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Property (*Bb*) and Tensor product

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Abstract. In this paper, we find necessary and sufficient conditions for Banach Space operator to satisfy the property (Bb). Then we obtain, if Banach Space operators $A \in B(X)$ and $B \in B(Y)$ satisfy property (Bb) implies $A \otimes B$ satisfies property (Bb) if and only if the B-Weyl spectrum identity $\sigma_{BW}(A \otimes B) = \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A)$ holds. Perturbations by Riesz operators are considered.

1. Introduction

Throughout this paper we denote by $B(X)$ the algebra of all bounded linear operators acting on a Banach space X. For $T \in B(X)$, let $T^*, \ker(T) = T^{-1}(0), \mathfrak{R}(T) = T(X), \sigma(T)$ and $\sigma_a(T)$ denote respectively the adjoint, the null space, the range, the spectrum and the approximate point spectrum of T. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by $\alpha(T) = \dim \ker(T)$ and $\beta(T) = codim\,\mathfrak{R}(T)$. If the range $\mathfrak{R}(T)$ of $T \in B(X)$ is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$) then T is upper semi-Fredholm (resp., lower semi-Fredholm) operator. Let $SF_+(X)$ (resp.,$SF_-(X)$) denote the semigroup of upper semi-Fredholm (resp., lower semi-Fredholm) operator on X. An operator $T \in B(X)$ is said to be semi-Fredholm if $T \in SF_+(X) \cup SF_-(X)$ and Fredholm if $T \in SF_+(X) \cap SF_-(X)$. If T is semi-Fredholm then the index of T is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. Recall that the ascent of an operator $T \in B(X)$ is the smallest non negative integer p:=$p(T)$ such that $T^{-p}(0) = T^{-(p+1)}(0)$. If there is no such integer, ie., $T^{-p}(0) \neq T^{-(p+1)}(0)$ for all p, then set $p(T) = \infty$. The descent of T is defined as the smallest non negative integer q:=q(T) such that $T^q(X) = T^{(q+1)}(X)$. If there is no such integer, ie., $T^q(X) \neq T^{(q+1)}(X)$ for all q, then set $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then they are equal [13, Proposition 38.6]. A bounded linear operator T acting on a Banach space X is Weyl if it is Fredholm of index zero and Browder if T is Fredholm of finite ascent and descent. For $T \in B(X)$, let , $E^0(T)$, and $\pi^0(T)$ denote, the eigenvalues of finite multiplicity and poles of T respectively . The Weyl spectrum $\sigma_w(T)$ and Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl }\},$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder }\}.$$

We have $\pi^0(T) := \sigma(T) \setminus \sigma_b(T)$. Set $\Delta(T) = \sigma(T) \setminus \sigma_w(T)$. According to Coburn [7], Weyl's theorem holds for T (abbreviation, $T \in Wt$) if $\Delta(T) = E^0(T)$ and that Browder's theorem holds for T (in symbol, $T \in Bt$) if $\sigma(T) \setminus \sigma_w(T) = \pi^0(T)$.

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An operator $T \in B(X)$ is called B-Fredholm, $T \in BF_+^-(X)$, if there exist a natural number n, for which the induced operator $T_n : T^n(X) \to T^n(X)$ is Fredholm in usual sense, and B-Weyl, $T \in BW_+^-(X)$, if $T \in BF_+^-(X)$ and $\text{ind}(T_n) = 0$. Let $E(T)$ be the set of all eigenvalues of T which are isolated in $\sigma(T)$ and $\sigma_{BW}(T) = \{\lambda \in C : T - \lambda \text{ is not B-Weyl }\}$. Set $\triangle^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$. According to [12], $T \in B(X)$ satisfies *property* (Bw) (in symbol $T \in (Bw)$) if $\Delta^g(T) = E^0(T)$. We say that T satisfies *property* (Bb) (in symbol, $T \in (Bb)$), a variant of generalized Browder's theorem, if $\Delta^g(T) = \pi^0(T)$. Property(Bb) is introduced and studied in [20] by the authors. Property (Bw) implies property (Bb) but converse is not true in general, see [20]. Let A be a unital algebra. We say that $x \in A$ is Drazin invertible of degree k if there exist an element $a \in A$ such that $x^k a x = x^k$, $axa = a$ and $xa = ax$. The Drazin spectrum of $a \in A$ is defined as $\sigma_D(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not Drazin invertible}\}$. It is well known that $T \in B(X)$ is Drazin invertible if and only if T has finite ascent and descent. Let $L_o(X)$ denote the set of all finite rank operators acting on an infinite dimensional Banach space X. The B-Browder spectrum $\sigma_{BB}(T)$ is defined in [8] as follows:

$$\sigma_{BB}(T) = \bigcap \{\sigma_D(T + F) : F \in L_o(X) \text{ and } TF = FT\}$$

