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# Common fixed point of $g$-approximative multivalued mapping in partially ordered metric space

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**Abstract.** In this paper, we introduce g-approximative multivalued mappings. Based on this definition, we gave some new definitions. Further, common fixed point results for g-approximative multivalued mappings satisfying generalized contractive conditions are obtained in the setup of ordered metric spaces. Our results generalize Theorems 2.6-2.9 given in ([1]).

## 1. Introduction and preliminaries

Contractive conditions play an important role in proving the existence of fixed points of single as well as multivalued mappings. One of the simplest and most useful results in the fixed point theory is the Banach-Caccioppoli contraction mapping principle [2]. This principle has been generalized in different directions in different spaces by mathematicians over the years. In 1968 Kannan [3] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point (see, e.g., [4] for a listing and comparison of many of these definitions). The concept of weak contractions in Hilbert spaces was defined by Alber and Guerre-Delabriere [5] in 1997. Weak inequalities of the above type have been used to establish fixed point results in a number of subsequent works, some of which are noted in [6].

The study of fixed points for multivalued contraction mappings using the Hausdorff metric was initiated by Nadler [7]. After this, fixed point theory has been developed further and applied to many disciplines to solve functional equations. Banach contraction principle has been extended in different directions either by using generalized contractions for multivalued mappings and hybrid pairs of single and multivalued mappings, or by using more general spaces. Dhage [8, 9] established hybrid fixed point theorems and obtained some applications of presented results. Hong and Shen [10] proved common fixed point results for generalized contractive multivalued operators in complete metric space. Also the monotone iterative technique is associated with several nonlinear problem [11]. This technique is also employed to prove the existence of fixed points for multivalued monotone operators (see, for example [12]). In [12], the problem of existence and approximation of coupled fixed points for mixed monotone multivalued operators were studied in ordered Banach spaces under the assumption that operators satisfy the condensing condition and upper demicontinuity.

Hong introduced the concepts of approximative values, comparable approximative values upper and lower comparable approximative values in [1]. These definition are very useful tool for proving existence

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of fixed point of multivalued operator in ordered metric space. Motivated by the work of [1], for a self map $g$ on a ordered metric space, we introduce $g$-approximative multivalued mappings and obtain coincidence and common fixed point results for a hybrid pair of multivalued and single valued mappings. Concepts of $g$-comparable approximative, $g$-upper comparable approximative and $g$-lower comparable approximative multivalued mappings are introduced. Employing these definitions, common fixed point results for generalized contractive multivalued mappings in the framework of ordered metric spaces are obtained. Consequently, Theorem 2.6- 2.9 in ([1]) are generalized.

Let $(X, d)$ be a metric space. For $x \in X$ and $A \subseteq X$, we denote $d(x, A) = \inf\{d(x, A) : y \in A\}$. The class of all nonempty bounded and closed subsets of $X$ is denoted by $CB(X)$. Let $H$ be the Hausdorff metric induced by the metric $d$ on $X$, that is,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

for every $A, B \in CB(X)$.

**Definition 1.1.** Let $X$ be a nonempty set. Then $(X, \leq, d)$ is called an ordered metric space iff:
(i) $d$ is a metric on $X$ and (ii) $\leq$ is a partial order on $X$.

**Definition 1.2.** Let $X$ be an ordered metric space. A mapping $g : X \to X$ is said to be (i) weakly $L-$ idempotent if $gx \leq g^2x$ for $x$ in $X$ (ii) weakly $R-$ idempotent if $g^2x \leq gx$ for $x$ in $X$. For example, a mapping $g : [0, 1] \to [0, 1]$ given by $g(x) = x^2$ is weakly $R-$ idempotent.

**Definition 1.3.** An ordered metric space is said to have a subsequential limit comparison property if for every nondecreasing sequence (nonincreasing sequence) $\{x_n\}$ in $X$ such that $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \leq x$ ($x \leq x_{n_k}$) respectively.

**Definition 1.4.** An ordered metric space is said to have a sequential limit comparison property if for every nondecreasing sequence (nonincreasing sequence) $\{x_n\}$ in $X$ such that $x_n \to x$ implies that $x_n \leq x$ ($x \leq x_n$) respectively.

