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A solution to the inverse problem for the Sturm-Liouville-type equation with a delay

Milenko Pikula[a], Vladimir Vladičić[b], Olivera Marković[c]

[a]*Faculty of Philosophy, University of East Sarajevo, Bosnia and Herzegovina and Business and Technical High School, Užice, Serbia*
[b]*Faculty of Philosophy, University of East Sarajevo, Bosnia and Herzegovina*
[c]*Teacher Training Faculty, Užice, University of Kragujevac, Serbia*

Abstract. The paper is devoted to study of the inverse problem of the boundary spectral assignment of the Sturm-Liouville with a delay.

$$-y''(x) + q(x)y(\alpha \cdot x) = \lambda y(x), \quad q \in AC[0,\pi],\ \alpha \in (0,1] \quad (1)$$

with separated boundary conditions:

$$y(0) = y(\pi) = 0 \quad (2)$$

$$y(0) = y'(\pi) = 0 \quad (3)$$

It is argued that if the sequence of eigenvalues is given $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ tasks (1-2) and (1-3) respectively, then the delay factor $\alpha \in (0,1)$ and the potential $q \in AC[0,\pi]$ are unambiguous. The potential q is composed by means of trigonometric Fourier coefficients. The method can be easily transferred to the case of $\alpha = 1$ i.e. to the classical Sturm-Liouville problem.

1. Introduction

Inverse problems in the spectral theory of operators, especially differential operators, have been studied since the 1930s until now. The monographs [2, 5] deal with this topic. A separate chapter of this study deals with the inverse tasks for the boundary problems of the generated equations with a delay. Papers [1, 3, 14, 15] are latest results in this field. Papers [4, 6, 7, 8, 9, 10, 11, 12, 13] are devoted to this issue. Results in these papers are obtained by solving the integral equation of Fredholm type. In order to make the solution unique, strict conditions for the given parameters are imposed. In this paper, a different approach is used and by means of the Fourier analysis method, a new solution to the inverse task has been found, which improves previous solutions.

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Email addresses: mpikula@paleol.net (Milenko Pikula), vvladicic@ffuis.edu.ba (Vladimir Vladičić), markovic@ucfu.kg.ac.rs (Olivera Marković)

2. Defining the task

We look at the boundary value problem

$$-y''(x) + q(x)y(\alpha \cdot x) = \lambda y(x), \quad q \in AC[0,\pi], \ \alpha \in (0,1) \tag{1}$$

with separated boundary conditions:

$$y(0) = y(\pi) = 0 \tag{2}$$

When $\alpha = 1$, the boundary value problem (1-2) is classical and as such it was studied in the mid-twentieth century.

The coefficient α and the function q are called the parameters of the boundary task (1-2). The question is: Which spectral characteristics of task (1-2) uniquely define the parameters α and q? This paper gives an answer to that question.

3. Solution to the direct spectral task

Equation (1) with the boundary condition $y(0) = 0$ is equivalent to the integral equation:

$$y(x,z) = \sin zx + \frac{1}{z}\int_0^x q(t)\sin z(x-t)y(\alpha t, z)dt; \quad z^2 = \lambda \tag{3}$$

Let us introduce labels

$$\int_{D_x^l} = \int_0^x \int_0^{\alpha t_1} \dots \int_0^{\alpha t_{l-1}}, \ Q(T_l) = \prod_{i=1}^{l} q(t_i), \ dT_l = \prod_{i=1}^{l} dt_i,$$

$$S_l(T_l) = \prod_{i=1}^{l-1} \sin z(\alpha t_i - t_{i+1})$$

Using the method of successive approximations to the equation (3) we obtain the solution in the form of

$$y(x,z) = \sin zx + \frac{1}{z}\int_0^x q(t)\sin z(x-t)\sin z\alpha t dt + \sum_{l=2}^{\infty}\frac{1}{z^l}\int_{D_x^l} Q(T_l)\sin z(x-t_1)S_l(T_l)\sin z\alpha t_l dT_l \tag{4}$$