An operator $T \in B(X)$ has the single valued extension property (SVEP) at $\lambda_0 \in \mathbb{C}$, if for every open disc D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \to X$ which satisfies $(T - \lambda)f(\lambda)=0$ for all $\lambda \in D_{\lambda_0}$ is the function $f \equiv 0$. We say that T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$. For more information, see [1].

The tensor product of two operators $A \in B(X)$ and $B \in B(Y)$ on $X \otimes Y$ is the operator $A \otimes B$ defined by

$$(A \otimes B)\Sigma_i x_i \otimes y_i = \Sigma_i A x_i \otimes B y_i$$

for every $\Sigma_i x_i \otimes y_i \in X \otimes Y$. Extensive study of preservation of Browder's theorem, Weyl's theorem ,a-Browder's theorem, a-Weyl's are found in [10, 11, 15, 16]

We studied necessary and sufficient conditions for Banach Space operator to satisfy the property (Bb) in first section of this paper . Then we obtain , if Banach space operators $A \in B(X)$ and $B \in B(Y)$ satisfy property (Bb) implies $A \otimes B$ satisfies property (Bb) if and only if the B-Weyl spectrum identity $\sigma_{BW}(A \otimes B) = \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A)$ holds.

2. property (*Bb*)

Theorem 2.1. *If T satisfies property (Bb), then T satisfies Browder's theorem.*

Proof. Suppose that T satisfies property (Bb) ie, $\Delta^g(T) = \pi^0(T)$. Let $\lambda \in \Delta(T)$. Then $T - \lambda$ is Fredholm of index zero and hence $T - \lambda$ is B-Fredholm of index zero. Thus $\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = \Delta^g(T)$. Hence $\lambda \in \pi^0(T)$

Conversely let $\lambda \in \pi^0(T)$. Since T satisfies property (Bb), $T - \lambda$ is B-Fredholm of index zero. Since $\alpha(T - \lambda) < \infty$, we conclude that $T - \lambda$ is Weyl . Thus $\lambda \in \triangle(T)$. This completes the proof. □

The following example shows that the converse of above theorem does not hold in general.

Example 2.2. *Let $T : l^2(N) \to l^2(N)$ be an injective quasinilpotent operator which is not nilpotant. we define S on Banach Space $X = l^2(N) \oplus l^2(N)$ by $S = I \oplus T$, where I is the identity operator on $l^2(N)$. Then $\sigma(S) = \sigma_w(S) = \{0, 1\}$ and $\sigma_{BW}(S) = \{0\}$. Also $E^0(S) = \pi^0(S) = \phi$. Clearly, S satisfies Browder's theorem but not (Bb).*

Theorem 2.3. *Let $T \in B(X)$. Then the following statements are equivalent.*

(i) $T \in (Bb)$;
(ii) $\sigma_{BW}(T) = \sigma_b(T)$;
(iii) $\sigma_{BW}(T) \cup E^0(T) = \sigma(T)$.

Proof. (i)$\Longrightarrow$ (*ii*). Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Since T satisfies (Bb), $\lambda \in \pi^0(T)$. Thus $\lambda \in \sigma(T) \setminus \sigma_b(T)$ and hence $\sigma_b(T) \subseteq \sigma_{BW}(T)$. Since the reverse inclusion is always true, we have $\sigma_b(T) = \sigma_{BW}(T)$.
(ii)$\Longrightarrow$ (*i*). Assume that $\sigma_b(T) = \sigma_{BW}(T)$ and we will establish that $\Delta^g(T) = \pi^0(T)$. Suppose $\lambda \in \Delta^g(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_b(T)$. Hence $\lambda \in \pi^0(T)$. Conversely suppose $\lambda \in \pi^0(T)$. Since $\sigma_{BW}(T) = \sigma_b(T)$, $\lambda \in \Delta^g(T)$.

