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Fixed points of a new type of contractive mappings and multifunctions

H. Alikhani[a], Sh. Rezapour[a], N. Shahzad[b]

[a]*Department of mathemetics, Azarbaijan Shahid Madani University, Azarshahr, Tabriz, Iran*
[b]*Department of Mathematics, King AbdulAziz University, P.O. Box 80203, Jeddah 21859, Saudi Arabia*

Abstract. In this paper, we introduce the concept of α-ψ-ξ-contractive mappings and β-ψ-ξ-contractive multifunctions and give some fixed point results for such mappings and multifunctions. We show that our fixed point result of α-ψ-ξ-contractive mappings is different from that of α-ψ-contractive mappings which has been proved recently by Samet, Vetro and Vetro.

1. Introduction

In recent years, there have appeared a number of fixed point results for multifunctions in metric spaces (see for example [2–4, 6, 7, 9]). In 2012, Samet, Vetro and Vetro introduced the concept of α-ψ-contractive type mappings ([8]). Their work generalized many ordered fixed point results (see [8]). Denote by Ψ the set of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$. Let (X, d) be a metric space, T a selfmap on X, $\alpha : X \times X \to [0,\infty)$ a function and $\psi \in \Psi$. Then, T is said to be α-ψ-contractive whenever $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$ ([8]). Also, we say that T is α-admissible whenever $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$ ([8]). Now, we say that T is an α-ψ-ξ-contractive selfmap whenever

$$\alpha(x, y)d(Tx, Ty) \leq \psi(h(x, y))$$

for all $x, y \in X$, where $h(x, y) = d(x, Ty) + d(y, Tx) + d(x, y) - \xi(x, y)$ and

$$\xi(x, y) = \max\{d(x, Ty), d(y, Tx)\}.$$

Now by using all obtained idea, we introduce the following notion. Let (X, d) be a metric space, $T : X \to 2^X$ a multifunction, $\beta : 2^X \times 2^X \to [0,\infty)$ a mapping and $\psi \in \Psi$. We say that T is β-admissible whenever $\beta(A, B) \geq 1$ implies $\beta(Tx, Ty) \geq 1$ for all $x \in A$ and $y \in B$, where A and B are subsets of X. Also, we say that a closed-valued multifunction T is β-ψ-ξ-contractive multifunction whenever

$$\beta(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, Ty) + d(y, Tx) + d(x, y) - \xi(x, y)) = \psi(h(x, y))$$

for all $x, y \in X$, where H is the Hausdorff generalized metric. Also, we say that the multifunction T is lower semi-continuous (briefly, LSC) at $x_0 \in X$ whenever for each sequence $\{x_n\}$ with $x_n \to x_0$ and every $y \in Tx_0$,

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Email addresses: h.alikhani09@gmail.com (H. Alikhani), rezapourshahram@yahoo.ca (Sh. Rezapour), naseer_shahzad@hotmail.com (N. Shahzad)

there exists a sequence $\{y_n\}$ such that $y_n \to y$ and $y_n \in Tx_n$ for all n ([5]). Let (X, d) be a metric space, C a nonempty subset of X and $x \in X$. An element $y_0 \in C$ is said to be a best approximation of x whenever $d(x, y_0) = d(x, C) = \inf_{y \in C} d(x, y)$. The set C is said to be a proximinal whenever every $x \in X$ has at least one best approximation in C ([1]). It is known that proximinal subsets are closed ([1]). Denote by $P(X)$ the set of all proximinal subsets of X.

2. Main Results

Now, we are ready to state and prove our main results.

Theorem 2.1. *Let (X, d) be a complete metric space and T a continuous, α-admissible and α-ψ-ξ-contractive selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$ for some $x_0 \in X$. Then T has a fixed point.*

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. Put $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_n = x_{n+1}$ for some n, then we have nothing to prove. Assume that $x_n \neq x_{n+1}$ for all n. Since T is α-admissible, it is easy to check that $\alpha(x_n, x_{n+1}) \geq 1$ for all n. Thus,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n) \leq \psi(h(x_{n-1}, x_n)) \\ &= \psi(d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}) + d(x_{n-1}, x_n) - \xi(x_{n-1}, x_n)) \\ &= \psi(d(x_{n-1}, x_{n+1}) + d(x_n, x_n) + d(x_{n-1}, x_n) - \max\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}) \\ &= \psi(d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n) - d(x_{n-1}, x_{n+1})) = \psi(d(x_{n-1}, x_n)) \end{aligned}$$

for all n. Hence, $d(x_{n+1}, x_n) \leq \psi^n(d(x_0, x_1))$ for all n. Fix $\varepsilon > 0$. Then, there exists a natural number N_ε such that $\sum_{n \geq N_\varepsilon} \psi^n(d(x_0, x_1)) < \varepsilon$. Let $m > n \geq N_\varepsilon$. By using the triangular inequality, we obtain

