alpha}, H^{u,v,\beta}) = \left\{ g \in H(\mathbb{D}) : D^{\alpha-\beta} g \in (H^p, H^u)[q \ominus v] \right\}, \quad 0 < p, q, u, v \le \infty, 0 < \alpha, \beta < \infty.$

Proof. (i) The statement (*i*) follows by Theorem 3 and the equalities

$$H^{p,q,\alpha} = H^{p,\infty,\alpha}[q] \quad \text{and} \quad H^{u,v,\beta} = H^{u,\infty,\beta}[v].$$

(ii) This statement follows from the equalities

$$H^{p,q,\alpha} = \{ f \in H(\mathbb{D}) : D^{-\alpha} f \in H^p[q] \}$$

and

$$H^{u,v,\beta} = \left\{ f \in H(\mathbb{D}) : D^{-\beta} f \in H^u[v] \right\},$$

(see Lemma 1), and Theorem 3.

□

As a corollary of Theorem 4 we have

Corollary 1. *Let* $0 < p, q, u, v \leq \infty, 0 < \alpha, \beta < \infty$. *Then*

$$(H^{p,\infty,\alpha}, H^{u,\infty,\beta})[q \ominus v] = \{g \in H(\mathbb{D}) : D^{\alpha-\beta}g \in (H^p, H^u)[q \ominus v]\}.$$

Theorem 5.

(i) $(H_0^{p,\infty,\alpha}, H^{u,v,\beta}) = (H^{p,\infty,\alpha}, H^{u,v,\beta}) = \left\{g \in H(\mathbb{D}) : D^{\alpha-\beta}g \in (H^p, H^u)[l^v]\right\}$;

(ii) $(H_0^{p,\infty,\alpha}, H_0^{u,\infty,\beta}) = (H^{p,\infty,\alpha}, H^{u,\infty,\beta}) = \left\{g \in H(\mathbb{D}) : D^{\alpha-\beta}g \in (H^p, H^u)[l^\infty]\right\}$;

(iii) $(H^{p,\infty,\alpha}, H_0^{u,\infty,\beta}) = \left\{g \in H(\mathbb{D}) : D^{\alpha-\beta}g \in (H^p, H^u)[c_0]\right\}$;

(iv) $(H^{p,q,\alpha}, H_0^{u,\infty,\beta}) = (H^{p,q,\alpha}, H^{u,\infty,\beta}) = \left\{g \in H(\mathbb{D}) : D^{\alpha-\beta}g \in (H^p, H^u)[l^\infty]\right\}, \quad 0 < q < \infty.$

Proof. We prove (i) only, the proofs of (ii) through (iv) being similar.

By Lemma 2

$$H_0^{p,\infty,\alpha} = \{f \in H(\mathbb{D}) : D^{-\alpha}f \in H^p[c_0]\},$$

and, by Lemma 1,

$$H^{u,v,\beta} = \{f \in H(\mathbb{D}) : D^{-\beta}f \in H^u[l^v]\}.$$

From this we conclude that

$$\begin{aligned}(H_0^{p,\infty,\alpha}, H^{u,v,\beta}) &= \{g \in H(\mathbb{D}) : D^{\alpha-\beta}g \in (H^p[c_0], H^u[l^v])\} \\ &= \{g \in H(\mathbb{D}) : D^{\alpha-\beta}g \in (H^p, H^u)[l^v]\},\end{aligned}$$

by Theorem 3.

The equality $(H^{p,\infty,\alpha}, H^{u,v,\beta}) = \{g \in H(\mathbb{D}) : D^{\alpha-\beta}g \in (H^p, H^u)[l^v]\}$ is proved in Theorem 4.
□

The next theorem is a consequence of the equality $(H^p, H^\infty) = H^{p'}$, $1 < p < \infty$, and Theorem 4.

Theorem 6. *Let* $1 < p < \infty$ *and* $p + p' = pp'$. *Then*

$$(H^{p,q,\alpha}, H^{\infty,v,\beta}) = H_\alpha^{p',q\ominus v,\beta}.$$

Corollary 2. *Let* $1 < p < \infty$ *and* $p + p' = pp'$. *Then*

(i) $(H_0^{p,\infty,\alpha}, H^{\infty,v,\beta}) = H_\alpha^{p',v,\beta}$;

(ii) $(H_0^{p,\infty,\alpha}, H_0^{\infty,\infty,\beta}) = H_\alpha^{p',\infty,\beta}$;

(iii) $(H^{p,\infty,\alpha}, H_0^{\infty,\infty,\beta}) = D^{-\alpha}H_0^{p',\infty,\beta}$;

(iv) $(H^{p,q,\alpha}, H_0^{\infty,\infty,\beta}) = H_\alpha^{p',\infty,\beta}, \quad q \neq \infty.$

Proof. The statements (i), (ii) and (iv) follow from Theorem 5 and Theorem 6. We prove the statement (iii).