Lemma 3.1. *Function (4) is an entire function of variable z, $\forall x \in [0,\pi]$.*

Proof. Since $\lim_{z\to 0} y(x,z) = 0$, $\forall x \in [0,\pi]$, point $z = 0$ is the apparent singularity. Members

$$u_l(x,z) = \frac{1}{z^l}\int_{D_l(x)} Q_l(T_l)\sin z(x-t_1)S_l(T_l)\sin z\alpha t_l dT_l$$

$l = 2,3,\dots$, of the series (4) are entire functions in C. □

Let us prove that the series (4) has uniform convergence on C, for each fixed $x \in [0,\pi]$.

Since $Q_l(T_l) = q(t_1)q(t_2)\cdots q(t_l)$, the equality is $Q_l(t_1,t_2,\dots,t_l) = Q_l(t_{k_1}t_{k_2}\cdots t_{k_l})$, where $(k_1,\cdots,k_l)$ is an arbitrary permutation of the set $\{1,2,\cdots,n\}$. From the theory of integral it is true that

$$\left|\int_0^x\int_0^{t_1}\dots\int_0^{t_{l-1}} Q(T_l)dT_l\right| = \left|\frac{1}{l!}\int_0^x\int_0^x\dots\int_0^x Q(T_l)dT_l\right| \le \frac{\|q\|_{L_1[0,\pi]}^l}{l!}.$$

It is easy to check the inequality

$$|\sin z(x-t_1)S_l(T_l)\sin z\alpha t_l| \le e^{x|Im(z)|}, \ x \in [0,\pi].$$

Therefore, in each ring $\delta \le |z| \le \Delta,\ 0 < \delta < \Delta < \infty$ it is true that

$$|u_l(x,z)| \le e^{\pi|Im(z)|} \cdot \left(\frac{||q||}{\delta}\right)^l \cdot \frac{1}{l!} \le \left(\frac{||q||}{\delta}\right)^l \frac{e^{\pi\Delta}}{l!}.$$

So, series (4) converges uniformly on each ring, and hence on C.

For $x = \pi$ from (4) we get the characteristic function

$$\begin{aligned} F(z,\alpha) &= \sin\pi z + \frac{1}{z}\int_0^\pi q(t)\sin z(\pi-t)\sin z\alpha t dt \\ &+ \frac{1}{z^2}\int_0^\pi\int_0^{\alpha t_1} q(t_1)q(t_2)\sin z(\pi-t_1)\sin z(\alpha t_1 - t_2)\sin z\alpha t_2 dt_2 dt_1 \\ &+ \sum_{l=3}^{\infty}\frac{1}{z^l}\int_{D_\pi^l} Q(T_l)\sin z(\pi-t_1)S_l(T_l)\sin z\alpha t_l dT_l \end{aligned} \tag{5}$$

Function F is an entire function of the complex variable z according to Lemma 3.1.

Theorem 3.2. *If $q \in AC[0,\pi]$ and $q' \in L_2[0,\pi]$ zeros $z_n(\alpha)$, $n \in N$ of function (5) have an asymptotic shape*

$$z_n(\alpha) = \pm\left(n + \frac{C_1(n,\alpha)}{n^2} + o\left(\frac{C_1(n,\alpha)}{n^2}\right)\right),\quad n \to \infty \tag{6}$$

where

$$C_1(n,\alpha) = \frac{q(\pi)}{\pi(1-\alpha^2)}(-1)^{n+1}\sin n\alpha\pi + \frac{(-1)^n}{\pi}O\left(\frac{1}{n^s}\right),\quad s > \frac{1}{2} \tag{7}$$

Proof. We use the elementary relation

$$I_1(z,\alpha) = \int_0^\pi q(t)\sin z(\pi-t)\sin z\alpha t dt = \frac{1}{2}\int_0^\pi q(t)\cos z(\pi-(1+\alpha)t)dt - \frac{1}{2}\int_0^\pi q(t)\cos z(\pi-(1-\alpha)t)dt.$$