Let $X$ be any nonempty set endowed with a partial order $\leq$ and let $g : X \to X$ be a given mapping. We define the set $\Delta_g \subseteq X \times X$ by

$$\Delta_g = \{(x, y) \in X \times X : gx \leq gy\}.$$

Note that for each $x \in X$, one has $(x, x) \in \Delta_g$.

**Example 1.5.** Let $X = \{0, 1, 2\}$ be endowed with usual order $\leq$ and $g$ be a self map on $X$ defined as $g0 = 0$, $g1 = 2$ and $g2 = 1$. Then the subset $\Delta_g$ of $X \times X$ is $\Delta_g = \{(0, 0), (0, 1), (0, 2), (1, 1), (2, 1), (2, 2)\}$.

**Definition 1.6.** Let $X$ be a metric space and $g : X \to X$. A subset $Y$ of $X$ is said to be $g$-approximative for some $x$ in $X$ if $Y \subset g(X)$ and the set

$$\Delta^g_Y(g(x)) = \{y \in Y : d(g(x), y) = d(Y, g(x))\}$$

is nonempty.

**Definition 1.7.** Let $X$ be a partially ordered set. A mapping $F : X \to 2^X$ (collection of all nonempty subsets of $X$ ) is said to be:

(i) $g$-approximative multivalued mapping (in short $g$- AV multivalued mapping), if $Fx$ is $g$-approximative for each $x \in X$, That is, $\Delta^g_{Fx}(g(x))$ is nonempty for each $x$ in $X$.

(ii) $g$-CAV multivalued mapping ($g$- comparable approximative multivalued mapping) if $F$ is $g$-approximative and for each $z \in X$, there exists $g(y) \in \Delta^g_{F(z)}(g(z))$ such that $gy$ is comparable to $gz$.

(iii) $g$-UCAV($g$- upper comparable approximative multivalued mapping) if $F$ is $g$-approximative and for each $z \in X$, there exists $g(y) \in \Delta^g_{F(z)}(g(z))$ such that $g(z) \leq g(y)$

(iv) $g$-LCAV($g$- lower comparable approximative multivalued mapping) if $F$ is $g$-approximative and for each $z \in X$, there exists $g(y) \in \Delta^g_{F(z)}(g(z))$ such that $g(y) \leq g(z)$.

If $F$ is a single-valued, then $g$-UCAV ($g$-LCAV) means that $Fx \geq gx$ ($Fx \leq gx$) for $x \in X$.

**Definition 1.8.** Let $g : X \longrightarrow X$ and $T : X \longrightarrow CB(X)$. A point $x$ in $X$ is said to be: (i) *fixed point of* $g$ if $g(x) = x$ : (ii) *fixed point of* $T$ if $x \in T(x)$ (iii) *coincidence point of a pair* $(g, T)$ if $gx \in Tx$ : (iv) *common fixed point of a pair* $(g, T)$ if $x = gx \in Tx$.

$F(g), C(g, T)$ and $F(g, T)$ denote set of all fixed points of $g$, set of all coincidence points of the pair $(g, T)$ and the set of all common fixed points of the pair $(g, T)$, respectively.

**Definition 1.9.** Let $f : X \longrightarrow X$, $T : X \longrightarrow CB(X)$, and $fTx \in CB(X)$. The pair $(f, T)$ is called (1)*commuting* if $Tfx = fTx$ for all $x \in X$ (2) *weakly compatible* [13] if they commute at their coincidence points, that is, $fTx = Tfx$ whenever $x \in C(f, T)$; (3) $(IT)-$ *commuting* at $x \in X$ if $fTx \subseteq Tfx$.

**Definition 1.10.** Let $T : X \longrightarrow CB(X)$. The map $f : X \longrightarrow X$, is said to be $T-$ *weakly commuting* at $x \in X$ if $f^2x \in Tfx$.

**Definition 1.11.** The map $f : X \to X$ is said to coincidently idempotent with respect to $T : X \to CB(X)$ if $f^2(x) = f(x)$ for $x$ in $C(f, T)$. The point $x$ is called point of coincident idempotency.

Now we present an example of hybrid pair $\{f, T\}$ for which $f$ is $T-$ weakly commuting at some $x \in C(f, T)$.