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k d(x_0, x_1) \leq \sum_{n \geq N_\varepsilon} \psi^n d(x_0, x_1) < \varepsilon.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Since T is continuous, $Tx^* = x^*$. This completes the proof. □

Now, we give the following example to show the difference of Theorem 2.1 and the first result of [8].

Example 2.1. *Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$. Define $Tx = \frac{4}{3}x$ for all $x \in \mathbb{R}$, $\psi(t) = \frac{3}{4}t$ for all $t \geq 0$ and $\alpha : X \times X \to [0, +\infty)$ by $\alpha(x, y) = 1$ whenever $y \leq \frac{7}{6}x$ and $\alpha(x, y) = 0$ otherwise. If $y > \frac{7}{6}x$, then $\alpha(x, y)d(Tx, Ty) = 0 \leq \psi(h(x, y))$. If $y \leq \frac{7}{6}x$, then $\max\left\{\left|x - \frac{4}{3}y\right|, \left|\frac{4}{3}x - y\right|\right\} = \left|x - \frac{4}{3}y\right|$. Thus, we have*

$$\begin{aligned} \alpha(x, y)d(Tx, Ty) &= \frac{4}{3}|x - y| \leq \frac{3}{2}\left|\frac{4}{3}x - y\right| = \frac{3}{4}\left(\left|\frac{4}{3}x - y\right| + \left|\frac{4}{3}x - y\right|\right) \leq \frac{3}{4}\left(\left|\frac{4}{3}x - y\right| + |x - y|\right) \\ &= \psi\left(\left|\frac{4}{3}x - y\right| + |x - y|\right) = \psi(d(x, Ty) + d(y, Tx) + d(x, y) - \xi(x, y)). \end{aligned}$$

Hence, T is an α-ψ-ξ-contractive selfmap. On the other hand, for $y \leq \frac{7}{6}x$ we have

$$\alpha(x, y)d(Tx, Ty) = \frac{4}{3}|x - y| \gneq \frac{3}{4}|x - y| = \psi(d(x, y)).$$

Therefore, T is not α-ψ-contractive.

Corollary 2.2. *Let $(X, d, \leq)$ be a complete ordered metric space and T a continuous and nondecreasing selfmap on X such that $d(Tx, Ty) \leq \lambda h(x, y)$ for all $x, y \in X$ with $x \leq y$ or $y \leq x$, where λ is an element in $[0, 1)$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$ or $Tx_0 \leq x_0$, then T has a fixed point.*

Corollary 2.3. *Let (X, d) be a complete metric space, $\lambda \in [0, 1)$ and T a continuous selfmap on X such that $T(A) \subset A$ for some subset A of X and $d(Tx, Ty) \leq \lambda h(x, y)$ for all $x, y \in A$. Then T has a fixed point.*

Proof. Define the mapping $\alpha : X \times X \to [0, +\infty)$ by $\alpha(x, y) = 1$ whenever $x \in A$ or $y \in A$ and $\alpha(x, y) = 0$ otherwise. Then, we have $\alpha(x, y)d(Tx, Ty) \leq kh(x, y)$ for all $x, y \in X$. Define $\psi(t) = kt$ for all $t \geq 0$. Thus, T is an α-ψ-ξ-contractive mapping. Let $x, y \in X$ be such that $\alpha(x, y) \geq 1$. Since $T(A) \subset A$, $Tx \in A$ or $Ty \in A$ and so $\alpha(Tx, Ty) \geq 1$. Hence, T is α-admissible. Since A is nonempty, $\alpha(x_0, Tx_0) = 1$ for all $x_0 \in A$. Now by using Theorem 2.1, T has a fixed point. □

Now, we give the following result for proximinal valued multifunctions.