We perform a partial integration and we get

$$\begin{aligned} \int_0^\pi q(t)\sin z(\pi-t)\sin z\alpha t dt &= \frac{1}{z^2}(\xi_1(\alpha)\sin z\alpha\pi - \xi_2(\alpha)\sin z\pi) \\ &-\frac{1}{2z^2}\int_0^\pi\left[\frac{q'(t)}{1+\alpha}\sin z(\pi-(1+\alpha)t) - \frac{q'(t)}{1-\alpha}\sin z(\pi-(1-\alpha)t)\right]dt \end{aligned} \tag{8}$$

where

$$\xi_1(\alpha) = \frac{q(\pi)}{1-\alpha^2},\quad \xi_2(\alpha) = \frac{\alpha q(0)}{1-\alpha^2} \tag{9}$$

We will use the label

$$I_2(z,q',\alpha) = \int_0^\pi\left[\frac{q'(t)}{1+\alpha}\sin z(\pi-(1+\alpha)t) - \frac{q'(t)}{1-\alpha}\sin z(\pi-(1-\alpha)t)\right]dt \tag{10}$$

If we write (5) in the form of

$$F(z,\alpha) = \sin\pi z + \frac{1}{z}I_1(z,\alpha) + \Delta(z,\alpha),$$

it is easy to check that

$$\Delta(z,\alpha) = O\left(\frac{1}{z^4}\right)\quad Re(z) \to \infty \tag{11}$$

Using (8), (10) and (11) we obtain

$$F(z,\alpha)=\sin\pi z+\frac{1}{z^2}(\xi_1(\alpha)\sin z\alpha\pi-\xi_2(\alpha)\sin\pi z)-\frac{1}{2z^2}I_2(z,q',\alpha)+O\left(\frac{1}{z^4}\right),\quad Re(z)\to\infty \tag{$*$}$$

Since $q'\in L_2[0,\pi]$ is, it is true that $I_2(z,q',\alpha)=O(\frac{1}{n^s})$, $s>\frac{1}{2}$. The function F is odd, and from $F(z_n)=0$ it follows that $F(-z_n)=0$. Suppose the zeros z_n of function F have an asymptotic shape

$$z_n(\alpha)=n+\frac{C_1(n,\alpha)}{n^2}+o\left(\frac{C_1(n,\alpha)}{n^2}\right),\quad n\to\infty \tag{12}$$

If we include $\sin\pi z_n=(-1)^n\frac{\pi C_1}{n^2}+o\left(\frac{C_1}{n^2}\right)$ and $\sin\alpha\pi z_n=\sin n\alpha\pi+O\left(\frac{1}{n^2}\right)$ in $(*)$ we will get

$$0=F(z_n,\alpha)=\frac{1}{n^2}\left[(-1)^n\pi C_1(n,\alpha)+\xi_1(\alpha)\sin n\alpha\pi-\frac{1}{2}I_2(n,q',\alpha)\right]+o\left(\frac{I_2(n,q',\alpha)}{n^2}\right),\quad n\to\infty$$

Consequently,

$$C_1(n,\alpha)=\frac{\xi_1(\alpha)}{\pi}(-1)^{n+1}\sin n\alpha\pi+\frac{(-1)^n}{2\pi}I_2(n,q',\alpha) \tag{13}$$

This completes the proof. □

Eigenvalues $\lambda_n(\alpha)$ of the task (1-2) are squares of zeros function $F(z,\alpha)$. It is therefore

$$\lambda_n(\alpha)=n^2+\frac{1}{n}\left[\frac{2\xi_1(\alpha)}{\pi}(-1)^{n+1}\sin n\alpha\pi+\frac{(-1)^n}{\pi}I_2(n,q',\alpha)\right]+o\left(\frac{I_2(n,q',\alpha)}{n}\right) \tag{14}$$

