Filomat 27:7 (2013), 1351–1356
DOI 10.2298/FIL1307351X

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Sharp Coefficient Estimates for a Certain General Class of Spirallike Functions by Means of Differential Subordination

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Abstract. In the present paper, the authors derive several sharp estimates for the Taylor-Maclaurin coefficients of functions in a certain general class $\mathcal{S}^{\beta}(A,B)$ of spirallike functions in the open unit disk $\mathbb{U}$, which is defined here by using the principle of differential subordination The results presented here would generalize those given in the earlier work of R. J. Libera.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and are normalized by $f(0) = f'(0) - 1 = 0$. The class of functions in $\mathcal{A}$, which are also univalent in $\mathbb{U}$, is denoted (as usual) by $\mathcal{S}$.

Following Robertson [14], we introduce below the familiar class of starlike functions of order α in $\mathbb{U}$.

Definition 1. Let the function $f(z)$ be in the normalized analytic function class $\mathcal{A}$. Also let $\alpha \in [0,1)$ and

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{2}$$

We then say that the function $f(z)$ is starlike of order α in $\mathbb{U}$. We denote by $\mathcal{S}^{*}(\alpha)$ the class of all starlike functions of order α in $\mathbb{U}$.

2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C50

Keywords. Analytic functions; Univalent functions; Spirallike functions; Differential subordination; Coefficient estimates; Coefficient bounds; Taylor-Maclaurin coefficients; Schwarz lemma; Bieberbach conjecture (or de Branges theorem); Parseval's theorem.

Received: 15 March 2013; Accepted: 05 May 2013

Communicated by Dragan S. Djordjević

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This work was supported by the *National Natural Science Foundation of the People's Republic of China* (Grant Nos. 11261022 and 11061015), the *Jiangxi Provincial Natural Science Foundation of the People's Republic of China* (Grant No. 2010GZS0096), and the *Natural Science Foundation of the Department of Education of the Jiangxi Province and the People's Republic of China* (Grant No. GJJ12177).

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Spaček [16] extended the class of starlike functions by introducing the class of spirallike functions of type β in $\mathbb{U}$ and gave the following analytical characterization of spirallikeness functions of type β in $\mathbb{U}$.

Theorem A. (see Spaček [16]) *Let the function $f(z)$ be in the normalized analytic function class $\mathcal{A}$. Also let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $f(z)$ is a spirallike function of type β in $\mathbb{U}$ if and only if*

$$\Re\left(e^{i\beta}\,\frac{zf'(z)}{f(z)}\right) > 0 \qquad \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}\right). \tag{3}$$

Henceforth we denote the class of spiralike functions of type β in $\mathbb{U}$ by $\hat{\mathcal{S}}_\beta$.

Libera [12] unified and extended the function classes $\mathcal{S}^*(\alpha)$ and $\hat{\mathcal{S}}_\beta$ by introducing the analytic function class $\hat{\mathcal{S}}_\alpha^\beta$ in $\mathbb{U}$ as follows.

Definition 2. (see Libera [12]) Let the function $f(z)$ be in the normalized analytic function class $\mathcal{A}$. Also let

$$\alpha \in [0,1) \qquad \text{and} \qquad \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

We then say that $f \in \hat{\mathcal{S}}_\alpha^\beta$ if and only if

$$\Re\left(e^{i\beta}\,\frac{zf'(z)}{f(z)}\right) > \alpha\cos\beta \qquad \left(z \in \mathbb{U};\ 0 \leqq \alpha < 1;\ -\frac{\pi}{2} < \beta\frac{\pi}{2}\right). \tag{4}$$

Obviously, we find from Definitions 1 and 2 that

$$\hat{\mathcal{S}}_\alpha^0 = \mathcal{S}^*(\alpha) \qquad \text{and} \qquad \hat{\mathcal{S}}_0^\beta = \hat{\mathcal{S}}_\beta.$$

Libera [12] also proved some coefficient bounds for functions in the function class $\hat{\mathcal{S}}_\alpha^\beta$, which we recall here as Theorem B below.

Theorem B. (see Libera [12]) *If the function $f \in \hat{\mathcal{S}}_\alpha^\beta$ is given by* (1), *then*

$$|a_n| \leqq \prod_{j=0}^{n-2}\left(\frac{|2(1-\alpha)e^{-i\beta}\cos\beta + j|}{j+1}\right) \qquad (n \in \mathbb{N}\setminus\{1\};\ \mathbb{N} := \{1,2,3,\cdots\}). \tag{5}$$

The coefficient estimates in (5) *are sharp.*

In this sequel to many recent developments stemming from the aforementioned and other related works (see, for example, [1] to [8], [15], [17], [18], [20], [21] and [22]; see also [11] and [19]), we introduce a substantially general subclass of the normalized analytic function class $\mathcal{A}$.