Theorem 2.4. *Let (X, d) be a complete metric space and $T : X \to P(X)$ a LSC, β-admissible and β-ψ-ξ-contractive multifunction such that $\beta(A, Tx_0) \geq 1$ for some $A \subset X$ and $x_0 \in A$. Then T has a fixed point.*

Proof. Choose $A \subset X$ and $x_0 \in A$ such that $\beta(A, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ by $x_{n+1} \in Tx_n$ and $d(x_n, x_{n+1}) = d(x_n, Tx_n)$ for all $n \geq 0$. If $x_n = x_{n+1}$ for some n, then we have nothing to prove. Assume that $x_n \neq x_{n+1}$ for all n. Since T is β-admissible, $x_0 \in A$, $x_1 \in Tx_0$ and $\beta(A, Tx_0) \geq 1$, we have $\beta(Tx_0, Tx_1) \geq 1$. By continuing this process it is easy to show that $\beta(Tx_{n-1}, Tx_n) \geq 1$ for all n. Thus,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, Tx_n) \leq H(Tx_{n-1}, Tx_n) \leq \beta(Tx_{n-1}, Tx_n)H(Tx_{n-1}, Tx_n) \leq \psi(h(x_{n-1}, x_n)) \\ &= \psi(d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}) + d(x_{n-1}, x_n) - \xi(x_{n-1}, x_n)) = \psi(d(x_{n-1}, x_n)) \end{aligned}$$

and so $d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1))$ for all n. Fix $\varepsilon > 0$. Then there exists a natural number N_ε such that $\sum_{n \geq N_\varepsilon} \psi^n(t) < \varepsilon$. Let $m > n \geq N_\varepsilon$. Then,

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k d(x_0, x_1) \leq \sum_{n \geq N_\varepsilon} \psi^n d(x_0, x_1) < \varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence. Choose $x^* \in X$ such that $x_n \to x^*$. Let $y \in Tx^*$. Since T is LSC, there exists a sequence $\{y_n\}$ such that $y_n \in Tx_n$ for all n and $y_n \to y$. Hence, $d(x^*, Tx^*) \leq d(x^*, y) \leq d(x^*, x_{n+1}) + d(x_{n+1}, z) + d(z, y_n) + d(y_n, y)$ for all $z \in Tx_n$. This implies that

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + \inf_{z \in Tx_n} d(x_{n+1}, z) + \inf_{z \in Tx_n} d(z, y_n) + d(y_n, y) \\ &= d(x^*, x_{n+1}) + d(x_{n+1}, Tx_n) + d(Tx_n, y_n) + d(y_n, y) \\ &= d(x^*, x_{n+1}) + d(y_n, y), \end{aligned}$$

and so $d(x^*, Tx^*) \leq d(x^*, x_{n+1}) + d(y_n, y)$ for all n. Thus, we get $x^* \in Tx^*$. □

Now, we give the following example to show that there are multifunctions which satisfy the assumptions of Theorem 2.4.

Example 2.2. *Let $X = [0, \infty)$, $a > 0$, $d(x, y) = |x - y|$ for all $x, y \in X$, H the Hausdorff metric, T a proximinal-valued multifunction on X defined by $Tx = [x, a]$ whenever $x \leq a$ and $Tx = [a, x]$ whenever $x > a$ and $\beta : 2^X \times 2^X \to [0, +\infty)$ a mapping defined by $\beta(C, D) = 1$ whenever $C \cap D = \{a\}$ and $\beta(C, D) = 0$ otherwise. Suppose that A and B are subsets of X such that $A \cap B = \{a\}$. Then, $\beta(Tx, Ty) = 1$ whenever $x \leq a < y$ or $y \leq a < x$. If $x \leq a < y$, then $\rho(Tx, Ty) = a - x$ and $\rho(Ty, Tx) = y - a$, where $\rho(A, B) = \sup_{a \in A} d(a, B)$. Hence, $H(Tx, Ty) = \max\{a - x, y - a\}$. If $a - x > y - a$, then $\max\{a - x, y - a\} = a - x$. Also, we have*

$$\begin{aligned} \beta(Tx, Ty)H(Tx, Ty) = (a - x) &< (y - a) + (y - a) + (a - x) \\ &= (a - x) + (y - a) + (y - x) - \max\{a - x, y - a\}. \end{aligned}$$

Now, by using the Archimedean property, there exists $k \in [0,1)$ such that

$$(a-x) \leq k((a-x)+(y-a)+(y-x)-\max\{a-x, y-a\}).$$

If we define $\psi(t) = kt$, then

$$\begin{aligned}\beta(Tx,Ty)H(Tx,Ty) = (a-x) &\leq \psi((a-x)+(y-a)+(y-x)-\max\{a-x,y-a\}) \\ &= \psi(d(x,Ty)+d(y,Tx)+d(x,y)-\xi(x,y)).\end{aligned}$$

Therefore, by providing a similar proof for another cases, one can show that T is a β-ψ-ξ-contractive multifunction. It is easy to see that T is β-admissible and LSC. Let $a \leq c$ and $A = [a,c]$. Then, $Ta = \{a\}$ and $\beta(A, Ta) = 1$. Thus, the multifunction T satisfies the assumptions of Theorem 2.4. Note that, each element of the interval $[0,\infty)$ is a fixed point of T.

