Filomat 27:7 (2013), 1165–1171
DOI 10.2298/FIL1307165C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Coefficient bounds for new subclasses of bi-univalent functions

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Abstract. In the present investigation, we consider two new subclasses $\mathcal{N}_{\Sigma}^{\mu}(\alpha,\lambda)$ and $\mathcal{N}_{\Sigma}^{\mu}(\beta,\lambda)$ of bi-univalent functions defined in the open unit disk $\mathcal{U}=\{z:|z|<1\}$. Besides, we find upper bounds for the second and third coefficients for functions in these new subclasses.

1. Introduction and definitions

Let $\mathcal{A}$ denote the family of functions f of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1}$$

which are analytic in the open unit disk $\mathcal{U}=\{z:|z|<1\}$. Further, by $\mathcal{S}$ we will denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. The Koebe one-quarter theorem [2] ensures that the image of $\mathcal{U}$ under every $f\in\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. So, every $f\in\mathcal{S}$ has an inverse function f^{-1} satisfying

$$f^{-1}(f(z))=z \quad (z\in\mathcal{U})$$

and

$$f(f^{-1}(w))=w \quad \left(|w|<r_0(f);\ r_0(f)\geq\frac{1}{4}\right)$$

where

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+....$$

A function $f\in\mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$.
Let Σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1).

2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C50.
Keywords. Analytic and univalent function, bi-univalent function, coefficient bounds.
Received: 13 February 2012; Accepted: 29 July 2012
Communicated by Prof. Dr. H. M. Srivastava
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Research supported by Ataturk University Rectorship under "The Scientific and Research Project of Ataturk University", Project No: 2012/173.
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According to Brannan and Taha [1], a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_\Sigma[\alpha]$ of strongly bi-starlike functions of order $\alpha(0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ z \in \mathcal{U})$$

and

$$\left|\arg\left(\frac{zg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ z \in \mathcal{U}),$$

where g is the extension of f^{-1} to $\mathcal{U}$. For the function class $\mathcal{S}^*_\Sigma[\alpha]$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$.

In recent years, Srivastava et al. [5], Frasin and Aouf [3], Qing et al. [6–8] discussed estimate on the coefficients $|a_2|$ and $|a_3|$ for subclasses of bi-univalent function.

The main aim of the present investigation is to introduce and study two new subclasses $\mathcal{N}^\mu_\Sigma(\alpha, \lambda)$ and $\mathcal{N}^\mu_\Sigma(\beta, \lambda)$ of the function class Σ and obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ.

In order to establish our main results, we shall require the following lemma.

Lemma 1.1. *[4] If $p \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions p analytic in $\mathcal{U}$ for which $\Re p(z) > 0$, $p(z) = 1 + c_1z + c_2z^2 + \dots$ for $z \in \mathcal{U}$.*

2. Coefficient bounds for the function class $\mathcal{N}^\mu_\Sigma(\alpha, \lambda)$

Definition 2.1. *A function $f(z)$ given by (1) is said to be in the class $\mathcal{N}^\mu_\Sigma(\alpha, \lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma \text{ and } \left|\arg\left((1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right| < \frac{\alpha\pi}{2} \tag{2}$$

$$(0 < \alpha \leq 1,\ \mu \geq 0,\ z \in \mathcal{U})$$

and

$$\left|\arg\left((1-\lambda)\left(\frac{g(w)}{w}\right)^\mu + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)\right| < \frac{\alpha\pi}{2} \tag{3}$$

$$(0 < \alpha \leq 1,\ \mu \geq 0,\ w \in \mathcal{U})$$

where the function g is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots. \tag{4}$$

Note that for $\lambda = \mu = 1$, the class $\mathcal{N}^1_\Sigma(\alpha, 1)$ introduced and studied by Srivastava et al. [5] and for $\mu = 1$, the class $\mathcal{N}^1_\Sigma(\alpha, \lambda)$ was considered by Frasin and Aouf [3].