(ii)$\Longrightarrow$ (*iii*). Let $\lambda \in \Delta^g(T)$. Since $\sigma_{BW}(T) = \sigma_b(T)$, $\lambda \in \sigma(T) \setminus \sigma_b(T)$, ie., $\lambda \in \pi^0(T)$ which implies that $\lambda \in E^0(T)$. Thus $\sigma_{BW}(T) \cup E^o(T) \supseteq \sigma(T)$. Since $\sigma_{BW}(T) \cup E^o(T) \subseteq \sigma(T)$, always we must have $\sigma_{BW}(T) \cup E^o(T) = \sigma(T)$.
(iii)$\Longrightarrow$ (*ii*). Suppose $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Since $\sigma_{BW}(T) \cup E^0(T) = \sigma(T)$, $\lambda \in E^0(T)$. In particular λ is an isolated point of $\sigma(T)$. Then by [4, Theorem 4.2] that $\lambda \notin \sigma_D(T)$ and this implies that $\lambda \in \pi(T)$ and so $a(T-\lambda) = d(T-\lambda) < \infty$. So, it follows from [1, Theorem 3.4] that $\beta(T-\lambda) = \alpha(T-\lambda) < \infty$. Hence $\lambda \in \pi^0(T)$. Therefore, $\lambda \notin \sigma_b(T)$. Since the other inclusion is always verified, we have $\sigma_{BW}(T) = \sigma_b(T)$. This completes the proof. □

Theorem 2.4. *Let $T \in B(X)$. IF T satisfies property (Bb). Then the following statements are equivalent.*

(i) $T \in (Bw)$;
(ii) $\sigma_{BW}(T) \cap E^0(T) = \emptyset$;
(iii) $E^o(T) = \pi^0(T)$.

Proof. (i)$\Longrightarrow$ (*ii*). Suppose (i) holds, that is, $\Delta^g(T) = E^0(T)$. then it follows that $\sigma_{BW}(T) \cap E^0(T) = \emptyset$.
(ii)$\Longrightarrow$ (*iii*). Suppose $\sigma_{BW}(T) \cap E^0(T) = \emptyset$ and let $\lambda \in E^0(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Since $T \in (Bb)$, we must have $\lambda \in \pi^0(T)$ and hence $E^0(T) \subseteq \pi^0(T)$. Since the reverse inclusion is trivial, we have $E^0(T) = \pi^0(T)$.
(iii)$\Longrightarrow$ (*i*). Since T satisfies property (Bb) and $E^o(T) = \pi^0(T)$, we conclude that $T \in (Bw)$. □

3. property(*Bb*) and Tensor product

Let $SF_+(X)$ denote the set of upper semi B-Fredholm operators and let $\sigma_{SBF_+}(T) = \{\lambda \in \mathbb{C} : \lambda \notin SF_+(X)\}$. We write $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SBF_+}(T) \text{ or } \operatorname{ind}(T-\lambda) > 0\}$.

The quasinilpotent part $H_0(T-\lambda I)$ and the analytic core $K(T-\lambda I)$ of $T-\lambda I$ are defined by

$$H_0(T-\lambda I) := \{x \in X : \lim_{n\longrightarrow\infty} \|(T-\lambda I)^n x\|^{\frac{1}{n}} = 0\}.$$

and

$$\begin{aligned} K(T-\lambda I) = \{&x \in X : \text{there exists a sequence } \{x_n\} \subset X \text{ and } \quad \delta > 0 \\ &\text{for which} \quad x = x_0, (T-\lambda I)x_{n+1} = x_n \text{and} \quad \|x_n\| \leq \delta^n \|x\| \text{for all} \quad n = 1, 2, \cdots\}. \end{aligned}$$

We note that $H_0(T-\lambda I)$ and $K(T-\lambda I)$ are generally non-closed hyper-invariant subspaces of $T-\lambda I$ such that $(T-\lambda I)^{-p}(0) \subseteq H_0(T-\lambda I)$ for all $p = 0, 1, \cdots$ and $(T-\lambda I)K(T-\lambda I) = K(T-\lambda I)$. Recall that if $\lambda \in iso(\sigma(T))$, then $H_0(T-\lambda I) = \chi_T(\{\lambda\})$, where $\chi_T(\{\lambda\})$ is the glocal spectral subspace consisting of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus \{\lambda\} \longrightarrow X$ that satisfies $(T-\mu)f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus \{\lambda\}$, see, Duggal [9].