Corollary 2.5. *Let (X,d) be a complete metric space, $\lambda \in [0,1)$, $T : X \to P(X)$ a LSC multifunction and C a nonempty subset of X such that $Tx \subset C$ for all $x \in C$. Suppose that $H(Tx,Ty) \leq \lambda h(x,y)$ for all $x, y \in C$. Then T has a fixed point.*

Proof. Define $\beta : 2^X \times 2^X \to [0,+\infty)$ by $\beta(A,B) = 1$ whenever $A \subset C$ or $B \subset C$ and $\beta(A,B) = 0$ otherwise. Define $\psi(t) = kt$ for all $t \geq 0$. Then, we have

$$\beta(Tx,Ty)H(Tx,Ty) \leq \psi(h(x,y))$$

for all $x, y \in X$. Hence, T is a β-ψ-ξ-contractive multifunction. If $A, B \subset X$ and $\beta(A,B) \geq 1$, then $A \subset C$ or $B \subset C$. Without loss of generality, suppose that $A \subset C$. Then, $Tx \subset C$ for all $x \in A$ and so $\beta(Tx,Ty) \geq 1$ for all $y \in B$. Therefore, T is β-admissible. If $x \in C$, then $Tx \subset C$ and so $\beta(C,Tx) = 1$. Now by using Theorem 2.4, T has a fixed point. □

Theorem 2.6. *Let (X,d) be a complete metric space and $T : X \to P(X)$ a β-admissible and β-ψ-ξ-contractive multifunction such that $\beta(A,Tx_0) \geq 1$ for some $A \subset X$ and $x_0 \in A$. Also, suppose that $\beta(Tx_{n-1},Tx) \geq 1$ for all n whenever $\{x_n\}$ is a sequence in X such that $\beta(Tx_{n-1},Tx_n) \geq 1$ for all n and $x_n \to x$. Then T has a fixed point.*

Proof. Choose $A \subset X$ and $x_0 \in A$ such that $\beta(A,Tx_0) \geq 1$. Define the sequence $\{x_n\}$ by $x_{n+1} \in Tx_n$ and $d(x_n,x_{n+1}) = d(x_n,Tx_n)$ for all $n \geq 0$. If $x_n = x_{n+1}$ for some n, then we have nothing to prove. Assume that $x_n \neq x_{n+1}$ for all n. By using a similar technique in proof of Theorem 2.4, one can deduce that $\{x_n\}$ is a Cauchy sequence. Choose $x^* \in X$ such that $x_n \to x^*$. Since $\beta(Tx_{n-1},Tx_n) \geq 1$ for all n, by using the assumption we obtain $\beta(Tx_{n-1},Tx^*) \geq 1$ for all n. Hence,

$$d(x^*,Tx^*) \leq d(x^*,z) + d(z,Tx^*)$$

for all $z \in Tx_{n-1}$. But, we have

$$\begin{aligned}d(x^*,Tx^*) &\leq d(x^*,Tx_{n-1}) + H(Tx_{n-1},Tx^*) \\ &\leq d(x^*,x_n) + \beta(Tx_{n-1},Tx^*)H(Tx_{n-1},Tx^*) \leq d(x^*,x_n) + \psi(h(x_{n-1},x^*)) \\ &\leq d(x^*,x_n) + \psi(d(x_{n-1},Tx^*) + d(x^*,Tx_{n-1}) + d(x_{n-1},x^*) - \xi(x_{n-1},x^*))\end{aligned}$$

for all n. If $\xi(x_{n-1},x^*) = d(x_{n-1},Tx^*)$, then we have

$$d(x^*,Tx^*) \leq d(x^*,x_n) + \psi(d(x_n,x^*) + d(x_{n-1},x^*))$$

and if $\xi(x_{n-1},x^*) = d(x^*,Tx_{n-1})$, then we have

$$d(x^*,Tx^*) \leq d(x^*,x_n) + \psi(d(x_{n-1},Tx^*) + d(x_{n-1},x^*)).$$

These implies that $d(x^*,Tx^*) = 0$ and so $x^* \in Tx^*$. □

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