We first recall the principle of differential subordination between two analytic functions. Indeed, for two functions $f(z)$ and $g(z)$, which are analytic in $\mathbb{U}$, we say that the function $f(z)$ is *subordinate* to $g(z)$ in $\mathbb{U}$, and write (see, for details, [13])

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $\mathfrak{w}(z)$, which is analytic in $\mathbb{U}$ with

$$\mathfrak{w}(0) = 0 \qquad \text{and} \qquad |\mathfrak{w}(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(\mathfrak{w}(z)\big) \qquad (z \in \mathbb{U}).$$

In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, then the above subordination is equivalent to the following conditions:

$$f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 3. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}^{\beta}(A, B)$ if it satisfies the following subordination condition:

$$(1+\tan\beta)\left(\frac{zf'(z)}{f(z)}\right) - i\tan\beta \prec \frac{1+Az}{1+Bz} \qquad \left(-1 \leqq B < A \leqq 1;\ z \in \mathbb{U};\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}\right).$$

Remark 1. There are many choices of the parameters A and B which would provide interesting subclasses of analytic functions. For example, by setting

$$A = 1 - 2\alpha \quad (0 \leqq \alpha < 1) \qquad \text{and} \qquad B = -1$$

in Definition 3, we easily observe that the general function class $\mathcal{S}^{\beta}(A, B)$ becomes the function class $\hat{\mathcal{S}}^{\beta}_{\alpha}$, which is involved in Definition 2 and Theorem B above.

After the proof of the celebrated Bieberbach conjecture [9] (that is, the de Branges theorem [10]) on the sharp coefficient bounds for functions in the univalent function class $\mathcal{S}$, the coefficient estimate problem for various other interesting subclasses of the normalized analytic function class $\mathcal{A}$ has aroused great interest. We choose to recall here the investigations by (for example) Altintaş *et al.* ([1] to [8]), Silverman ([15]), Srivastava *et al.* ([17] and [18]), Xu *et al.* ([20], [21] and [22]) and other authors (see also [11] and [19]). In our present investigation, we obtain some sharp coefficient bounds for functions in the class $\mathcal{S}^{\beta}(A, B)$. Our main result would extend the corresponding result obtained earlier by Libera [12].

2. A Set of Sharp Coefficient Bounds

The following lemma will be required in our investigation.

Lemma. *Let the parameters A, B and β, as well as the integer n, constrained by $-1 \leqq B < A \leqq 1$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ and $n \in \mathbb{N} \setminus \{1\}$ be fixed. Suppose also that*

$$[A - (n-1)B]^2 \cos^2\beta + (n-2)^2(B^2 \sin^2\beta - 1) \geqq 0. \tag{6}$$

Then

$$\begin{aligned} &\frac{1}{(n-1)^2}\left[(A-B)^2\cos^2\beta + \sum_{k=2}^{n-1} \left|(A-kB)^2\cos^2\beta + (k-1)^2(B^2\sin^2\beta - 1)\right| \right. \\ &\qquad \left. \cdot \prod_{j=0}^{k-2}\left(\frac{|(A-B)e^{-i\beta}\cos\beta - jB|^2}{(j+1)^2}\right)\right] \\ &= \prod_{j=0}^{n-2}\left(\frac{|(A-B)e^{-i\beta}\cos\beta - jB|^2}{(j+1)^2}\right). \end{aligned} \tag{7}$$

Proof. We make use of the principle of mathematical induction on $m \in \mathbb{N}$. Indeed, for $n = 2$, the assertion (5) of the above Lemma holds true trivially. If we assume (for $n \in \mathbb{N} \setminus \{1, 2\}$) that

$$\begin{aligned} &\frac{1}{(n-2)^2}\left[(A-B)^2\cos^2\beta + \sum_{k=2}^{n-2} |(A-kB)^2\cos^2\beta + (k-1)^2(B^2\sin^2\beta - 1)| \right. \\ &\qquad \left. \cdot \prod_{j=0}^{k-2}\left(\frac{|(A-B)e^{-i\beta}\cos\beta - jB|^2}{(j+1)^2}\right)\right] \\ &= \prod_{j=0}^{n-3}\left(\frac{|(A-B)e^{-i\beta}\cos\beta - jB|^2}{(j+1)^2}\right), \end{aligned}$$