Theorem 2.2. *Let $f(z)$ given by (1) be in the class $\mathcal{N}^\mu_\Sigma(\alpha, \lambda)$, $0 < \alpha \leq 1$, $\lambda \geq 1$ and $\mu \geq 0$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda+\mu)^2 + \alpha(\mu + 2\lambda - \lambda^2)}} \tag{5}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\lambda+\mu)^2} + \frac{2\alpha}{2\lambda+\mu}.$$

Proof. It follows from (2) and (3) that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}=[p(z)]^{\alpha} \tag{6}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}=[q(w)]^{\alpha} \tag{7}$$

where $p(z)=1+p_1z+p_2z^2+...$ and $q(w)=1+q_1w+q_2w^2+...$ in $\mathcal{P}$.

Now, equating the coefficients in (6) and (7), we have

$$(\lambda+\mu)\,a_2 = \alpha p_1, \tag{8}$$

$$(2\lambda+\mu)\,a_3+(\mu-1)\left(\lambda+\frac{\mu}{2}\right)a_2^2 = \alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2, \tag{9}$$

$$-(\lambda+\mu)\,a_2 = \alpha q_1 \tag{10}$$

and

$$-(2\lambda+\mu)\,a_3+(3+\mu)\left(\lambda+\frac{\mu}{2}\right)a_2^2=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{11}$$

From (8) and (10), we arrive at

$$p_1=-q_1 \tag{12}$$

and

$$2(\lambda+\mu)^2a_2^2=\alpha^2\left(p_1^2+q_1^2\right). \tag{13}$$

Now, from (9), (11) and (13), we get that

$$\begin{aligned}(\mu+1)(2\lambda+\mu)\,a_2^2 &= \alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2+q_1^2\right)\\ &= \alpha(p_2+q_2)+\frac{(\alpha-1)}{\alpha}(\lambda+\mu)^2a_2^2.\end{aligned}$$

Therefore, we have

$$a_2^2=\frac{\alpha^2(p_2+q_2)}{(\lambda+\mu)^2+\alpha(\mu+2\lambda-\lambda^2)}. \tag{14}$$

Applying Lemma 1.1 for (14), we obtain

$$|a_2|\le\frac{2\alpha}{\sqrt{(\lambda+\mu)^2+\alpha(\mu+2\lambda-\lambda^2)}}$$

which gives us desired estimate on $|a_2|$ as asserted in (5).

Next, in order to find the bound on $|a_3|$, by subtracting (11) from (10), we have

$$2(2\lambda+\mu)\,a_3-2(2\lambda+\mu)\,a_2^2=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2-\left(\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2\right). \tag{15}$$

It follows from (12), (13) and (15) that

$$a_3 = \frac{\alpha^2\left(p_1^2+q_1^2\right)}{2\left(\lambda+\mu\right)^2} + \frac{\alpha\left(p_2-q_2\right)}{2\left(2\lambda+\mu\right)} \tag{16}$$

Applying Lemma 1.1 for (16), we readily get

$$|a_3| \leq \frac{4\alpha^2}{\left(\lambda+\mu\right)^2} + \frac{2\alpha}{2\lambda+\mu}.$$

This completes the proof of Theorem 2.2. □

If we take $\mu = 1$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *[3] Let $f(z)$ given by (1) be in the class $\mathcal{N}_\Sigma^1(\alpha,\lambda)$, $0<\alpha\leq 1$ and $\lambda \geq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\left(\lambda+1\right)^2+\alpha\left(1+2\lambda-\lambda^2\right)}} \text{ and } |a_3| \leq \frac{4\alpha^2}{\left(\lambda+1\right)^2} + \frac{2\alpha}{2\lambda+1}.$$

If we write $\lambda = \mu = 1$ in Theorem 2.2, we get the following corollary.

Corollary 2.4. *[5] Let $f(z)$ given by (1) be in the class $\mathcal{N}_\Sigma^1(\alpha,1)$, $0<\alpha\leq 1$. Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha+2}} \text{ and } |a_3| \leq \frac{\alpha\left(3\alpha+2\right)}{3}.$$

If we choose $\lambda = \mu + 1 = 1$ in Theorem 2.2, we obtain well-known the class $\mathcal{N}_\Sigma^0(\alpha,1) = \mathcal{S}_\Sigma^*[\alpha]$ of strongly bi-starlike functions of order α and get following corollary.