**Example 1.12.** Let $X = [0, \infty)$ with usual metric. Define $f : X \to X$, $T : X \to CB(X)$ by

$$fx = \begin{cases} 0, & 0 \leq x < 1 \\ x + 1, & 1 \leq x < \infty \end{cases}$$

and

$$Tx = \begin{cases} \{x\}, & 0 \leq x < 1 \\ [1, x + 2], & 1 \leq x < \infty \end{cases}.$$

It can be easily verified that $f$ is $T-$ *weakly commuting* at $x = 0 \in C(f, T)$.

**Example 1.13.** Let $X = R$ with usual metric. Define $f : X \to X$, $T : X \to CB(X)$ by

$$fx = \begin{cases} -1, & x \leq 0 \\ -\dfrac{2}{x}, & 0 < x \end{cases}$$

and

$$Tx = \begin{cases} \{x\}, & x \leq -1 \\ [x, 1], & -1 < x \leq 1 \\ [1, x], & 1 < x < \infty \end{cases}.$$

Here $C(f, T) = \{-1\}$ and $f$ is coincidently idempotent with respect to $T$.

Let $\alpha \in (0, +\infty]$. $F$ denotes the class of mappings $f : [0, \alpha) \to \mathbb{R}$ which satisfy the following conditions:

**(i)** $f(0) = 0$ and $f(t) > 0$ for each $t \in (0, \alpha)$,

**(ii)** $f$ is continuous,

**(iii)** $f$ is nondecreasing on $[0, \alpha)$.

A mapping $f$ is said to be sublinear if $f(t_1 + t_2) \leq f(t_1) + f(t_2)$, whenever $t_1, t_2, t_1 + t_2 \in (0, \alpha)$. We define $F_s = \{f : [0, \alpha) \to \mathbb{R} : f$ is sublinear and $f \in F\}$.

$\Psi$ denotes the family of mappings $\psi : [0, \alpha) \to [0, +\infty)$ which satisfy the following conditions:

(a) $\psi(t) < t$ for each $t \in (0, \alpha)$,

(b) $\psi$ is nondecreasing and right upper semi-continuous,

(c) For each $t \in (0, \alpha)$, $\lim\limits_{n\to\infty} \psi^n(t) = 0$.

By means for the functions $f$ and $\psi$ given in $F$ and $\Psi$ respectively, a generalized contractive condition was defined in [9]. Let $\Phi$ denotes the class of mappings $\psi : [0, \alpha) \to [0, +\infty)$ for which $\psi(t) < t$ and $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t$ in $(0, \alpha)$.

**Definition 1.14.** For two subset $A, B$ of $X$, we say $A \leq_1 B$ if for each $x \in X$, there exists $y \in Y$ such that $x \leq y$ and $A \leq B$ if each $x \in A$, $y \in B$ implies that $x \leq y$.

A multivalued mapping $F : X \to 2^X$ is said to be $g$-nondecreasing ($g$-nonincreasing) if $gx \leq gy$ implies that $Fx \leq_1 Fy$ ($Fy \leq_1 Fx$) for all $x, y \in X$. $F$ is said to be $g$- monotone if $F$ is $g$-nondecreasing or $g$-nonincreasing. Moreover in what follows $(X, \leq)$ will be a partially ordered set such that there exists a complete metric $d$ on $X$. Let $D = \sup\{d(x, y) : x, y \in X\}$. Set $\alpha = d$ if $d = \infty$ and $\alpha > d$ if $d < \infty$.

## 2. Common fixed point theorems

In this section we obtain common fixed point theorems.

**Theorem 2.1.** Suppose that $g$ be a nondecreasing self map on $X$ and $F : X \to 2^X$ is $g$-UCAV and the following holds

$$f(H(Fx, Fy)) \leq \psi\left(f(M_g(x, y))\right) \tag{1}$$

for any $(x, y) \in \Delta_g$, where $f \in F_s$ and $\psi \in \Phi$ and

$$M_g(x, y) = \max\left\{d(gx, gy), d(gx, Fx), d(gy, Fy), \frac{d(gx, Fy) + d(gy, Fx)}{2}\right\}.$$

If $X$ has a limit comparison property and $g(X)$ is closed, then $F$ and $g$ have a coincidence point $x$ in $X$. Moreover $F$ and $g$ have common fixed point if one of the following conditions holds:

**(i)** Pair $(F, g)$ is $IT-$ commuting at some $x \in C(F, g)$ and $\lim\limits_{n\to\infty} g^n x = u$, for some $u \in X$ and $g$ is continuous at $u$.