Let us introduce the function $\widetilde{q}$ in the segment $[-\pi,\pi]$ as follows:

$$\widetilde{q}(\theta,\alpha)=\begin{cases}0, & \theta\in[-\pi,-\alpha\pi)\\ \frac{1}{(1+\alpha)^2}q'\left(\frac{\pi-\theta}{1+\alpha}\right), & \theta\in(-\alpha\pi,\alpha\pi)\\ \frac{1}{(1+\alpha)^2}q'\left(\frac{\pi-\theta}{1+\alpha}\right)-\frac{1}{(1-\alpha)^2}q'\left(\frac{\pi-\theta}{1-\alpha}\right), & \theta\in(\alpha\pi,\pi)\end{cases} \tag{15}$$

Since $q'\in L_2[0,\pi]$ is, this equation (15) should be understood almost everywhere, i.e. at all points where the derivative function q exists. Then there is a

$$I_2(n,q'\alpha)=\int_{-\pi}^{\pi}\widetilde{q}(\theta)\sin n\theta d\theta,\quad \widetilde{q}\in L_2[-\pi,\pi]$$

and (14) becomes

$$\lambda_n(\alpha)=n^2+\frac{1}{n}\left[\frac{2\xi_1(\alpha)}{\pi}(-1)^{n+1}\sin n\alpha\pi+\frac{(-1)^n}{\pi}\int_{-\pi}^{\pi}\widetilde{q}(\theta)\sin n\theta d\theta\right]+o\left(\frac{\int_{-\pi}^{\pi}\widetilde{q}(\theta)\sin n\theta d\theta}{n}\right) \tag{16}$$

We ask: Does the given sequence of eigenvalues λ_n, $n=0,1,2,\dots$ uniquely determine the parameter of delay α.

Theorem 3.3. *If a sequence of eigenvalues of the task (1-2) with the asymptotic shape of the type (16), the coefficient of delay $\alpha\in(0,1)$ is unambiguously determined.*

Proof. Let us consider sequences $\Delta_2\lambda_n$ and $\Delta_1\lambda_n$ defined with

$$\Delta_2\lambda_n = [\lambda_{n+2} - (n+2)^2](n+2) - [\lambda_{n-2} - (n-2)^2](n-2) \tag{17}$$

and

$$\Delta_1\lambda_n = [\lambda_{n+1} - (n+1)^2](n+1) - [\lambda_{n-1} - (n-1)^2](n-1) \tag{18}$$

Based on (16) it applies that

$$\Delta_2\lambda_n = \frac{2\xi_1}{\pi}(-1)^{n+1}[\sin(n+2)\alpha\pi - \sin(n-2)\alpha\pi] + O\left(\frac{1}{n^s}\right)$$

$$\Delta_1\lambda_n = \frac{2\xi_1}{\pi}(-1)^{n}[\sin(n+1)\alpha\pi - \sin(n-1)\alpha\pi] + O\left(\frac{1}{n^s}\right)$$

i.e.

$$\Delta_2\lambda_n = (-1)^{n+1}\frac{4\xi_1}{\pi}\cos n\alpha\pi \sin 2\alpha\pi + O\left(\frac{1}{n^s}\right)$$

$$\Delta_1\lambda_n = (-1)^{n}\frac{4\xi_1}{\pi}\cos n\alpha\pi \sin \alpha\pi + O\left(\frac{1}{n^s}\right)$$

Let us choose the subsequence n_k of the sequence n, $n \in N$ where $\Delta_1\lambda_{n_k} \neq 0$. Then the sequence

$$\Delta\lambda_{n_k} = \frac{\Delta_2\lambda_{n_k}}{\Delta_1\lambda_{n_k}} = -\cos\alpha\pi + O\left(\frac{1}{n^s}\right), \quad k = 1, 2, \ldots$$

is well defined. It also applies that

$$\cos\alpha\pi = -\lim_{k\to\infty}\Delta\lambda_{n_k} = d$$

Since the $d \in (-1, 1)$ it follows that

$$\alpha = \frac{1}{\pi}\arccos d \quad \text{and} \quad \alpha \in (0, 1) \tag{19}$$

This proves the determination of coefficient α.