Lemma 3.1. *Let $A \in B(X)$ and $B \in B(Y)$. Then*

$$\begin{aligned} \sigma_{BW}(A \otimes B) &\subseteq \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A) \subseteq \sigma_w(A)\sigma(B) \cup \sigma_w(B)\sigma(A) \\ &\subseteq \sigma_b(A)\sigma(B) \cup \sigma_b(B)\sigma(A) = \sigma_b(A \otimes B). \end{aligned}$$

Proof. Since $\sigma_{BW}(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T)$, the inclusion

$$\sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A) \subseteq \sigma_w(A)\sigma(B) \cup \sigma_w(B)\sigma(A) \subseteq \sigma_b(A)\sigma(B) \cup \sigma_b(B)\sigma(A)$$

is evident. Also we have $\sigma_b(A)\sigma(B) \cup \sigma_b(B)\sigma(A) = \sigma_b(A \otimes B)$ is true so it is enough to prove the inclusion $\sigma_{BW}(A \otimes B) \subseteq \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A)$. Let $\lambda \notin \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A)$. Since $\sigma_{SBF_+}(A \otimes B) \subseteq \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A)$, we have $\lambda \neq 0$. For every factorization $\lambda = \mu\nu$ such that $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$ we have that $\mu \in \sigma(A) \setminus \sigma_{BW}(A)$ and $\mu \in \sigma(B) \setminus \sigma_{BW}(B)$. That is $\mu \in BF_+(A)$ and $\nu \in BF_+(B)$, such that $\operatorname{ind}(A-\mu) \leq 0$ and $\operatorname{ind}(B-\nu) \leq 0$. In particular $\lambda \notin \sigma_{SBF_+}(A \otimes B)$. Now we have to prove that $\operatorname{ind}(A \otimes B - \lambda) \leq 0$. If $\operatorname{ind}(A \otimes B - \lambda) > 0$, then $\alpha(A\otimes B-\lambda)\leq\infty$ and so $\beta(A\otimes B-\lambda)\leq\infty$. Let $E=\{(\mu_i,\nu_i)\in\sigma(A)\sigma(B):1\leq i\leq p,\mu_i\nu_i=\lambda\}$. Then we have by [14, Theorem 3.5] that

$$\text{ind}\,(A\otimes B-\lambda)=\sum_{j=n+1}^{p} ind(A-\mu_j)\dim H_0(B-\nu_j)+\sum_{j=1}^{n}\text{ind}\,(B-\nu_j)\,\dim H_0(A-\mu_j).$$

Since ind $(A-\mu_i)<0$ and ind$(B-\nu_i)<0$, we have a contradiction. Hence we have $\lambda\notin\sigma_{BW}(A\otimes B)$. This completes the proof. □

Lemma 3.2. *Let $A\in B(X)$ and $B\in B(Y)$. If $A\otimes B$ satisfies property (Bb), then $\sigma_{BW}(A\otimes B)=\sigma_{BW}(A)\sigma(B)\cup\sigma_{BW}(B)\sigma(A)$.*

Proof. It follows from Theorem 2.3 that $A\otimes B$ satisfies property (Bb) if and only if $\sigma_{BW}(A\otimes B)=\sigma_b(A\otimes B)$. Thus the required result is an immediate consequence of Lemma 3.1. □

The following theorem gives a sufficient condition for the equality $\sigma_{BW}(A\otimes B)=\sigma_{BW}(A)\sigma(B)\cup\sigma_{BW}(B)\sigma(A)$ to hold. The equality $\sigma_{SBF_+}(A\otimes B)=\sigma_{SBF_+}(A)\sigma(B)\cup\sigma_{SBF_+}(B)\sigma(A)$ follows as in lemma 2 of [11] is useful for our proof of Theorem 3.3

Theorem 3.3. *If A and B satisfy property (Bb), then the following conditions are equivalent:*

(i) *$A\otimes B$ satisfies property (Bb);*
(ii) *$\sigma_{BW}(A\otimes B)=\sigma_{BW}(A)\sigma(B)\cup\sigma_{BW}(B)\sigma(A)$;*
(iii) *A has SVEP at points $\mu\in BF_+(A)$ and $\nu\in BF_+(B)$ such that $\lambda=\mu\nu\notin\sigma_{BW}(A\otimes B)$.*

Proof. $(i)\Longrightarrow(ii)$. is clear from Lemma 3.2.
$(ii)\Longrightarrow(i)$. Let (ii) satisfied. since A and B satisfy Bb, it follows that

$$\sigma_{BW}(A\otimes B)=\sigma_{BW}(A)\sigma(B)\cup\sigma_{BW}(B)\sigma(A)=\sigma_b(A)\sigma(B)\cup\sigma_b(B)\sigma(A)=\sigma_b(A\otimes B).$$

