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Filter exhaustiveness and $\mathcal{F}$-α-convergence of function sequences

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Abstract. In this work, we generalize the concepts of exhaustiveness, α-convergence, Cauchy sequence, pointwise convergence and uniform convergence for sequences of functions on metric spaces in terms of filters. We investigate some properties of these new concepts. We also examine the relations between the new concepts and the classical concepts.

1. Introduction

For a sequence of functions, the notion of continuous convergence which is stronger than the pointwise convergence was introduced in the first half of the twentieth century (see [8, 17, 25]) and called as α-convergence later (see [10, 14]. These two concepts are equivalent for a sequence of functions, but are not equivalent for a net of functions ([14]). Gregoriades and Papanastassiou [14] defined a new concept, that is, the exhaustiveness for sequences and nets of functions. Later, Caserta and Kočinac ([9]) defined the notions of statistical exhaustiveness and statistical α-convergence, and presented the relations between statistical α-convergence, statistical pointwise convergence and statistical uniform convergence. Boccuto et al. [6] studied ideal exhaustiveness and $(\mathcal{I}\alpha)$-convergence for lattice group-valued functions. In [2], some results were given with respect to ideal exhaustiveness and ideal α-convergence for sequences of functions defined from metric spaces into $\mathbb{R}$.

In this paper, we study sequences of functions defined from a metric space to a metric space. As a generalization, we introduce the concepts of *$\mathcal{F}$-α-convergence* and *$\mathcal{F}$-exhaustiveness* where $\mathcal{F}$ is a filter on $\mathbb{N}$. We also generalize some concepts related to sequences of functions in terms of the filter.

First, we recall some basic concepts related to filters (see [12, 26]). A family $\mathcal{F}$ of subsets of $\mathbb{N}$ (i.e., $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$) is called a *filter* on $\mathbb{N}$ if $\mathcal{F}$ satisfies the following conditions:

(1) $\emptyset \notin \mathcal{F}$,
(2) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$,
(3) If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.

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By (3), we have $\mathbb{N} \in \mathcal{F}$. A filter is said to be *free* if the intersection of all its members is empty, and *fixed* otherwise. If $\mathcal{F}$ is a filter on $\mathbb{N}$, then the set $\mathcal{I}(\mathcal{F}) = \{\mathbb{N} \setminus A : A \in \mathcal{F}\}$ is an *ideal* on $\mathbb{N}$; and conversely, if $\mathcal{I}$ is an ideal on $\mathbb{N}$, then the set $\mathcal{F}(\mathcal{I}) = \{\mathbb{N} \setminus A : A \in \mathcal{I}\}$ is a filter on $\mathbb{N}$. Filter and ideal are dual concepts. So, the notions defined by an ideal are equivalent to the ones defined by a filter. For example, the concepts of $\mathcal{F}$-convergence and $\mathcal{I}$-convergence are equivalent.
Let $\mathcal{F}$ be a filter. A subset A of $\mathbb{N}$ is called *$\mathcal{F}$-stationary* if it has nonempty intersection with each member of the filter $\mathcal{F}$. We denote the collection of all $\mathcal{F}$-stationary sets by $\mathcal{F}^*$. In brief, for an $A \subseteq \mathbb{N}$ we have

$$A \in \mathcal{F}^* \Longleftrightarrow A \notin \mathcal{I}(\mathcal{F}),$$

where $\mathcal{I}(\mathcal{F})$ is the ideal corresponding to $\mathcal{F}$.

A filter $\mathcal{F}$ is said to be a *P-filter*, if for every sequence $(K_n)_{n\in\mathbb{N}}$ of the sets in $\mathcal{F}$ there is a $K \in \mathcal{F}$ such that $|K \setminus K_n| < \infty$ for each $n \in \mathbb{N}$ (see [4, 5]). P-filters are duals of P-ideals.

Definition 1.1. (see [3, 15, 16]) A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, ρ_X) is said to be *$\mathcal{F}$-convergent* to $x \in X$ if for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} : \rho_X(x_n, x) < \varepsilon\}$ belongs to $\mathcal{F}$. In this case, we write $\mathcal{F} - \lim x_n = x$ or $x_n \xrightarrow{\mathcal{F}} x$.

Now we present some examples of filters.

