Filomat 27:7 (2013), 1285–1290
DOI 10.2298/FIL1307285S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Rodrigues formula for the Dunkl-classical symmetric orthogonal polynomials

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Abstract. We find Rodrigues type formula for the Dunkl-classical symmetric orthogonal polynomials.

1. Introduction

Different authors (see [2],[3], [5], [8], among others), in various contexts dealt with Rodrigues' formula. In this work, we are concerned with Rodrigues type formula for the Dunkl-classical symmetric orthogonal polynomials which have been introduced in [1].

We begin by reviewing some preliminary results needed for the sequel. The vector space of polynomials with coefficients in $\mathbb{C}$ (the field of complex numbers) is denoted by $\mathcal{P}$ and by $\mathcal{P}'$ its dual space, whose elements are called forms. The set of all nonnegative integers will be denoted by $\mathbb{N}$. The action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ is denoted by $\langle u, f\rangle$. In particular, we denote by $(u)_n := \langle u, x^n\rangle, n \in \mathbb{N}$, the moments of u. For any form u, any $a \in \mathbb{C} - \{0\}$ and any polynomial h let $Du = u'$, hu, $h_a u$, δ_0 and $x^{-1}u$ be the forms defined by: $\langle u', f\rangle := -\langle u, f'\rangle$, $\langle hu, f\rangle := \langle u, hf\rangle$, $\langle h_a u, f\rangle =: \langle u, h_a f\rangle = \langle u, f(ax)\rangle$ $\langle \delta_0, f\rangle := f(0)$, and $\left\langle x^{-1}u, f\right\rangle := \langle u, \theta_0 f\rangle$ where $\left(\theta_0 f\right)(x) = \dfrac{f(x) - f(0)}{x}$, $f \in \mathcal{P}$.

Then, it is straightforward to prove that for $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, we have

$$x^{-1}(xu) = u - (u)_0\delta_0 \ , \tag{1}$$

$$(fu)' = f'u + fu' \ . \tag{2}$$

We will only consider sequences of polynomials $\{P_n\}_{n\geq 0}$ such that $\deg P_n \leq n, n \in \mathbb{N}$. If the set $\{P_n\}_{n\geq 0}$ spans $\mathcal{P}$, which occurs when $\deg P_n = n, n \in \mathbb{N}$, then it will be called a polynomial sequence (PS). Along the text, we will only deal with PS whose elements are monic, that is, monic polynomial sequences (MPS). It is always possible to associate to $\{P_n\}_{n\geq 0}$ a unique sequence $\{u_n\}_{n\geq 0}, u_n \in \mathcal{P}'$, called its dual sequence, such that $\langle u_n, P_m\rangle = \delta_{n,m}$, $n, m \geq 0$, where $\delta_{n,m}$ is the Kronecker's symbol [6].

The MPS $\{P_n\}_{n\geq 0}$ is orthogonal with respect to $u \in \mathcal{P}'$ when the following conditions hold: $\langle u, P_n P_m\rangle =$

2010 *Mathematics Subject Classification.* Primary 42C05; Secondary 33C45
Keywords. Orthogonal polynomials, Dual sequence, Dunkl operator, Rodrigues formula.
Received: 24 June 2012; Accepted: 6 May 2013
Communicated by Dragan S. Djordjevic
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$r_n\delta_{n,m}$, $n, m \geq 0$, $r_n \neq 0$, $n \geq 0$ [2]. In this case, we say that $\{P_n\}_{n\geq 0}$ is a monic orthogonal polynomial sequence (MOPS) and the form u is said to be regular. Necessarily, $u = \lambda u_0, \lambda \neq 0$. Furthermore, we have

$$u_n = \left(\langle u_0, P_n^2\rangle\right)^{-1} P_n u_0, n \geq 0, \tag{3}$$

and the MOPS $\{P_n\}_{n\geq 0}$ fulfils the second order recurrence relation

$$\begin{array}{l} P_0(x) = 1 \quad , \quad P_1(x) = x - \beta_0 \\ P_{n+2} = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x) \quad , \gamma_{n+1} \neq 0, \quad n \geq 0. \end{array} \tag{4}$$