$(ii)\Longrightarrow(iii)$. Let $\lambda\in\sigma(A\otimes B)\setminus\sigma_{BW}(A\otimes B)$. Since A and B satisfy Bb, we have $\lambda\in\sigma(A\otimes B)\setminus\sigma_b(A\otimes B)$. Then for every factorization $\lambda=\mu\nu$ of λ , we have $\mu\in SBF_+(A)$ and $\nu\in SBF_+(B)$ we have that $p(A-\mu)$ and $q(B-\nu)$ are finite. Hence , A and B have SVEP at μ and ν, respectively.
$(iii)\Longrightarrow(ii)$. Suppose (iii) holds. We have to prove that $\sigma_b(A\otimes B)\subseteq\sigma_{BW}(A\otimes B)$. Let $\lambda\in\sigma(A\otimes B)\setminus\sigma_{BW}(A\otimes B)$. Then $\lambda\in BF_+(A\otimes B)$ and ind$(A\otimes B)\leq 0$. Then by the hypothesis and by equality $\sigma_{SBF_+}(A\otimes B)=\sigma_{SBF_+}(A)\sigma(B)\cup\sigma_{SBF_+}(B)\sigma(A)$, we conclude that $\mu\notin\sigma_b(A\otimes B)$ and $\nu\notin\sigma_b(A\otimes B)$. Thus $\lambda\notin\sigma_b(A\otimes B)$. □

Theorem 3.4. *Let $A\in B(X)$ and $B\in B(Y)$. If A^* and B^* have SVEP, then $A\otimes B$ satisfies property (Bb).*

Proof. The hypothesis A^* and B^* have SVEP implies

$$\sigma_w(A)=\sigma_{BW}(A),\qquad \sigma_w(B)=\sigma_{BW}(B)$$

and

$$A,B\quad\text{and}\quad A\otimes B\quad\text{satisfy Browder's theorem.}$$

Hence, Browder's theorem transfer from A and B to $A\otimes B$. Thus,

$$\begin{aligned}\sigma_b(A\otimes B)&=\sigma_w(A\otimes B)=\sigma(A)\sigma_w(B)\cup\sigma_w(A)\sigma(B)\\&=\sigma(A)\sigma_{BW}(B)\cup\sigma_{BW}(A)\sigma(B)=\sigma_{BW}(A\otimes B)\end{aligned}$$

Therefore,

$$\pi^0(A\otimes B)=\sigma(A\otimes B)\setminus\sigma_w(A\otimes B)=\sigma(A\otimes B)\setminus\sigma_{BW}(A\otimes B),$$

i.e., $A\otimes B$ satisfies property (Bb).
□

An operator $T \in B(X)$ is polaroid if every $\lambda \in iso\sigma(T)$ is a pole of the resolvent operator $(T-\lambda I)^{-1}$. $T \in B(X)$ polaroid implies T^* polaroid. It is well known that if T or T^* has SVEP and T is polaroid, then T and T^* satisfy Weyl's theorem.

Theorem 3.5. *Suppose that the operators $A \in B(X)$ and $B \in B(Y)$ are polaroid.*

(i) If A^* and B^* have SVEP, then $A \otimes B$ satisfies property (Bw).
(ii) If A and B have SVEP, then $A^* \otimes B^*$ satisfies property (Bw).

Proof. (i) The hypothesis A^* and B^* have SVEP implies

$$\sigma_w(A) = \sigma_{BW}(A), \qquad \sigma_w(B) = \sigma_{BW}(B)$$

and

$$A, B \quad \text{and} \quad A \otimes B \quad \text{satisfy Browder's theorem.}$$

Hence, Browder's theorem transfer from A and B to $A \otimes B$. Thus,

$$\begin{aligned}\sigma_b(A \otimes B) &= \sigma_w(A \otimes B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)\\ &= \sigma(A)\sigma_{BW}(B) \cup \sigma_{BW}(A)\sigma(B) = \sigma_{BW}(A \otimes B)\end{aligned}$$