Next, let us take the subsequence $2k_l - 1$, $(l \in N)$ of the sequence of odd integers such that $\sin(2k_l - 1)\alpha\pi \neq 0$ $(\forall l \in N)$. Then from (16) it easily follows that

$$\xi_1 = \lim_{l\to\infty}\frac{\pi(2k_l - 1)}{2\sin(2k_l - 1)\alpha\pi}(\lambda_{2k_l-1} - (2k_l - 1)^2),$$

so that

$$q(\pi) = (1 - \alpha^2)\xi_1.$$

Thus, the asymptotic sequence λ_n uniquely determines the numbers α and $q(\pi)$.

Let us analyze the direct task (1-2) further in order to observe the determination of potential q based on the given sequence of numbers λ_n, $n \in N_0$ which has the asymptotic shape of type (16).

The entire function (5) can be represented by its zeros $\pm z_n$, $n \in N_0$ in the form of an infinite product.

$$F(z, \alpha) = Az\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n}\right) = \frac{A}{\pi}\prod_{n=1}^{\infty}\frac{n^2}{\lambda_n}\cdot\pi z\prod_{n=1}^{\infty}\left[\left(1 - \frac{z^2}{n^2}\right) + \frac{\lambda_n - n^2}{n^2}\right]$$

where A is indefinite parameter. If we put $B = \frac{A}{\pi}\prod_{n=1}^{\infty}\frac{n^2}{\lambda_n}$, then we have

$$F(z, \alpha) = B\pi z\left[\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) + \sum_{j=1}^{\infty}\prod_{n\neq j}\left(1 - \frac{z^2}{n^2}\right)\frac{\lambda_j - j^2}{j^2} + \sum_{l=2}^{\infty}\sum_{j_1<j_2<\cdots<j_l}\prod_{n\neq j_1,\ldots,j_l}\left(1 - \frac{z^2}{n^2}\right)\prod_{i=1}^{l}\frac{\lambda_{j_i} - j_i^2}{j_i^2}\right] =$$

$$= B\left[\sin\pi z + \left(\sum_{n=1}^{\infty}\frac{\lambda_n - n^2}{n^2 - z^2}\right)\sin\pi z + \psi(z,\alpha)\right], \quad z \in C \setminus Z \tag{20}$$

From (5) and (20) it follows that $B = 1$, i.e.

$$A = \pi\prod_{n=1}^{\infty}\frac{\lambda_n}{n^2} \tag{21}$$

We will continue to use these labels

$$\left.\begin{aligned}
&\xi(z,\alpha) = \sum_{l=2}^{\infty}\sum_{1\le j_1<j_2<\cdots<j_l}\prod_{i=1}^{l}\frac{\lambda_{j_i} - j_i^2}{j_i^2 - z^2}, \quad z \in C \setminus Z\\
&\psi(z,\alpha) = \xi(z,\alpha)\sin\pi z\\
&S_1(\alpha) = \sum_{n=1}^{\infty}(\lambda_n - n^2)\\
&S_1^*(\alpha) = \sum_{n=1}^{\infty}\left(\lambda_n - n^2 - \frac{C_1(n,\alpha)}{n}\right)\\
&b_m(\widetilde{q},\pi) = \int_{-\alpha\pi}^{\pi}\widetilde{q}(\theta)\sin m\theta d\theta\\
&a_m(\widetilde{q},\pi) = \int_{-\alpha\pi}^{\pi}\widetilde{q}(\theta)\cos m\theta d\theta\\
&\beta_m(\widetilde{q},k) = k\int_{-\alpha\pi}^{\pi}b_m(\widetilde{q},\theta)\mathrm{sh}k\theta d\theta\\
&\alpha_m(\widetilde{q},k) = k\int_{-\alpha\pi}^{\pi}a_m(\widetilde{q},\theta)\mathrm{ch}k\theta d\theta
\end{aligned}\right\} \tag{22}$$