By using Theorem 5 we get

$$\begin{aligned}(H^{p,\infty,\alpha}, H_0^{\infty,\infty,\beta}) &= \{g \in H(\mathbb{D}) : D^{\alpha-\beta}g \in (H^p, H^\infty)[c_0]\} \\ &= \{g \in H(\mathbb{D}) : D^{\alpha-\beta}g \in H^{p'}[c_0]\} \\ &= D^{-\alpha}H_0^{p',\infty,\beta},\end{aligned}$$

by Lemma 2. □

Theorem 7. $(H^{\infty,q,\alpha}, H^{\infty,v,\beta}) = H_{\alpha}^{1,q\ominus v,\beta}$.

Theorem is a consequence of Theorem 4 and the following theorem

Theorem 8. *Let* $0 < \alpha, \beta < \infty$. *Then*

$$(H^{\infty,\infty,\alpha}, H^{\infty,\infty,\beta}) = H_{\alpha}^{1,\infty,\beta}.$$

Proof. We will use the equalities $(H_0^{\infty,\infty,\alpha})^a = H_{\alpha+1}^{1,1,1}$, $(H_0^{\infty,\infty,\beta})^a = H_{\beta+1}^{1,1,1}$, $(H_{\alpha+1}^{1,1,1})^a = H^{\infty,\infty,\alpha}$ and $(H_{\beta+1}^{1,1,1})^a = H^{\infty,\infty,\beta}$. All these results follows from Theorem 1 and Theorem 2, since the Abel dual of separable mixed norm space $H^{p,q,\alpha}$ coincides with the space $(H^{p,q,\alpha}, H^{\infty})$, (see [7], [8], [5]). See also [1].

Using this we find that

$$\begin{aligned}(H_0^{\infty,\infty,\alpha}, H_0^{\infty,\infty,\beta}) &\subset ((H_0^{\infty,\infty,\beta})^a, (H_0^{\infty,\infty,\alpha})^a) = (H_{\beta+1}^{1,1,1}, H_{\alpha+1}^{1,1,1})\\ &\subset ((H_{\alpha+1}^{1,1,1})^a, (H_{\beta+1}^{1,1,1})^a) = (H^{\infty,\infty,\alpha}, H^{\infty,\infty,\beta}).\end{aligned}$$

Now let $\lambda \in (H^{\infty,\infty,\alpha}, H^{\infty,\infty,\beta})$. Then $\lambda \in (H_0^{\infty,\infty,\alpha}, H_0^{\infty,\infty,\beta})$ since λ maps polynomials into polynomials and these are dense in $H_0^{\infty,\infty,\alpha}$ and in $H_0^{\infty,\infty,\beta}$. Thus,

$$(H_0^{\infty,\infty,\alpha}, H_0^{\infty,\infty,\beta}) = (H^{\infty,\infty,\alpha}, H^{\infty,\infty,\beta}) = (H_{\beta+1}^{1,1,1}, H_{\alpha+1}^{1,1,1}) = H_{\alpha}^{1,\infty,\beta}.$$

For the last equality see Theorem 10 below. □

Corollary 3.

(i) $(H_0^{\infty,\infty,\alpha}, H^{\infty,v,\beta}) = H_{\alpha}^{1,v,\beta}$;

(ii) $(H_0^{\infty,\infty,\alpha}, H_0^{\infty,\infty,\beta}) = H_{\alpha}^{1,\infty,\beta}$;

(iii) $(H^{\infty,\infty,\alpha}, H_0^{\infty,\infty,\beta}) = D^{-\alpha} H_0^{1,\infty,\beta}$;

(iv) $(H^{\infty,q,\alpha}, H_0^{\infty,\infty,\beta}) = H_{\alpha}^{1,\infty,\beta}, \quad q \neq \infty$.