Evidently, $A \otimes B$ is polaroid by Lemma 2 of [10]; combining this with $A \otimes B$ satisfies Browder's theorem, it follows that $A \otimes B$ satisfies Wt, i.e., $\sigma(A \otimes B) \setminus \sigma_w(A \otimes B) = E^0(A \otimes B)$. But then

$$E^0(A \otimes B) = \sigma(A \otimes B) \setminus \sigma_w(A \otimes B) = \sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B),$$

i.e., $A \otimes B$ satisfies property (Bw).
(ii) In this case $\sigma(A) = \sigma(A^*), \sigma(B) = \sigma(B^*), \sigma_w(A^*) = \sigma_{BW}(A^*), \sigma_w(B^*) = \sigma_{BW}(B^*)$, $\sigma(A \otimes B) = \sigma(A^* \otimes B^*)$, polaroid property transfer from A, B to $A^* \otimes B^*$, and Browder's theorem transfer from A, B to $A \otimes B$. Hence

$$\begin{aligned}\sigma_b(A^* \otimes B^*) &= \sigma_b(A \otimes B) = \sigma_w(A \otimes b) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)\\ &= \sigma(A^*)\sigma_w(B^*) \cup \sigma_w(A^*)\sigma(B^*)\\ &= \sigma(A^*)\sigma_{BW}(B^*) \cup \sigma_{BW}(A^*)\sigma(B^*)\\ &= \sigma_{BW}(A^* \otimes B^*).\end{aligned}$$

Thus, since $A^* \otimes B^*$ polaroid and $A \otimes B$ satisfies Browder's theorem imply $A^* \otimes B^*$ satisfy Wt,

$$E^0(A^* \otimes B^*) = \sigma(A^* \otimes B^*) \setminus \sigma_w(A^* \otimes B^*) = \sigma(A^* \otimes B^*) \setminus \sigma_{BW}(A^* \otimes B^*),$$

i.e., $A^* \otimes B^*$ satisfies property (Bw). □

4. Perturbations

Let $[A, Q] = AQ - QA$ denote the commutator of the operators A and Q. If $Q_1 \in B(X)$ and $Q_2 \in B(Y)$ are quasinilpotent operators such that $[Q_1, A] = [Q_2, B] = 0$ for some operators $A \in B(X)$ and $B \in B(Y)$, then

$$(A + Q_1) \otimes (B + Q_2) = (A \otimes B) + Q,$$

where $Q = Q_1 \otimes B + A \otimes Q_2 + Q_1 \otimes Q_2 \in B(X \otimes Y)$ is a quasinilpotent operator. If in the above, Q_1 and Q_2 are nilpotents then $(A + Q_1) \otimes (B + Q_2)$ is the perturbation of $A \otimes B$ by a commuting nilpotent operator.
A bounded operator T on X is called finite isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T of finite multiplicity, i.e $iso\sigma(T) \subseteq E^0(T)$. Recall that an operator $T \in B(X)$ satisfies generalized Browder's theorem (in symbol, $T \in gBt$) if $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$. Note that from Theorem 2.1 of [3] that an operator T, $T \in Bt$ if and only if $T \in gBt$. The following lemma from [12] is useful in the proof of the following results.

Lemma 4.1. *Let $T \in B(X)$. Then the following statements are equivalent:*

(i) T satisfies property (Bw);

(ii) generalized Browder's theorem holds for T and $\pi(T) = E^0(T)$.

Proposition 4.2. *Let $Q_1 \in B(X)$ and $Q_2 \in B(Y)$ be quasinilpotent operators such that $[Q_1, A] = [Q_2, B] = 0$ for some operators $A \in B(X)$ and $B \in B(Y)$. If $A \otimes B$ is finitely isoloid, then $A \otimes B$ satisfies property (Bw) implies $(A + Q_1) \otimes (B + Q_2)$ satisfies property (Bw).*

Proof. Recall that $\sigma((A+Q_1)\otimes(B+Q_2)) = \sigma(A\otimes B)$, $\sigma_w((A+Q_1)\otimes(B+Q_2)) = \sigma_w(A\otimes B)$, $\sigma_{BW}((A+Q_1)\otimes(B+Q_2)) = \sigma_{BW}(A \otimes B)$ and that the perturbation of an operator by a commuting quasinilpotent has SVEP if and only if the operator has SVEP. If $A \otimes B$ satisfies property (Bw), then

$$\begin{aligned} E^0(A \otimes B) &= \sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B) \\ &= \sigma((A + Q_1) \otimes (B + Q_2)) \setminus \sigma_{BW}((A + Q_1) \otimes (B + Q_2)). \end{aligned}$$