Using (21) the equation (20) becomes

$$\sum_{n=1}^{\infty}\frac{\lambda_n - n^2}{n^2 - z^2} = \frac{\left(\pi\prod_{n=1}^{\infty}\frac{\lambda_n}{n^2}\right)z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n}\right)}{\sin\pi z} - 1 - \xi(z,\alpha), \quad z \in C \setminus Z \tag{23}$$

By analogy with the Levitan transformation in [9], we write

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\lambda_n - n^2}{n^2 - z^2} &= \sum_{n=1}^{\infty}\frac{\lambda_n - n^2 - \frac{C_1(n,\alpha)}{n}}{n^2 - z^2} + \sum_{n=1}^{\infty}\frac{C_1(n,\alpha)}{n(n^2 - z^2)}\\
&= \sum_{n=1}^{\infty}\frac{C_1(n,\alpha)}{n(n^2 - z^2)} - \frac{1}{z^2}S_1^*(\alpha) + \frac{1}{z^2}\sum_{n=1}^{\infty}\left(\lambda_n - n^2 - \frac{C_1(n,\alpha)}{n}\right)\frac{n^2}{n^2 - z^2}
\end{aligned}$$

Since $\widetilde{q} \in L_2[-\pi,\pi]$, it is true that

$$\int_{-\alpha\pi}^{\pi}\widetilde{q}(\theta)\sin n\theta d\theta = O\left(\frac{1}{n^s}\right), \quad s > \frac{1}{2}$$

so,

$$\lambda_n - n^2 - \frac{C_1(n,\alpha)}{n} = O\left(\frac{1}{n^2}\right)$$

Therefore, it is

$$\frac{1}{z^2}\sum_{n=1}^{\infty}\left(\lambda_n - n^2 - \frac{C_1(n,\alpha)}{n}\right)\frac{n^2}{n^2-z^2} \sim \frac{1}{z^2}\sum\frac{1}{n^2-z^2} = \frac{1}{z^2}\left(\frac{1}{2z^2} - \frac{\pi}{2z}\frac{\cos\pi z}{\sin\pi z}\right) = O\left(\frac{\operatorname{ctg}\pi z}{z^3}\right),\ \ z\to\infty \tag{24}$$

Furthermore, based on the known relations

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos nx}{n^2-z^2} = \frac{\pi}{2z}\frac{\cos zx}{\sin\pi z} - \frac{1}{2z^2}$$

by means of integration we get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n(n^2-z^2)} = \frac{\pi}{2z^2\sin\pi z}\sin zx - \frac{x}{2z^2} \tag{25}$$

Based on (24) and (25) we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\lambda_n - n^2}{n^2-z^2} \;=\; & -\frac{\xi_1(\alpha)\pi}{2z^2}\frac{\sin\alpha\pi z}{\sin\pi z} + \frac{\alpha\pi\xi_1(\alpha)}{2z^2} - \frac{1}{2z^2}\int_{-\alpha\pi}^{\pi}\theta\widetilde{q}(\theta)d\theta \\ & -\frac{1}{z^2\sin\pi z}b_z(\widetilde{q},\pi) - \frac{1}{z^2}S_1^*(\alpha) + O\left(\frac{\operatorname{ctg}\pi z}{z^3}\right),\ \ (z\to\infty)\end{aligned} \tag{26}$$

The first forced regularized trace of the operator (1-2) is obtained analogously to the calculation of the first trace (see (9)) and it is true that

$$S_1^*(\alpha) = \frac{\alpha\pi\xi_1(\alpha)}{2} + \xi_2(\alpha) + \frac{1}{\pi}\int_{-\pi}^{\pi}\theta\widetilde{q}(\theta)d\theta \tag{27}$$