**(ii)** Pair $(F, g)$ is $IT-$ commuting at some $x \in C(F, g)$ and $g^2 x = gx$.

**(iii)** $g$ is $F-$ weakly commuting at some $C(F, g)$ and $g$ is coincidently idempotent with respect to $T$.

**(iv)** $g$ is continuous at $x$ for some $x \in C(F, g)$ for some $u \in X$; $\lim g^n u = x$.

**(v)** $g(C(g, F))$ is singleton subset of $C(g, F)$.

*Proof.* Let $x_0 \in X$. If $gx_0 \in Fx_0$, then the result is proved. If not, then we proceed as follows: As $F$ is $g$-UCAV, $Fx_0 \subset g(X)$, $\Delta^g_{F(x_0)}(g(x_0))$ is nonempty so there exists $gx_1 \in Fx_0$ with $gx_1 \neq gx_0$ such that $d(gx_1, gx_0) = d(Fx_0, gx_0)$ for some $x_1 \in X$ and $gx_1 \geq gx_0$. Similarly, there exists $gx_2 \in Fx_1$ with $gx_1 \neq gx_2$ such that

$d(gx_1, gx_2) = d(Fx_1, gx_1)$ for some $x_2 \in X$, and $gx_2 \geq gx_1$. We continue to construct a sequence $\{x_n\}$ for which either $gx_{n-1} \in Fx_{n-1}$ or there exists $gx_n \in Fx_{n-1}$ with $gx_n \neq gx_{n-1}$ and $gx_n \geq gx_{n-1}$ such that

$$d(gx_n, gx_{n-1}) = d(Fx_{n-1}, gx_{n-1}), \text{ for } n = 1, 2, \cdots \tag{2}$$

for some $x_n$ in $X$. On the other hand,

$$d(Fx_{n-1}, gx_{n-1}) = \sup_{x \in Fx_{n-2}} d(x, Fx_{n-1}) \leq H(Fx_{n-1}, Fx_{n-2}), \tag{3}$$

implies that

$$d(gx_n, gx_{n-1}) \leq H(Fx_{n-1}, Fx_{n-2}), \text{for } n = 2, 3, \cdots . \tag{4}$$

Since $f$ is nondecreasing, then we have

$$\begin{aligned} f(d(gx_n, gx_{n-1})) &\leq f(H(Fx_{n-1}, Fx_{n-2})) \\ &\leq \psi(f(M_g(x_{n-1}, x_{n-2}))), \end{aligned}$$

in which

$$\begin{aligned} &M_g(x_{n-1}, x_{n-2}) \\ &= \max\left\{d(gx_{n-1}, gx_{n-2}), d(Fx_{n-1}, gx_{n-1}), d(Fx_{n-2}, gx_{n-2}), \frac{d(Fx_{n-2}, gx_{n-1}) + d(Fx_{n-1}, gx_{n-2})}{2}\right\} \\ &= \max\left\{d(gx_{n-1}, gx_{n-2}), d(gx_n, gx_{n-1}), d(gx_{n-1}, gx_{n-2}), \frac{d(gx_{n-1}, gx_{n-1}) + d(gx_n, gx_{n-2})}{2}\right\} \\ &= \max\left\{d(gx_{n-1}, gx_{n-2}), d(gx_n, gx_{n-1})\right\}. \end{aligned}$$

If $d(gx_{n-1}, gx_n) > d(gx_{n-1}, gx_{n-2})$, then we have

$$\begin{aligned} f(d(gx_n, gx_{n-1})) &\leq f(H(Fx_{n-1}, Fx_{n-2})) \\ &\leq \psi(f(M_g(x_{n-1}, x_{n-2}))) \\ &\leq \psi(f(\max\{d(gx_{n-1}, gx_{n-2}), d(gx_{n-1}, gx_n)\})) \\ &\leq \psi(f(d(gx_n, gx_{n-1}))) \\ &< f(d(gx_n, gx_{n-1})), \end{aligned}$$

a contraction. So we have $d(gx_{n-1}, gx_{n-2}) \geq d(gx_{n-1}, gx_n)$. This yields

$$f(d(gx_n, gx_{n-1})) \leq \psi(f(d(gx_{n-1}, gx_{n-2}))).$$

Repeating this process, we have

$$\begin{aligned} f(d(gx_n, gx_{n-1})) &\leq \psi(f(d(gx_{n-1}, gx_{n-2}))) \\ &\leq \psi^2(f(d(gx_{n-2}, gx_{n-3}))) \\ &\ \ \vdots \\ &\leq \psi^{n-1}(f(dgx_0, gx_1)). \end{aligned}$$