We prove that $E^0(A\otimes B) = E^0((A + Q_1)\otimes(B + Q_2))$. Observe that if $\lambda \in iso\sigma(A\otimes B)$, then $A^* \otimes B^*$ has SVEP at λ; equivalently, $(A^* + Q_1^*)\otimes(B^* + Q_2^*)$ has SVEP at λ. Let $\lambda \in E^0(A\otimes B)$; then $\lambda \in \sigma((A + Q_1)\otimes(B + Q_2)) \setminus \sigma_{BW}((A + Q_1) \otimes (B + Q_2))$. Since $(A^* + Q_1^*) \otimes (B^* + Q_2^*)$ has SVEP at λ, it follows that $\lambda \notin \sigma_{BW}((A + Q_1) \otimes (B + Q_2))$ and $\lambda \in iso\sigma((A + Q_1) \otimes (B + Q_2))$. Thus, $\lambda \in E^0((A + Q_1) \otimes (B + Q_2))$. Hence $E^0(A \otimes B) \subseteq E^0((A + Q_1) \otimes (B + Q_2))$. Conversely, if $\lambda \in E^0((A+Q_1)\otimes(B+Q_2))$, then $\lambda \in iso\sigma(A\otimes B))$, and this, since $A\otimes B$ is finitely isoloid implies that $\lambda \in E^0(A \otimes B)$, Hence $E^0((A + Q_1) \otimes (B + Q_2)) \subseteq E^0(A \otimes B)$. □

From [6], we recall that an operator $R \in B(X)$ is said to be Riesz if $R - \lambda I$ is Fredholm for every non-zero complex number λ, that is, $\Pi(R)$ is quasi-nilpotent in $C(X)$ where $C(X) := B(X)/K(X)$ is the Calkin algebra and Π is the canonical mapping of $B(X)$ into $C(X)$. Note that for such operator,$\pi^0(R) = \sigma(R) \setminus \{0\}$, and its restriction to one of its closed subspace is also a Riesz operator, see [6]. The situation for perturbations by commuting Riesz operators is a bit more delicate. The equality $\sigma(T) = \sigma(T + R)$ always hold for operators $T, R \in B(X)$ such that R is Riesz and $[T, R] = 0$; the tensor product $T\otimes R$ is not a Riesz operator (the Fredholm spectrum $\sigma_e(T\otimes R) = \sigma(T)\sigma_e(R)\cup\sigma_e(T)\sigma(R) = \sigma_e(T)\sigma(R) = \{0\}$ for a particular choice of T only). However, σ_w (also, σ_{BW}) is stable under perturbation by commuting Riesz operators [17, 18], and so T satisfies Browder's theorem if and only if $T + R$ satisfies Browder's theorem. Thus, if $T, R \in B(X)$ (such that R is Riesz and $[T, R] = 0$), then $\pi^0(T) = \sigma(T) \setminus \sigma_w(T) = \sigma(T + R) \setminus \sigma_w(T + R) = \pi^0(T + R)$, where $\pi^0(T)$ is the set of $\lambda \in iso\sigma(T)$ which are finite rank poles of the resolvent of T . If we now suppose additionally that T satisfies property (Bw), then

$$E^0(T) = \sigma(T) \setminus \sigma_{BW}(T) = \sigma(T + R) \setminus \sigma_{BW}(T + R) \tag{1}$$

and a necessary and sufficient condition for $T + R$ to satisfy property (Bw) is that $E^0(T) = E^0(T + R)$. One such condition, namely T is finitely isoloid.

Proposition 4.3. *Let $T, R \in B(X)$, where R is Riesz, and T is finitely isoloid. Then T satisfies property (Bw) implies $T + R$ satisfies property (Bw).*

Proof. Observe that if T satisfies property (Bw), then identity (1) holds. Let $\lambda \in E^0(T)$. Then, $\lambda \in E^0(T)\cap\sigma(T) = E^0(T + R - R) \cap \sigma(T + R) \subseteq iso\sigma(T + R)$, and so $T^* + R^*$ has SVEP at λ. Since $\lambda \in \sigma(T + R) \setminus \sigma_{BW}(T + R)$, $T^* + R^*$ has SVEP at λ implies that $T + R - \lambda I$ is Fredholm of index 0 and so $\lambda \in E^0(T + R)$. Hence, $E^0(T) \subseteq E^0(T + R)$. Now let $\lambda \in E^0(T + R)$. Then $\lambda \in E^0(T + R) \cap \sigma(T + R) = E^0(T + R) \cap \sigma(T) \subseteq iso\sigma(T)$, which by the finite isoloid property of T implies $\lambda \in E^0(T)$. Thus, $E^0(T + R) \subseteq E^0(T)$. □