On the basis of (20), (21), (26) and (27), we come to the relation

$$\left[\pi\prod_{n=1}^{\infty}\frac{\lambda_n}{n^2}\cdot z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n}\right) - \sin\pi z\right](-2z^2) - 2\xi_2\cdot\sin\pi z = \int_{-\pi}^{\pi}\widetilde{q}(\theta,\alpha)\sin z\theta d\theta + O\left(\frac{\cos\pi z}{z}\right), (z\to\infty) \tag{28}.$$

Relation (28) has a central role in this paper. Namely, it connects the given sequence of eigenvalues λ_n of the operator (1-2) with the function $\int_{-\pi}^{\pi}\widetilde{q}(\theta,\alpha)\sin z\theta d\theta$ which is the generator of Fourier coefficients of the auxiliary function $\widetilde{q}$ in the segment $[-\pi,\pi]$.

Putting $z_k = 4k + \frac{1}{2}$ and letting $k\to\infty$, from (28) we get

$$\xi_2 = -\lim_{k\to\infty}\left(4k+\frac{1}{2}\right)^2\left[\pi\prod_{n=1}^{\infty}\frac{\lambda_n}{n}\left(4k+\frac{1}{2}\right)\prod_{n=1}^{\infty}\left(1-\frac{(4k+\frac{1}{2})^2}{\lambda_n}\right) - 1\right]$$

As α is already determined, we have

$$q(0) = \frac{1-\alpha^2}{\alpha}\xi_2.$$

Thus, the left side of (28) is a known entire function in the whole complex plane. Its asymptotic behavior will have a key role on straight lines $z = m + ik$, $k\to+\infty$, $m\in N_0$.

Let us introduce the following real sequences

$$u_{m,k} = \prod_{n=1}^{\infty}\left(1-\frac{m^2-k^2}{\lambda_n}\right) + \sum_{l=1}^{\infty}(-1)^l \sum_{1\le k_1<\ldots<k_{2l}} \prod_{s=1}^{2l}\frac{2mk}{\lambda_{k_s}} \prod_{n\neq k_1\ldots k_{2l}}\left(1-\frac{m^2-k^2}{\lambda_n}\right) \tag{29}$$

$$v_{m,k} = \sum_{l=1}^{\infty}(-1)^{l-1} \sum_{1\le k_1<\ldots<k_{2l-1}} \prod_{s=1}^{2l-1}\frac{2mk}{\lambda_{k_s}} \prod_{n\neq k_1\ldots k_{2l-1}}\left(1-\frac{m^2-k^2}{\lambda_n}\right) \tag{30}$$

For $z = m + ik$, the left side of (28) becomes

$$\begin{aligned} U_{m,k} + iV_{m,k} \quad = \quad & \left\{(-2\pi)\prod_{n=1}^{\infty}\frac{\lambda_n}{n^2}[(m^3-3mk^2)u_{m,k} - (3mk^2-k^3)v_{m,k}] + 2(-1)^{m+1}mk\,\mathrm{sh}k\pi\right\} \\ & +i\left\{(m^3-3mk^2)v_{m,k} + (3mk^2-k^3)u_{m,k} + 2(m^2-k^2-\xi_2)(-1)^m\,\mathrm{sh}k\pi\right\} \end{aligned} \tag{31}$$

Using tags (22) the right side of (28) takes the form of

$$\mathrm{ch}k\pi\int_{-\pi}^{\pi}\widetilde{q}(\theta,\alpha)\sin m\theta d\theta - \beta_m(\widetilde{q},k) + i\left(\mathrm{sh}k\pi\int_{-\pi}^{\pi}\widetilde{q}(\theta,\alpha)\cos m\theta d\theta - \alpha_m(\widetilde{q},k)\right) + O\left(\frac{m-ik}{m^2+k^2}\mathrm{ch}k\pi\right) \tag{32}$$

From (31) and (32) it follows that

$$\int_{-\pi}^{\pi}\widetilde{q}(\theta,\alpha)\sin m\theta d\theta = \lim_{k\to\infty}\frac{U_{m,k}}{\mathrm{ch}k\pi}, \quad m\in N \tag{33}$$

$$\int_{-\pi}^{\pi}\widetilde{q}(\theta,\alpha)\cos m\theta d\theta = \lim_{k\to\infty}\frac{V_{m,k}}{\mathrm{sh}k\pi}, \quad m\in N_0 \tag{34}$$