A form u is said symmetric if and only if $(u)_{2n+1} = 0, n \geq 0$, or, equivalently, in (4) $\beta_n = 0, n \geq 0$.
Let us introduce the Dunkl operator

$$T_\mu(f) = f' + 2\mu H_{-1}f, \quad (H_{-1}f)(x) = \frac{f(x) - f(-x)}{2x}, \quad f \in \mathcal{P}, \mu \in \mathbb{C}.$$

This operator was introduced and studied for the first time by Dunkl [4]. Note that T_0 is reduced to the derivative operator D. The transposed ${}^tT_\mu$ of $T\mu$ is ${}^tT_\mu = -D - H_{-1} = -T_\mu$, leaving out a light abuse of notation without consequence. Thus we have

$$\langle T_\mu u, f\rangle = -\langle u, T_\mu f\rangle, \quad u \in \mathcal{P}', \quad f \in \mathcal{P}, \quad \mu \in \mathbb{C}.$$

In particular, this yields $\langle T_\mu u, x^n\rangle = -\mu_n(u)_{n-1}, n \geq 0$, where $(u)_{-1} = 0$ and

$$\mu_n = n + \mu(1 - (-1)^n), \quad n \geq 0. \tag{5}$$

It is easy to see that

$$T_\mu(fu) = fT_\mu u + f'u + 2\mu\,(H_{-1}f)\,(h_{-1}u)\,, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}', \tag{6}$$

$$h_a \circ T_\mu = aT_\mu \circ h_a \quad \text{in } \mathcal{P}', \quad a \in \mathbb{C} - \{0\}. \tag{7}$$

Remark 1.1 When u is a symmetric form, (6) becomes

$$T_\mu(fu) = fT_\mu u + \left(T_\mu f\right)u, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}', \tag{8}$$

Now, consider a MPS $\{P_n\}_{n\geq 0}$ and let

$$P_n^{[1]}(x, \mu) = \frac{1}{\mu_{n+1}}\left(T_\mu P_{n+1}\right)(x), \quad \mu \neq -n - \frac{1}{2}, \quad n \geq 0. \tag{9}$$

Definition 1.1. *[1, 7] A MOPS $\{P_n\}_{n\geq 0}$ is called Dunkl-classical or T_μ-classical if $\{P_n^{[1]}(., \mu)\}_{n\geq 0}$ is also a MOPS. In this case, the form u_0 is called Dunkl-classical or T_μ-classical form.*

2. Rodrigues type formula

The following was proved in [7]

Theorem 2.1. *For any symmetric MOPS $\{P_n\}_{n\geq 0}$, the following statements are equivalent*
(a) The sequence $\{P_n\}_{n\geq 0}$ is Dunkl-classical.
(b) There exist two polynomials Φ (monic) and Ψ with $\deg\Phi \leq 2$ and $\deg\Psi = 1$ such that the associated regular form u_0 satisfies

$$T_\mu\,(\Phi u_0) + \Psi u_0 = 0 \tag{10}$$

$$\Psi'(0) - \frac{1}{2}\Phi''(0)\mu_n \neq 0, \quad n \geq 0. \tag{11}$$

Proposition 2.2. *If* $\{P_n\}_{n\geq 0}$ *is Dunkl-classical symmetric MOPS, then* $\left\{P_n^{[m]}(.,\mu) = \dfrac{T_\mu^m P_{n+m}}{\prod_{k=1}^m \mu_{n+k}}\right\}_{n\geq 0}$*,* $m \geq 1$ *is also a Dunkl-classical symmetric MOPS and we have*

$$T_\mu\left(\Phi u_0^{[m]}(\mu)\right) + \left(\Psi - mT_\mu\Phi\right)u_0^{[m]}(\mu) = 0, \tag{12}$$

$$u_0^{[m]}(\mu) = k_m\Phi^m u_0, m \geq 1 \tag{13}$$

where Φ *and* Ψ *are the same polynomials as in (10),* $\left\{u_n^{[m]}(\mu)\right\}_{n\geq 0}$ *is the dual sequence of* $\left\{P_n^{[m]}(.,\mu)\right\}_{n\geq 0}$ *and* k_m *is defined by the condition* $\left(u_0^{[m]}(\mu)\right)_0 = 1$.