then we readily observe that

$$\frac{1}{(n-1)^2}\Bigg[(A-B)^2\cos^2\beta+\sum_{k=2}^{n-1}|(A-kB)^2\cos^2\beta+(k-1)^2(B^2\sin^2\beta-1)|$$
$$\prod_{j=1}^{k-2}\left(\frac{|(A-B)e^{-i\beta}\cos\beta-jB|^2}{(j+1)^2}\right)$$
$$=\frac{1}{(n-1)^2}\Bigg[(A-B)^2\cos^2\beta+\sum_{k=2}^{n-2}|(A-kB)^2\cos^2\beta+(k-1)^2(B^2\sin^2\beta-1)|$$
$$\cdot\prod_{j=1}^{k-2}\left(\frac{|(A-B)e^{-i\beta}\cos\beta-jB|^2}{(j+1)^2}\right)+|(A-(n-1)B)^2\cos^2\beta+(n-2)^2(B^2\sin^2\beta-1)|$$
$$\cdot\prod_{j=1}^{n-3}\left(\frac{|(A-B)e^{-i\beta}\cos\beta-jB|^2}{(j+1)^2}\right)\Bigg]$$
$$=\prod_{j=1}^{n-3}\left(\frac{|(A-B)e^{-i\beta}\cos\beta-jB|^2}{(j+1)^2}\right)\cdot\left[\left(\frac{n-2}{n-1}\right)^2+\frac{[(A-(n-1)B)^2\cos^2\beta+(n-2)^2(B^2\sin^2\beta-1)]}{(n-1)^2}\right]$$
$$=\prod_{j=1}^{n-3}\left(\frac{|(A-B)e^{-i\beta}\cos\beta-jB|^2}{(j+1)^2}\right)\cdot\left(\frac{|(A-B)e^{-i\beta}\cos\beta-(n-2)B|^2}{(n-1)^2}\right)$$
$$=\prod_{j=1}^{n-2}\left(\frac{|(A-B)e^{-i\beta}\cos\beta-jB|^2}{(j+1)^2}\right),$$

which evidently completes the proof of the assertion (7) of the above Lemma by the principle of mathematical induction on $n\in\mathbb{N}\setminus\{1\}$.

Theorem. *Let* $f\in\mathcal{S}^\beta(A,B)$, $-1\leqq B<A\leqq 1$, $-\frac{\pi}{2}<\beta<\frac{\pi}{2}$ *and* $n\in\mathbb{N}\setminus\{1\}$. *Suppose also that*

$$[A-(n-1)B]^2\cos^2\beta+(n-2)^2(B^2\sin^2\beta-1)\geqq 0 \tag{8}$$

and that the function $f(z)$ *is given by* (1). *Then*

$$|a_n|\leqq\prod_{j=0}^{n-2}\left(\frac{|(A-B)e^{-i\beta}\cos\beta-jB|}{j+1}\right)\qquad(n\in\mathbb{N}\setminus\{1\}). \tag{9}$$

The coefficient estimates in (9) *are sharp.*

Proof. Since $f\in\mathcal{S}^\beta(A,B)$, there exists a Schwarz function $\mathfrak{w}(z)$, which is already introduced in Section 1, such that

$$(1+i\tan\beta)\left(\frac{zf'(z)}{f(z)}\right)-i\tan\beta=\frac{1+A\mathfrak{w}(z)}{1+B\mathfrak{w}(z)}\qquad(-1\leqq B<A\leqq 1;\ z\in\mathbb{U}).$$

Hence

$$\sum_{k=2}^{\infty}(1-k)(1+i\tan\beta)a_kz^k=\left((B-A)z+\sum_{k=2}^{\infty}[kB(1+i\tan\beta)-(Bi\tan\beta+A)]a_kz^k\right)\mathfrak{w}(z). \tag{10}$$

This last relation (10) may be written (for $n\in\mathbb{N}$) as follows:

$$\sum_{k=2}^{n}(1-k)(1+i\tan\beta)a_kz^k+\sum_{k=n+1}^{\infty}b_kz^k=\left((B-A)z+\sum_{k=2}^{n-1}[kB(1+i\tan\beta)-(Bi\tan\beta+A)]a_kz^k\right)\mathfrak{w}(z). \tag{11}$$

The second series on the left-hand side of (11) is given by

$$\sum_{k=n+1}^{\infty} b_k z^k = \sum_{k=n+1}^{\infty} (1-k)(1+i\tan\beta)a_k z^k - \left(\sum_{k=n}^{\infty}[kB(1+i\tan\beta)-(Bi\tan\beta+A)]a_k z^k\right)\mathfrak{w}(z) \qquad (n\in\mathbb{N}), \quad (12)$$

which obviously is convergent in $\mathbb{U}$.