For $m, n \in \mathbb{N}, n > m$, we obtain

$$d(gx_n, gx_m) \leq \sum_{i=m}^{n-1} d(gx_i, gx_{i+1}).$$

This implies

$$\begin{aligned} f(d(gx_n, gx_m)) &\leq f(d(gx_n, gx_{n-1}) + \cdots + d(gx_{m+1}, gx_m)) \\ &\leq f(d(gx_n, gx_{n-1})) + \cdots + f(d(gx_{m+1}, gx_m)) \\ &\leq \psi^{n-1}(f(d(gx_0, gx_1))) + \cdots + \psi^m(f(d(gx_0, gx_1))) \\ &\leq \sum_{i=m}^{n-1} \psi^i(f(d(gx_0, gx_1))). \end{aligned}$$

On taking limit as $n, m \to \infty$ and using $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, it follows that $\{g(x_n)\}$ is Cauchy sequence in $X$. Since $X$ is complete and $g(X)$ is closed so we have $\lim_{n\to\infty} gx_n = gx$ for some $x$ in $X$. Now we prove that $d(Fx, gx) = 0$. Suppose that this is not true, then $d(Fx, gx) > 0$. For large enough $n$, we claim that the following equation holds

$$\begin{aligned} M_g(x, x_{n+1}) &= \max\left\{d(gx, gx_{n+1}), d(Fx, gx), d(Fx_{n+1}, gx_{n+1}), \frac{d(Fx, gx_{n+1}) + d(Fx_{n+1}, gx)}{2}\right\} \\ &= d(Fx, gx). \end{aligned}$$

Indeed, since $\lim_{n\to\infty} gx_n = gx$ and $\lim_{n\to\infty} d(Fx_{n+1}, gx_{n+1}) = 0$, it follows that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{2}[d(Fx, gx_{n+1}) + d(Fx_{n+1}, gx)] \\ \leq\ &\lim_{n\to\infty} \frac{1}{2}[d(Fx, gx) + d(gx, gx_{n+1}) + d(Fx_{n+1}, gx_{n+1}) + d(gx_{n+1}, gx)] \\ =\ &\frac{1}{2} d(Fx, gx). \end{aligned}$$

So there exists $n_0 \in \mathbb{N}$ such that $M_g(x, x_{n+1}) = d(Fx, gx)$ for every $n > n_0$. Note that

$$f(d(Fx, gx_{n+2})) \leq f(H(Fx, Fx_{n+1})) \leq \psi(f(M_g(x, x_{n+1}))),$$

which on taking limit as $n \to \infty$ gives

$$f(d(Fx, gx)) \leq \psi(f(d(Fx, gx))) < f(d(Fx, gx)),$$

a contradiction. So $d(Fx, gx) = 0$, we have $gx \in Fx$. Suppose now that (i) holds. Then $\lim_{n\to\infty} g^n x = u$ where $u \in X$. Since $g$ is continuous at $u$, so we have that $u$ is fixed points of $g$. By given assumption, $g^n x \in C(F, g^{n-1})$ for all $n \geq 1$ and $g^n x \in F(g^{n-1}x)$. Now we prove that $d(Fu, gu) = 0$. Suppose that this is not true, then $d(Fu, gu) > 0$. Using (1), since $f$ is nondecreasing and sublinear, we obtain,

$$\begin{aligned} f(d(gu, Fu)) &\leq f(d(gu, g^n x)) + f(d(g^n x, Fu)) \\ &\leq f(d(gu, g^n x)) + f(H(F(g^{n-1}x), F(u))) \\ &\leq f(d(gu, g^n x)) + \psi(f(M_g(g^{n-1}x, u))). \end{aligned} \tag{5}$$