Theorem 4.4. *Let $A \in B(X)$ and $B \in B(Y)$ be finitely isoloid operators which satisfy property (Bw). If $R_1 \in B(X)$ and $R_2 \in B(Y)$ are Riesz operators such that $[A, R_1] = [B, R_2] = 0$, $\sigma(A + R_1) = \sigma(A)$ and $\sigma(B + R_2) = \sigma(B)$, then $A \otimes B$ satisfies property (Bw) implies $(A + R_1) \otimes (B + R_2)$ satisfies property (Bw) if and only if generalized Browder's theorem transforms from $A + R_1$ and $B + R_2$ to their tensor product.*

Proof. The hypotheses imply (by Proposition 4.3) that both $A+R_1$ and $B+R_2$ satisfy property (Bw). Suppose that $A\otimes B$ satisfies property (Bw). Then $\sigma(A\otimes B)\setminus\sigma_{BW}(A\otimes B) = E^0(A\otimes B)$. Evidently $A\otimes B$ satisfies generalized Browder's theorem, and so the hypothesis A and B satisfy property (Bw) implies that generalized Browder's theorem transfers from A and B to $A\otimes B$. Furthermore, since , $\sigma(A+R_1) = \sigma(A)$, $\sigma(B+R_2) = \sigma(B)$, and σ_{BW} is stable under perturbations by commuting Riesz operators,

$$\begin{aligned}\sigma_{BW}(A\otimes B) &= \sigma(A)\sigma_{BW}(B)\cup\sigma_{BW}(A)\sigma(B)\\ &= \sigma(A+R_1)\sigma_{BW}(B+R_2)\cup\sigma_{BW}(A+R_1)\sigma(B+R_2).\end{aligned}$$

Suppose now that generalized Browder's theorem transfers from $A+R_1$ and $B+R_2$ to $(A+R_1)\otimes(B+R_2)$. Then

$$\sigma_{BW}(A\otimes B) = \sigma_{BW}((A+R_1)\otimes(B+R_2))$$

and

$$E^0(A\otimes B) = \sigma((A+R_1)\otimes(B+R_2))\setminus\sigma_{BW}((A+R_1)\otimes(B+R_2)).$$

Let $\lambda\in E^0(A\otimes B)$. Then $\lambda\neq 0$, and hence there exist $\mu\in\sigma(A+R_1)\setminus\sigma_{BW}(A+R_1)$ and $\nu\in\sigma(B+R_2)\setminus\sigma_{BW}(B+R_2)$ such that $\lambda=\mu\nu$. As observed above, both $A+R_1$ and $B+R_2$ satisfy property (Bw); hence $\mu\in E^0(A+R_1)$ and $\nu\in E^0(B+R_2)$. This, since $\lambda\in\sigma(A\otimes B)=\sigma((A+R_1)\otimes(B+R_2))$, implies $\lambda\in E^0((A+R_1)\otimes(B+R_2))$. Conversely, if $\lambda\in E^0((A+R_1)\otimes(B+R_2))$, then $\lambda\neq 0$ and there exist $\mu\in E^0(A+R_1)\subseteq iso\sigma(A)$ and $\nu\in E^0(B+R_2)\subseteq iso\sigma(B)$ such that $\lambda=\mu\nu$. Recall that $E^0((A+R_1)\otimes(B+R_2))\subseteq E^0(A+R_1)E^0(B+R_2)$. Since A and B are finite isoloid, $\mu\in E^0(A)$ and $\nu\in E^0(B)$. Hence, since $\sigma((A+R_1)\otimes(B+R_2))=\sigma(A\otimes B)$, $\lambda=\mu\nu\in E_a^0(A\otimes B)$. To complete the proof, we observe that if the implication of the statement of the theorem holds, then (necessarily) $(A+R_1)\otimes(B+R_2)$ satisfies generalized Browder's theorem. This, since $A+R_1$ and $B+R_2$ satisfy generalized Browder's theorem, implies generalized Browder's theorem transfers from $A+R_1$ and $B+R_2$ to $(A+R_1)\otimes(B+R_2)$. □

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