Corollary 2.5. *Let $f(z)$ given by (1) be in the class $\mathcal{S}_\Sigma^*[\alpha]$, $0<\alpha\leq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{1+\alpha}} \text{ and } |a_3| \leq \alpha(4\alpha+1)\,.$$

3. Coefficient bounds for the function class $\mathcal{N}_\Sigma^\mu(\beta,\lambda)$

Definition 3.1. *A function $f(z)$ given by (1) is said to be in the class $\mathcal{N}_\Sigma^\mu(\beta,\lambda)$ if the following conditions are satisfied*

$$f \in \Sigma \text{ and } \Re\left((1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > \beta \tag{17}$$

$$(0 \leq \beta < 1,\ \mu \geq 0,\ \lambda \geq 1,\ z \in \mathcal{U})$$

and

$$\Re\left((1-\lambda)\left(\frac{g(w)}{w}\right)^\mu + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right) > \beta \tag{18}$$

$$(0 \leq \beta < 1,\ \mu \geq 0,\ \lambda \geq 1,\ w \in \mathcal{U})$$

where the function g is defined by (4).

The class which is satisfy the conditon (17) except $f \in \Sigma$ also was studied with other aspects by Zhu [9].

Note that for $\lambda = \mu = 1$, the class $\mathcal{N}_{\Sigma}^{1}(\beta, 1)$ introduced and studied by Srivastava et al. [5] and for $\mu = 1$, the class $\mathcal{N}_{\Sigma}^{1}(\beta, \lambda)$ introduced and worked by Frasin and Aouf [3].

Next, we derive

Theorem 3.2. *Let $f(z)$ given by (1) be in the class $\mathcal{N}_{\Sigma}^{\mu}(\beta, \lambda)$, $0 \leq \beta < 1$, $\lambda \geq 1$ and $\mu \geq 0$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{4(1-\beta)}{(\mu+1)(2\lambda+\mu)}}, \frac{2(1-\beta)}{\lambda+\mu}\right\} \tag{19}$$

and

$$|a_3| \leq \begin{cases} \min\left\{\dfrac{4(1-\beta)}{(\mu+1)(2\lambda+\mu)}, \dfrac{4(1-\beta)^2}{(\lambda+\mu)^2} + \dfrac{2(1-\beta)}{2\lambda+\mu}\right\}, & 0 \leq \mu < 1 \\ \dfrac{2(1-\beta)}{2\lambda+\mu}, & \mu \geq 1 \end{cases}. \tag{20}$$

Proof. It follows from (17) and (18) that there exist $p, q \in \mathcal{P}$ such that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = \beta + (1-\beta)p(z) \tag{21}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = \beta + (1-\beta)q(w) \tag{22}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + ...$ and $q(w) = 1 + q_1 w + q_2 w^2 + ...$ As in the proof of Theorem 2.2, by suitably comparing coefficients in (21) and (22), we get

$$(\lambda+\mu)a_2 = (1-\beta)p_1, \tag{23}$$

$$(2\lambda+\mu)a_3 + (\mu-1)\left(\lambda+\frac{\mu}{2}\right)a_2^2 = (1-\beta)p_2, \tag{24}$$

$$-(\lambda+\mu)a_2 = (1-\beta)q_1 \tag{25}$$

and

$$-(2\lambda+\mu)a_3 + (3+\mu)\left(\lambda+\frac{\mu}{2}\right)a_2^2 = (1-\beta)q_2. \tag{26}$$

Now, considering (23) and (25), we obtain

$$p_1 = -q_1 \tag{27}$$

and

$$2(\lambda+\mu)^2 a_2^2 = (1-\beta)^2\left(p_1^2 + q_1^2\right). \tag{28}$$

Also, from (24) and (26), we have

$$(\mu+1)(2\lambda+\mu)a_2^2 = (1-\beta)(p_2+q_2). \tag{29}$$

Therefore, from the equalities (28) and (29) we find that

$$|a_2|^2 \leq \frac{(1-\beta)^2}{2(\lambda+\mu)^2}\left(|p_1|^2 + |q_1|^2\right)$$

and

$$|a_2|^2 \leq \frac{(1-\beta)}{(\mu+1)(2\lambda+\mu)}\left(|p_2|+|q_2|\right)$$

respectively, and applying Lemma 1.1, we obtain

$$|a_2| \leq \frac{2(1-\beta)}{\lambda+\mu} \tag{30}$$

and

$$|a_2| \leq 2\sqrt{\frac{1-\beta}{(\mu+1)(2\lambda+\mu)}} \tag{31}$$

respectively. If we compare the right sides of the inequalities (30) and (31) we obtain desired estimate on $|a_2|$ as asserted in (19).