Where

$$\begin{aligned} M_g(g^{n-1}x, u) &= \max\left\{d(g^n x, gu), d(Fg^{n-1}x, g^n x), d(Fu, gu), \frac{d(Fu, g^n x) + d(Fg^{n-1}, gu)}{2}\right\} \\ &= \max\left\{d(g^n x, gu), d(g^n x, g^n x), d(Fu, gu), \frac{d(Fu, g^n x) + d(g^n x, gu)}{2}\right\}. \end{aligned}$$

On taking limit as $n \to \infty$, we have

$$M_g(g^{n-1}x, u) = d(Fu, gu)$$

which further implies

$$\begin{aligned} f(d(gu, Fu)) &\leq f(d(gu, g^n x)) + \psi(f(d(Fu, gu))) \\ &< f(d(gu, g^n x)) + f(d(Fu, gu)) \end{aligned}$$

On taking limit as $n \to \infty$,

$$f(d(gu, Fu)) < f(d(Fu, gu)) \tag{4}$$

a contradiction, so $d(gu, Fu) = 0$ and hence $gu \in Fu$. Consequently $u = gu \in Fu$. Hence $u$ is a common fixed point of $F$ and $g$. Suppose now that (ii) holds. As $x \in C(F, g)$, so $g^2 x \in gFx \subset Fgx$. Now $gx = g^2 x \in Fgx$ implies that that $gx$ is a common fixed point of $F$ and $g$. Suppose now that (iii) holds. The result is obvious. Suppose that (iv ) holds. As $x \in C(g, F)$ and for some $u \in X$, $\lim_{n\to\infty} g^n u = x$. By the continuity of $g$ at $x$, we get $x = gx \in Fx$. Hence $x$ is common fixed point of $F$ and $g$. Finally, suppose that (v) holds. Let $g(C(F, g)) = \{x\}$. Then $\{x\} = \{gx\} = Fx$. Hence $x$ is common fixed point of $F$ and $g$. □

Similarly, we have following theorem.

**Theorem 2.2.** Suppose that $g$ be a nondecreasing self map on $X$ and $F : X \to 2^X$ is $g$-LCAV and the following holds

$$f(H(Fx, Fy)) \leq \psi\left(f(M_g(x, y))\right)$$

for any $(x, y) \in \Delta_g$, where $f \in F_s$ and $\psi \in \Phi$ and

$$M_g(x, y) = \max\left\{d(gx, gy), d(gx, Fx), d(gy, Fy), \frac{d(gx, Fy) + d(gy, Fx)}{2}\right\}$$

If $X$ has sequential limit comparison property and $g(X)$ is closed, then $F$ and $g$ have a coincidence point $x$ in $X$. Moreover $F$ and $g$ have common fixed point if any one of conditions (i)-(v) holds as in Theorem 2.1.

**Example 2.3.** Let $X = \{0\} \cup [1, \infty)$ with usual metric. Define $g : X \to X$, $F : X \to 2^X$ by

$$gx = \begin{cases} 0, & x = 0 \\ x + 1, & 1 \leq x < \infty \end{cases}$$

and

$$Fx = \begin{cases} \{x\}, & x = 0 \\ [1, x + 2], & 1 \leq x < \infty \end{cases}.$$

We can see that function of $F$ and $g$ are satisfy condition of Theorem 2. It is clear that $F$ is $g$-UCAV, also $g(X)$ is closed and $X$ has a property of limit comparison. we can see easly that $g$ is $F$- weakly commuting at $x = 0$. Besides, $g$ is concidently idempotent with respect to $F$ at $x = 0$. In this case, These functions satisfy condition of (iii) in Theorem 2.1. Also we can define $f(t) = t, \psi(t) = \frac{t}{2}$, then $f \in F_s$ and $\psi \in \Psi$. If $x = y = 0$, we have $gx = gy = 0$ and $Fx = \{x\}$, $Fy = \{y\}$

$$\begin{aligned} f(H(Fx, Fy)) &= H(Fx, Fy) = \max\left\{\sup_{z\in\{x\}}\{d(z, \{y\})\}, \sup_{t\in\{y\}}\{d(\{x\}, t)\}\right\} \\ &= \max\left\{\sup_{z\in\{0\}} \inf_{t\in\{0\}} d(z, t), \sup_{p\in\{0\}} \inf_{k\in\{0\}} d(p, k)\right\} = 0 \\ &= \frac{\max\left\{d(gx, gy), d(gx, Fx), d(gy, Fy), \frac{d(gx, Fy) + d(gy, Fx)}{2}\right\}}{2} \\ &= \psi(f(M_g(x, y))). \end{aligned}$$