For the proof, the following lemma is needed.

Lemma 2.3. *[7] If* $\{P_n\}_{n\geq 0}$ *is Dunkl-classical symmetric MOPS, then*

$$u_0^{[1]}(\mu) = k\Phi u_0 \tag{14}$$

where k *is a normalization factor and* Φ *is the same polynomials as in (10).*

Proof of Proposition 2.2. Suppose $m = 1$. The form u_0 satisfies (10). Multiplying both sides by Φ and on account of (8) and (14), we get

$$T_\mu\left(\Phi u_0^{[1]}(\mu)\right) + \left(\Psi - T_\mu\Phi\right)u_0^{[1]}(\mu) = 0.$$

Therefore, (12) and (13) are valid for $m = 1$. By induction, we easily obtain the general case. □

The main result of this paper follows:

Theorem 2.4. *The symmetric MOPS* $\{P_n\}_{n\geq 0}$ *is Dunkl-classical if and only if there exist a monic polynomial* Φ*,* $\deg\Phi \leq 2$ *and a sequence* $\{\Lambda_n\}_{n\geq 0}$*,* $\Lambda_n \neq 0$*,* $n \geq 0$ *such that*

$$P_n u_0 = \Lambda_n T_\mu^n\left(\Phi^n u_0\right), \quad n \geq 0. \tag{15}$$

We may call (15) *a (functional) Rodrigues type formula for the Dunkl-classical symmetric orthogonal polynomials.*

Proof. Necessity. Consider $\left\langle T_\mu^n u_0^{[n]}, P_m\right\rangle = (-1)^n\left\langle u_0^{[n]}, T_\mu^n P_m\right\rangle, \quad n, m \geq 0$. For $0 \leq m \leq n-1, n \geq 1$, we have $T_\mu^n P_m = 0$. For $m \geq n$, put $m = n+k$, $k \geq 0$. Then

$$\left\langle u_0^{[n]}, T_\mu^n P_{n+k}\right\rangle = \left(\prod_{\nu=1}^n \mu_{k+\nu}\right)\left\langle u_0^{[n]}, P_k^{[n]}\right\rangle = \left(\prod_{\nu=1}^n \mu_\nu\right)\delta_{0,k}$$

following the definitions. Consequently

$$T_\mu^n u_0^{[n]} = (-1)^n\left(\prod_{\nu=1}^n \mu_\nu\right)u_n, \quad n \geq 0.$$

But from (3) so that, in accordance with (13), we obtain (15) where

$$\Lambda_n = (-1)^n\frac{\langle u_0, P_n^2\rangle}{\prod_{\nu=1}^n \mu_\nu}k_n, n \geq 0. \tag{16}$$

Sufficiency. Making $n = 1$ in (15), we have $P_1 u_0 = \Lambda_1 T_\mu\left(\Phi u_0\right)$ and (11) is satisfied since u_0 is regular. Therefore, the sequence $\{P_n\}_{n\geq 0}$ is Dunkl-classical according to Theorem 2.1. □

The next proposition summarizes some properties of the the generalized Hermite polynomials $\{H_n^\mu(x)\}_{n\geq 0}$ and the generalized Gegenbauer ones $\{S_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ (see [2]). It will be used in the sequel.

Proposition 2.5. *1) The sequence $\{H_n^{\mu}(x)\}_{n\geq 0}$ is orthogonal with respect to $\mathcal{H}(\mu)$, this last form satisfies*

$$D\left(x\mathcal{H}(\mu)\right) + \left(2x^2 - (2\mu+1)\right)\mathcal{H}(\mu) = 0. \tag{17}$$

In addition, $\{H_n^{\mu}(x)\}_{n\geq 0}$ verifies (4) with

$$\beta_n = 0,\ \gamma_{n+1} = \frac{1}{2}\left(n+1+\mu\left(1+(-1)^n\right)\right), \quad 2\mu \neq -2n-1, \quad n \geq 0. \tag{18}$$