Since, by hypothesis, $|\mathfrak{w}(z)|<1 \ (z\in\mathbb{U})$, it is not difficult to find by appealing to Parseral's theorem that

$$(n-1)^2|a_n|^2 \leqq \sum_{k=1}^{n-1}[(A-kB)^2\cos^2\beta+(k-1)^2(B^2\sin^2\beta-1)]\cdot|a_k|^2 \qquad (a_1:=1). \quad (13)$$

We now apply the principle of mathematical induction on $n\in\mathbb{N}\setminus\{1\}$. For $n=2$, it follows from (13) that

$$|a_2| \leqq (A-B)\cos\beta,$$

which is precisely the case $n=2$ of the assertion (9). Suppose now that (9) holds true for $k=n-1$ for some fixed n, that is, that the following inequality holds true:

$$|a_{n-1}| \leqq \prod_{j=0}^{n-3}\left(\frac{|(A-B)e^{-i\beta}\cos\beta-jB|}{j+1}\right) \qquad (n\in\mathbb{N}\setminus\{1\}). \quad (14)$$

Then, by using (13), (14) and the above Lemma, we deduce for $k=n$ that

$$\begin{aligned}
|a_n|^2 &\leqq \frac{1}{(n-1)^2}\sum_{k=1}^{n-1}[(A-kB)^2\cos^2\beta+(k-1)^2(B^2\sin^2\beta-1)]\cdot|a_k|^2 \\
&\leqq \frac{1}{(n-1)^2}\left[(A-B)^2\cos^2\beta+\sum_{k=2}^{n-1}|(A-kB)^2\cos^2\beta+(k-1)^2(B^2\sin^2\beta-1)||a_k|^2\right] \\
&\leqq \frac{1}{(n-1)^2}\Bigg[(A-B)^2\cos^2\beta+\sum_{k=2}^{n-1}|(A-kB)^2\cos^2\beta+(k-1)^2(B^2\sin^2\beta-1)| \\
&\qquad \cdot\prod_{j=0}^{k-2}\left(\frac{|(A-B)e^{-i\beta}\cos\beta-jB|^2}{(j+1)^2}\right)\Bigg] \\
&= \prod_{j=0}^{n-2}\left(\frac{|(A-B)e^{-i\beta}\cos\beta-jB|^2}{(j+1)^2}\right).
\end{aligned}$$

This establishes the inequality (9) asserted by the Theorem.

In order to see that the coefficient estimates asserted by the Theorem are sharp, it suffices to consider the following function:

$$f(z)=\frac{z}{(1+Bz)^{[(B-A)e^{-i\beta}\cos\beta]/B}}.$$

This completes the proof of the Theorem.

Remark 2. For $A=1-2\alpha \ (0\leqq\alpha<1)$, $B=-1$ and $n\in\mathbb{N}\setminus\{1\}$, it is readily seen from the hypothesis (6) that

$$[A-(n-1)B]^2\cos^2\beta+(n-2)^2(B^2\sin^2\beta-1)=4(1-\alpha)(n-1-\alpha)\cos^2\beta \geqq 0.$$

Remark 3. In view of Remarks 1 and 2, if we set $A=1-2\alpha \ (0\leqq\alpha<1)$ and $B=-1$ in the above Theorem, we are led easily to Theorem B.

3. Concluding Remarks and Observations

By using the principle of differential subordination between two analytic functions, we have introduced here a certain general class $\mathcal{S}^{\beta}(A,B)$ of spirallike functions in the open unit disk $\mathbb{U}$. For the Taylor-Maclaurin coefficients of functions belonging to this general class $\mathcal{S}^{\beta}(A,B)$, we have derived several sharp estimates which are asserted by the Theorem in the preceding section. The main results presented in this paper are shown to generalize those given in the earlier work of Libera [12] (see Remark 3 above).

We conclude this paper by observing that, if the condition (8) in Theorem 1 is dropped, we do not know whether or not our Theorem will remain true. This observation leads us to the following *Open Problems*.

Open Problem 1. If the function $f \in \mathcal{S}^{\beta}(A,B)$ is given by (1) and the parameters A, B and β are constrained by $-1 \leqq B < A \leqq 1$ and $-\dfrac{\pi}{2} < \beta < \dfrac{\pi}{2}$, then prove or disprove that

$$|a_n| \leqq \prod_{j=0}^{n-2} \left(\frac{|(A-B)e^{-i\beta}\cos\beta - jB|}{j+1} \right) \qquad (n \in \mathbb{N} \setminus \{1\}). \tag{15}$$

Open Problem 2. If the coefficient estimates in (15) do hold true, then prove or disprove that these estimates are sharp.

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