if $x = 0, y \in [1, \infty)$, we have $gx = 0, gy = y + 1$ and $Fx = \{x\}, Fy = [1, y + 2]$

$$\begin{aligned} f(H(Fx, Fy)) &= H(Fx, Fy) = \max\left\{\sup_{z\in\{0\}} d(z, [1, y + 2]), \sup_{t\in[1,y+2]} d(\{0\}, t)\right\} \\ &= 1 \end{aligned}$$

also since $x < y$ then we have

$$M_g(x, y) = \max\left\{y + 1, x, 1, \frac{y - x}{2}\right\} = y + 1.$$

So we satisfy contractive condition. Finally, If $x, y \in [1, \infty)$, we have $gx = x + 1, gy = y + 1$ and $Fx = [1, x + 2]$, $Fy = [1, y + 2]$ and we can see easly that the contractive condition is satified. Hence, satisfy all condition of Theorem 2.1. It is clear that $0 = x = gx \in Fx$ that is, $x = 0$ is common fixed point of $F$ and $g$.

**Corollary 2.4.** Suppose that $g$ be a nondecreasing self map on $X$ and $F : X \to X$ and $g : X \to X$ are self mappings which satisfy

$$f(d(Fx, Fy)) \leq \psi(f(M_g(x, y)))$$

for any $(x, y) \in \Delta_g$, where $f \in F_s$, $\psi \in \Phi$ and

$$M_g(x, y) = \max\left\{d(gx, gy), d(Fx, gx), d(Fy, gy), \frac{d(Fx, gy) + d(Fy, gx)}{2}\right\}.$$

Then $F, g$ have a unique coincidence point $x \in X$. Moreover $F$ and $g$ have unique common fixed point if any one of conditions (i)-(v) holds as in Theorem 2.1.

*Proof.* Theorem 2.1 ensures the existence of coincidence point. To prove the uniqueness, let $y$ be another coincidence point of $F$ and $g$. If $x \neq y$, then $d(gx, gy) > 0$. Thus,

$$M_g(x, y) = \max\left\{d(gx, gy), d(Fx, gx), d(Fy, gy), \frac{d(Fx, gy) + d(Fy, gx)}{2}\right\} = d(gx, gy).$$

This yields

$$\begin{aligned} f(d(gx, gy)) &= f(d(Fx, Fy)) \leq \psi(f(M_g(x, y))) \\ &= \psi(f(d(gx, gy))) \\ &< f(d(gx, gy)), \end{aligned}$$

a contradiction, therefore $d(gx, gy) = 0$. The results follows. □

**Theorem 2.5.** Suppose that $g$ be a nondecreasing self map on $X$ and $F : X \to 2^X$ is $g$-AV and the following holds

$$f(H(Fx, Fy)) \leq \psi\left(f(M_g(x, y))\right)$$

for any $(x, y) \in \Delta_g$, where $f \in F_s$ and $\psi \in \Phi$ and

$$M_g(x, y) = \max\left\{d(gx, gy), d(gx, Fx), d(gy, Fy), \frac{d(gx, Fy) + d(gy, Fx)}{2}\right\}.$$

If $g(X)$ is closed and there exists $x_0 \in X$ such that $\{gx_0\} \leq Fx_0$, then $F$ and $g$ have a coincidence point $x \in X$. Further, an iterative sequence $\{gx_n\}$ with $gx_n \in Fx_{n-1}$ converges to $gx$, where $x \in C(F, g)$. Moreover $F$ and $g$ have common fixed point if any one of conditions (i)-(v) holds as in Theorem 2.1.