Proof. We should only prove (iii). By using Theorem 5 we obtain

$$\begin{aligned}(H^{\infty,\infty,\alpha}, H_0^{\infty,\infty,\beta}) &= \{g \in H(\mathbb{D}) : D^{\alpha-\beta} g \in (H^{\infty}, H^{\infty})[c_0]\}\\ &= \{g \in H(\mathbb{D}) : D^{\alpha-\beta} g \in M_+[c_0]\}.\end{aligned}$$

Since $\|w_n \star D^{\alpha-\beta} g\|_{M_+} \approx \|w_n \star D^{\alpha-\beta} g\|_{H^1}$, (see Section 2), we get

$$\begin{aligned}(H^{\infty,\infty,\alpha}, H_0^{\infty,\infty,\beta}) &= \{g \in H(\mathbb{D}) : D^{\alpha-\beta} g \in H^1[c_0]\}\\ &= D^{-\alpha} H_0^{1,\infty,\beta},\end{aligned}$$

by Lemma 2. □

Corollary 4. $(\mathcal{B}, H^{\infty,v,\beta}) = H^{1,v,\beta}$ *and* $(\mathcal{B}, H_0^{\infty,\infty,\beta}) = H_0^{1,\infty,\beta}$.

As usual $\mathcal{B} = H_1^{\infty,\infty,1}$ denote the Bloch space and $\mathcal{B}_0$ is the little Bloch space.

If $v = \infty$, more is true

Theorem 9. $(\mathcal{A}, H^{\infty,\infty,\beta}) = (\mathcal{B}, H^{\infty,\infty,\beta}) = H^{1,\infty,\beta}$.

In particular,

$$(\mathcal{A}, \mathcal{B}) = (\mathcal{B}, \mathcal{B}) = H_1^{1,\infty,1}.$$

Proof. It suffices to show that $(\mathcal{A}, H^{\infty,\infty,\beta}) \subset H^{1,\infty,\beta}$. Now we give the proof of this inclusion.

$$\begin{aligned}(\mathcal{A}, H^{\infty,\infty,\beta}) &\subset ((H^{\infty,\infty,\beta})^a, \mathcal{A}^a) = (H^{1,1,1}_{\beta+1}, M^+)\\ &\subset (H^{1,1,1}_{\beta+1}, H^{1,\infty,1+\beta}_{1+\beta}) = (H^{1,1,1}, H^{1,\infty,1+\beta})\\ &= H_1^{1,\infty,1+\beta} = H^{1,\infty,\beta}.\end{aligned}$$

Here, we used the fact that $\mathcal{A}^a = M_+$, (see [9]).
□

Note that Theorem 9 represents an extension of Theorem 4.3 in [3].

As final remark we note the multipliers $(H^{p,q,\alpha}, H^{u,v,\beta})$, for $0 < p \le 1$, $p \le u \le \infty$, are characterized in [7]. See also [3].

Theorem 10. *([7]) Let* $0 < p \le 1,\ p \le u \le \infty, 0 < q, v \le \infty, 0 < \alpha, \beta < \infty$. *Then*

$$(H^{p,q,\alpha}, H^{u,v,\beta}) = \{g \in H(\mathbb{D}) : D^{\alpha+1/p-1} g \in H^{u,q\ominus v,\beta}\} = H^{u,q\ominus v,\beta}_{\alpha+1/p-1}.$$

In particular, $(H^{p,q,\alpha}, H^{\infty,v,\beta}) = H^{\infty,q\ominus v,\beta}_{\alpha+1/p-1}$.

Corollary 5. *([5]) Let* $0 < p \le 1$, $p \le u \le \infty$, $0 < v \le \infty$, $0 < \alpha, \beta < \infty$. *Then*

(i) $(H_0^{p,\infty,\alpha}, H^{u,v,\beta}) = H^{u,v,\beta}_{\alpha+1/p-1}$;

(ii) $(H_0^{p,\infty,\alpha}, H_0^{u,\infty,\beta}) = H^{u,\infty,\beta}_{\alpha+1/p-1}$;

(iii) $(H^{p,\infty,\alpha}, H_0^{u,\infty,\beta}) = D^{-\alpha-(1/p)+1} H_0^{u,\infty,\beta}$;

(iv) $(H^{p,q,\alpha}, H_0^{u,\infty,\beta}) = H^{u,\infty,\beta}_{\alpha+1/p-1}$, *if* $0 < q < \infty$.

We note that the statements (*i*), (*ii*) and (*iv*) also follow from Theorem 10 and Theorem 5. A different proof of these statements is given in [5].

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