2) The sequence $\{S_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ is orthogonal with respect to $\mathcal{GG}(\alpha,\beta)$, this last form satisfies

$$D\left(x(x^2-1)\mathcal{GG}(\alpha,\beta)\right) + \left(-2(\alpha+\beta+2)x^2 + 2(\beta+1)\right)\mathcal{GG}(\alpha,\beta) = 0. \tag{19}$$

In addition, $\{S_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ verifies (4) with

$$\begin{array}{c}\beta_n = 0,\ \gamma_{n+1} = \dfrac{(n+1+\delta_n)(n+1+2\alpha+\delta_n)}{4(n+\alpha+\beta+1)(n+\alpha+\beta+2)},\ \delta_n = (2\beta+1)\dfrac{1+(-1)^n}{2},\ n\geq 0 \\ \alpha \neq -n, \beta \neq -n, \alpha+\beta \neq -n, n \geq 1.\end{array} \tag{20}$$

Lemma 2.6. *If u_0 is a symmetric Dunkl-classical form , then $\tilde{u}_0 = h_{a^{-1}}u_0$ is also for every $a \neq 0$.*

Proof. It is easy to see that $\tilde{u}_0$ is symmetric. Applying the operator h_a to the functional equation (10) and using (7), we obtain

$$T_\mu\left(\tilde{\Phi}\tilde{u}_0\right) + \tilde{\Psi}\tilde{u}_0 = 0, \tag{21}$$

where $\tilde{\Phi}(x) = a^{-t}\Phi(ax), \quad \tilde{\Psi}(x) = a^{1-t}\Psi(ax), \quad t = \deg\Phi.$

We have $\tilde{\Psi}'(0) - \frac{1}{2}\tilde{\Phi}''(0)\mu_n = a^{2-t}\left(\Psi'(0) - \frac{1}{2}\Phi''(0)\mu_n\right) \neq 0$, by (11). Hence the desired result. □

Lemma 2.7. *If u_0 is a symmetric Dunkl-classical form then it satisfies (10) with*

$$\Phi(x) = ax^2 + c, \quad \Psi(x) = dx, \quad dc \neq 0.$$

Proof. From the statement b) of Theorem 2.1., we have Φ monic, $\deg\Phi \leq 2$ and $\deg\Psi = 1$. So, there exist $(a,b,c,d,e) \in \mathbb{C}^5$ such that $\Phi(x) = ax^2 + bx + c$, $\Psi(x) = dx + e$, $|a| + |b| + |c| \neq 0$ and $d \neq 0$. From (10), we have

$$\left\langle T_\mu\left(\Phi u_0\right) + \Psi u_0, x^n\right\rangle = 0, n \geq 0.$$

For $n = 0$, we obtain $d(u_0)_1 + e = 0$. Then $e = 0$ since u_0 is symmetric.
For $n = 2$, we get $-2b(u_0)_2 = 0$, then $b = 0$ because $(u_0)_2 = \gamma_1 \neq 0$.
Now, suppose that $c = 0$. We will necessarily have $a \neq 0$. Otherwise, we would have, from (10) and the last results

$$\left\langle T_\mu\left(ax^2u_0\right) + dxu_0, x^{2n+1}\right\rangle = 0, \quad n \geq 0$$

this gives $\left(d - a(2n+1+2\mu)\right)(u_0)_{2n+2} = 0$. Then we deduce that $(u_0)_2 = \dfrac{d}{a(1+2\mu)}$ and $(u_0)_{2n+2} = 0, n \geq 1$ which is a contradiction with the regularity of u_0. Hence $c \neq 0$ □

Using Lemmas 2.6 and 2.7, we distinguish two canonical cases for Φ: $\Phi(x) = 1, \quad \Phi(x) = x^2 - 1$. Any so-called canonical situation will be denoted by $\hat{u}$.

First case: $\Phi(x) = 1$.