*Proof.* If $gx_0 \in Fx_0$, then the proof is finished. Otherwise, for any $gx \in Fx_0$ one has $gx \geq gx_0$. As $F$ has $g$-approximative mutlivalued map, for $x_1 \in X$, there exists $gx_1 \in Fx_0$ with $gx_1 \geq gx_0$ and

$$d(gx_0, gx_1) = d(Fx_0, gx_0).$$

Similarly, for $x_2 \in X$, there exists $gx_2 \in Fx_1$ with $gx_2 \geq gx_1$ and

$$d(gx_1, gx_2) = d(Fx_1, gx_1).$$

We continue the process of constructing a sequence $\{gx_n\}$ such that for $x_n \in X$, one obtaines $gx_n \in Fx_{n-1}$ with $gx_n \geq gx_{n-1}$ such that

$$d(gx_{n-1}, gx_n) = d(Fx_{n-1}, gx_{n-1}) \quad n = 1, 2, \cdots .$$

On the other hand, we have

$$d(Fx_{n-1}, gx_{n-1}) = \sup_{x \in Fx_{n-2}} d(x, Fx_{n-1}) \leq H(Fx_{n-2}, Fx_{n-1}),$$

So,

$$d(gx_{n-1}, gx_n) \leq H(Fx_{n-2}, Fx_{n-1}) \quad \text{for } n = 2, 3, \cdots .$$

The rest of this proof is the same as that of Theorem 2.1. □

**Theorem 2.6.** Suppose that $g$ be a nondecreasing self map on $X$, $F : X \to 2^X$ is $g$-CAV and the following holds

$$f(H(Fx, Fy)) \leq \psi\left(f(M_g(x, y))\right)$$

for any $(x, y) \in \Delta_g$, where $f \in F_s$ and $\psi \in \Phi$ and

$$M_g(x, y) = \max\left\{d(gx, gy), d(gx, Fx), d(gy, Fy), \frac{d(gx, Fy) + d(gy, Fx)}{2}\right\}.$$

If $X$ has a subsequential limit comparison property and $g(X)$ is closed, then $F$ and $g$ have coincidence point. Moreover $F$ and $g$ have common fixed point if any one of conditions (i)-(v) holds as in Theorem 2.1.

*Proof.* Following similar arguments to those given in Theorem 2, and $F$ is $g$-CAV, we obtain a sequence $\{gx_n\}$ whose consecutive terms are comparable, satisfy (2) and (4) and following hold:

$$gx_{n+1} \in Fx_n, \lim_{n\to\infty} gx_n = gx.$$

Since $X$ has subsequential limit comparison property so $\{gx_n\}$ has subsequence $\{gx_{n_k}\}$ whose every term is comparable to $gx$. Now we prove $gx \in Fx$. Obviously,

$$\begin{aligned} d(gx_{n_k+2}, Fx) &\leq d(gx_{n_k+2}, gx_{n_k+1}) + d(gx_{n_k+1}, Fx) \\ &\leq d(gx_{n_k+2}, gx_{n_k+1}) + \sup_{t \in Fx_{n_k}} d(t, Fx) \\ &\leq d(gx_{n_k+2}, gx_{n_k+1}) + H(Fx_{n_k}, Fx) \end{aligned}$$

for $k = 0, 1, 2, \cdots$ . For $\varepsilon > 0$, there exists $k_0$ such that

$$f(d(gx_{n_k+2}, gx_{n_k+1})) < \varepsilon$$

for all $k > k_0$. As

$$\lim_{k\to\infty} f(d(gx_{n_k+2}, gx_{n_k+1})) = 0.$$

Since $gx_{n_k}$ is comparable to $gx$ for each $k$, therefore

$$\begin{aligned} f(d(gx_{n_k+2}, Fx)) &\leq f(d(gx_{n_k+2}, gx_{n_k+1}) + H(Fx_{n_k}, Fx)) \\ &\leq f(d(gx_{n_k+2}, gx_{n_k+1})) + f(H(Fx_{n_k}, Fx)) \\ &\leq \psi(f(M_g(x_{n_k}, x))) + \varepsilon. \\ &< f(M_g(x_{n_k}, x))) + \varepsilon \end{aligned}$$

Note that $f$ is continuous and $\lim\limits_{k\to\infty} d(gx_{n_k}, Fx) = d(gx, Fx)$, we obtain by letting $k \to \infty$,

$$f(d(gx, Fx)) < f(d(gx, Fx)) + \varepsilon.$$

This implies that $d(gx, Fx) = 0$, so we have $gx \in Fx$. By the similar arguments in Theorem 2, we can show the existence of a common fixed point. □

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