Next, in order to find the bound on $|a_3|$, by subtracting (26) from (24), we get

$$2(2\lambda+\mu)a_3 - 2(2\lambda+\mu)a_2^2 = (1-\beta)(p_2-q_2), \tag{32}$$

which, upon substitution of the value of a_2^2 from (28), yields

$$a_3 = \frac{(1-\beta)^2\left(p_1^2+q_1^2\right)}{2(\lambda+\mu)^2} + \frac{(1-\beta)(p_2-q_2)}{2(2\lambda+\mu)}. \tag{33}$$

Applying Lemma 1.1 for (33), we readily get

$$|a_3| \leq \frac{4(1-\beta)^2}{(\lambda+\mu)^2} + \frac{2(1-\beta)}{2\lambda+\mu}. \tag{34}$$

On the other hand, by using the equation (29) in (32), we obtain

$$a_3 = \frac{1-\beta}{2(2\lambda+\mu)}\left[\frac{\mu+3}{\mu+1}p_2 + \frac{1-\mu}{\mu+1}q_2\right]. \tag{35}$$

and applying Lemma 1.1 for (35), we get

$$|a_3| \leq \frac{1-\beta}{2\lambda+\mu}\left[\frac{\mu+3}{\mu+1} + \frac{|1-\mu|}{\mu+1}\right]. \tag{36}$$

Now, let us investigate the bound on $|a_3|$ according to μ.

Case1. We suppose that let $0 \leq \mu < 1$, thus from (36)

$$|a_3| \leq \frac{4(1-\beta)}{(2\lambda+\mu)(\mu+1)} \tag{37}$$

which is the first part of assertion (20).

Case2. We suppose that let $\mu \geq 1$, thus from (36) we readily see that

$$|a_3| \leq \frac{2(1-\beta)}{2\lambda+1} \tag{38}$$

which is the second part of assertion (20). When are compared the right sides of inequalities (34) and (38) we see that the right side of (38) smaller than the right side of (34).

This completes the proof of Theorem 3.2. □

If we write $\mu = 1$ in first parts of assertions (19) and (20) of Theorem 3.2, we have the following corollary.

Corollary 3.3. *Let $f(z)$ given by (1) be in the class $\mathcal{N}_{\Sigma}^{1}(\beta,\lambda)$, $0 \le \beta < 1$ and $\lambda \ge 1$. Then*

$$|a_2| \le \min\left\{\sqrt{\frac{2(1-\beta)}{2\lambda+1}}, \frac{2(1-\beta)}{\lambda+1}\right\} \text{ and } |a_3| \le \frac{2(1-\beta)}{2\lambda+1}.$$

If we choose $\lambda = \mu = 1$ in first parts of assertions (19) and (20) of Theorem 3.2, we arrive at the following corollary.

Corollary 3.4. *Let $f(z)$ given by (1) be in the class $\mathcal{N}_{\Sigma}^{1}(\beta,1)$, $0 \le \beta < 1$. Then*

$$|a_2| \le \begin{cases} \sqrt{\dfrac{2(1-\beta)}{3}}, & 0 \le \beta < \dfrac{1}{3} \\ 1-\beta, & \dfrac{1}{3} \le \beta < 1 \end{cases} \text{ and } |a_3| \le \frac{2(1-\beta)}{3}.$$

If we take $\lambda = \mu + 1 = 1$ in Theorem 3.2, we obtain well-known the class $\mathcal{N}_{\Sigma}^{0}(\beta,1) = \mathcal{S}_{\Sigma}^{*}(\beta)$ of bi-starlike functions of order β and get the following corollary.

Corollary 3.5. *Let $f(z)$ given by (1) be in the class $\mathcal{S}_{\Sigma}^{*}(\beta)$, $0 \le \beta < 1$. Then*

$$|a_2| \le \sqrt{2(1-\beta)} \text{ and } |a_3| \le \begin{cases} 2(1-\beta), & 0 \le \beta < \dfrac{3}{4} \\ (1-\beta)(5-4\beta), & \dfrac{3}{4} \le \beta < 1 \end{cases}.$$

Acknowledgements. Authors would like to thank Prof. Dr. Hari M. Srivastava for his careful reading and making some valuable comments which has essentially improved the presentation of this paper.

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