A more detailed analysis of the coefficients structure $a_m^- = \int_{-\pi}^{\pi}\widetilde{q}(\theta,\alpha)\cos m\theta d\theta$ and $b_m^- = \int_{-\pi}^{\pi}\widetilde{q}(\theta,\alpha)\sin m\theta d\theta$ may prove that it is true that $a_m^- = o\left(\frac{1}{\sqrt{m}}\right)$ and $b_m^- = o\left(\frac{1}{\sqrt{m}}\right)$, $m\to\infty$.

Let us put $a_m = \frac{1}{\pi}a_m^-$, $b_m = \frac{1}{\pi}b_m^-$. In the points of a continuous function $\widetilde{q}$ at $(-\pi,\pi)$ it is true that

$$\widetilde{q}(\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty}a_m\cos m\theta + b_m\sin m\theta \tag{35}$$

According to (15) this means that

$$q'(x) = (1+\alpha)^2\frac{a_0}{2} + \sum_{n=1}^{\infty}(1+\alpha)^2(-1)^m\cos m(1+\alpha)x + (1+\alpha)^2(-1)^{m+1}b_m\sin m(1+\alpha)x, \quad x\in\left(\frac{1-\alpha}{1+\alpha}\pi,\pi\right) \tag{36}$$

and

$$\begin{aligned} & q'(x) - \left(\frac{1+\alpha}{1-\alpha}\right)^2 q'\left(\frac{1+\alpha}{1-\alpha}x\right) = (1+\alpha)^2\frac{a_0}{2} + \\ & + \sum_{m=1}^{\infty}(1+\alpha)^2(-1)^m[a_m\cos m(1+\alpha)x - b_m\sin m(1+\alpha)x], \quad x\in\left(0,\frac{1-\alpha}{1+\alpha}\pi\right) \end{aligned} \tag{37}$$

Since $\left(0,\frac{1-\alpha}{1+\alpha}\pi\right) = \bigcup_{l=1}^{\infty}\left(\left(\frac{1-\alpha}{1+\alpha}\right)^{l+1}\pi,\left(\frac{1-\alpha}{1+\alpha}\right)^{l}\pi\right)$, for $l=1$, the function q' is defined on the interval $\left(\frac{1-\alpha}{1+\alpha}\pi,\pi\right)$, and in (37) $q'\left(\frac{1+\alpha}{1-\alpha}x\right)$ is known. So, with (36) the function $q'(x)$, $x\in\left(\left(\frac{1-\alpha}{1+\alpha}\right)^2\pi,\left(\frac{1-\alpha}{1+\alpha}\right)\pi\right)$ is defined.

Applying the procedure of expressing function values q' on the interval $\left(\left(\frac{1-\alpha}{1+\alpha}\right)^{l+1}\pi, \left(\frac{1-\alpha}{1+\alpha}\right)^{l}\pi\right)$ by its values on the interval $\left(\left(\frac{1-\alpha}{1+\alpha}\right)^{l}\pi, \left(\frac{1-\alpha}{1+\alpha}\right)^{l-1}\pi\right)$, $l \in N$ and the relation (36), we conclude that the function q' is uniquely determined in terms of metric in $L_2[0,\pi]$. As the values of the function q in points $x = 0$ and $x = \pi$ are known from the asymptotic sequence λ_n, thus the values in the interval $(0,\pi)$ are obtained by the integration series (36) and (37).

The previous discussion proves the main result. □

Inversion theorem 3.4. *Given the sequence of eigenvalues λ_n of the operator (1-2), its identification components α and q are uniquely determined.*

Comment. The described method of solving the inverse task for the operator $D(\alpha)$, $\alpha \in (0,1)$ is also applicable to the case $D = D(0)$, i.e. the classical Sturm-Liouville spectral task.

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