Let $\Psi(x) = dx$, it is possible to choose $d = 2$ by the dilatation $h_{\sqrt{\frac{2}{d}}}$, then

$$T_\mu\left(\hat{u}\right) + 2x\hat{u} = 0 \tag{22}$$

which is equivalent to

$$D(x\hat{u}) + \left(2x^2 - (2\mu+1)\right)\hat{u} = 0. \tag{23}$$

In fact, multiplying (22) by x, we obtain (23) by taking into account (8) and the fact $H_{-1}(x\hat{u}) = 0$. Conversely, multiplying (23) by x^{-1} and using (1), we obtain (22) since $\langle T_\mu(\hat{u}) + 2x\hat{u}, 1\rangle = 0$ and $H_{-1}(x\hat{u}) = 0$.
In other word, from (23), we have the moments $(\hat{u})_n, n \geq 0$ satisfy

$$2(\hat{u})_{n+2} = (n + 2\mu + 1)(\hat{u})_n, \quad n \geq 0,$$

and the set of solutions is a 1-dimensional linear space since $\hat{u}$ is symmetric.
Hence, in this case $\hat{u} = \mathcal{H}(\mu)$ by virtue of (17).
Second case: $\Phi(x) = x^2 - 1$.
Let $\Psi(x) = dx$. Putting $d = -2(\alpha+1), \alpha \neq -1$, we get

$$T_\mu\left((x^2-1)\hat{u}\right) - 2(\alpha+1)x\hat{u} = 0. \tag{24}$$

Since $H_{-1}(x(x^2-1)\hat{u}) = 0$, by applying the same process as we did in the first case, we prove that (24) is equivalent to

$$D\left(x(x^2-1)\hat{u}\right) + \left((-2\alpha - 2\mu - 3)x^2 + (2\mu+1)\right)\hat{u} = 0$$

And, we deduce that in this case $\hat{u} = \mathcal{GG}\left(\alpha, \mu - \frac{1}{2}\right)$ by comparing the last equation with (19).

As a conclusion, we can state:

Theorem 2.8. *(Compare with [1]) Up to a dilatation, the only Dunkl-classical symmetric MOPS are:*
(a) The generalized Hermite polynomials $\{H_n^\mu(x)\}_{n\geq 0}$ *for* $\mu \neq -n - \frac{1}{2}, n \geq 0$. *Moreover,*

$$T_\mu(\mathcal{H}(\mu)) + 2x\mathcal{H}(\mu) = 0. \tag{25}$$

(b) The generalized Gegenbauer polynomials $\{S_n^{(\alpha,\mu-\frac{1}{2})}(x)\}_{n\geq 0}$ *for* $\alpha \neq -n, \alpha + \mu \neq -n + \frac{1}{2}, \mu \neq -n + \frac{1}{2}, n \geq 1$. *Moreover,*

$$T_\mu\left((x^2-1)\mathcal{GG}\left(\alpha, \mu - \frac{1}{2}\right)\right) - 2(\alpha+1)x\mathcal{GG}\left(\alpha, \mu - \frac{1}{2}\right) = 0. \tag{26}$$

Finally, we characterize the generalized Hermite polynomials and the generalized Gegenbauer ones in terms of the Rodrigues type formula as follows:

Theorem 2.9. *We may write*

$$H_n^\mu(x)\mathcal{H}(\mu) = \left(\frac{-1}{2}\right)^n \prod_{\nu=1}^{n} \frac{\nu + 1 + \mu(1 + (-1)^\nu)}{\nu + \mu(1 - (-1)^\nu)} T_\mu^n(\mathcal{H}(\mu)), \quad n \geq 0. \tag{27}$$

$$S_n^{(\alpha,\mu-\frac{1}{2})}(x)\mathcal{GG}\left(\alpha, \mu - \frac{1}{2}\right) = \Lambda_n T_\mu^n\left((x^2-1)^n \mathcal{GG}\left(\alpha, \mu - \frac{1}{2}\right)\right), \quad n \geq 0 \tag{28}$$

with $\Lambda_n = \dfrac{\Gamma\left(\alpha + \mu + n + \frac{3}{2}\right)\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)\Gamma\left(\alpha + \mu + \frac{3}{2}\right)} \displaystyle\prod_{\nu=1}^{n} \frac{(\nu + \delta_\nu)(\nu + 2\alpha + \delta_\nu)}{(\nu + \mu(1 - (-1)^\nu))(2\nu + 2\alpha + 2\mu - 1)(2\nu + 2\alpha + 2\mu)}, \quad n \geq 0.$

Proof. Use Theorems 2.4 and 2.8, Proposition 2.5 and equation (16). □

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