1. **Fréchet Filter.** The family $\mathcal{F}_r = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite}\}$ is called the *Fréchet filter*. $\mathcal{F}_r$ is the minimum free filter with respect to the inclusion relation. Therefore, we can characterize free filters as follows: If $\mathcal{F} \supseteq \mathcal{F}_r$ then $\mathcal{F}$ is a free filter. $\mathcal{F}_r$-convergence coincides with the ordinary convergence.
2. **Statistical Convergence Filter.** If $\lim_{n\to\infty} (|A \cap [1, n]|)/n$ exists, where $|A|$ is the cardinality of the set $A \subseteq \mathbb{N}$, then the value of this limit is called the *asymptotic density* of the set A, and it is denoted by $d(A)$ (see [7, 22]). The family $\mathcal{F}_{st} = \{A \subseteq \mathbb{N} : d(A) = 1\}$ is a free P-filter, and it is called the *statistical convergence filter*. $\mathcal{F}_{st}$-convergence is called the *statistical convergence* (see [11, 13, 21]).
3. Let us consider the Euler function φ defined by

   $$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)...\left(1 - \frac{1}{p_m}\right)$$

   for $1 < n \in \mathbb{N}$, where $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_m^{\alpha_m}$ is the prime number decomposition of n, and $\varphi(1) = 1$ ([23]). Then

   $$d_\varphi(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{d|n} \varphi(d)\, \chi_A(d)$$

   is called the *φ-density* of the set A, provided that this limit exists ([19]; see also [1, 20]). The family $\mathcal{F}_\varphi = \left\{A \subseteq \mathbb{N} : d_\varphi(A) = 1\right\}$ is a free filter.

Lemma 1.2. *Let (X, ρ_X) be a metric space, $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, and $x \in X$. Let $\mathcal{F}$ be a P-filter on $\mathbb{N}$. If $\mathcal{F} - \lim x_n = x$ then there is a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}$ such that $\lim_{k\to\infty} x_{n_k} = x$.*

2. Filter exhaustiveness and $\mathcal{F}$-α-convergence

In this paper, (X, ρ_X) and (Y, ρ_Y) denote two metric spaces. Let $D \subseteq X$. Then $C(D, Y)$ denotes the family of all continuous functions from D into Y. By $S(x, \delta)$, we denote the open ball with center x and radius δ.

Using the notion of ideal $\mathcal{I}$ of $\mathbb{N}$; the generalizations of the concepts of exhaustiveness and α-convergence, namely, the concepts of $\mathcal{I}$-exhaustiveness and $\mathcal{I}$-α-convergence were defined in [6]; the concept of $\mathcal{I}$-pointwise convergence was defined in [4] and [18], the concept of $\mathcal{I}$-uniform convergence was introduced in [4]. Since we will study on filters, we will introduce the generalizations of the pointwise convergence, uniform convergence, α-convergence and exhaustiveness, via the notion of filter.

Definition 2.1. Given $D \subseteq X$, let $f, f_n : D \longrightarrow Y$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a filter on $\mathbb{N}$. Let $x \in D$. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{F}$*-exhaustive* at the point x provided that, for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\{n \in \mathbb{N} : \rho_Y(f_n(y), f_n(x)) < \varepsilon \text{ for all } y \in S(x,\delta)\} \in \mathcal{F}.$$

If the sequence $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive at every $x \in D$, then it is said to be $\mathcal{F}$*-exhaustive* on D.

Example 2.2. Let $\mathbb{P} = \{p_1 < p_2 < ... < p_k < ...\}$ be the set of all prime numbers, and $K = \{p_1, p_1.p_2, ..., p_1.p_2...p_k, ...\}$. The set K has φ-density 0 (see [24],[20]). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from $\mathbb{R}$ into $\mathbb{R}$ defined by

$$f_n(x) = \begin{cases} \dfrac{1}{x} & \text{if } n \in K \text{ and } x \neq 0 \\ \dfrac{1}{n}\arctan\left(\dfrac{1}{x}\right) & \text{if } n \notin K \text{ and } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

This sequence is $\mathcal{F}_\varphi$-exhaustive on $\mathbb{R}$, but not exhaustive at the point $x = 0$.
Let $x \in \mathbb{R}$. Indeed, for every $\varepsilon > 0$ there are a $\delta > 0$ and an $n(\varepsilon) = \left\lfloor \dfrac{\pi}{\varepsilon} \right\rfloor + 1 \in \mathbb{N}$ such that for every $y \in S(x,\delta)$ and for every $n \in \mathbb{N} \setminus (K \cup \{1, 2, ..., n(\varepsilon)\})$ we have

$$\left|f_n(y) - f_n(x)\right| = \left|\frac{1}{n}\arctan\left(\frac{1}{y}\right) - \frac{1}{n}\arctan\left(\frac{1}{x}\right)\right| \leq \frac{1}{n}\pi < \frac{1}{n(\varepsilon)}\pi < \varepsilon$$

(Here, $\lfloor . \rfloor$ denotes the greatest integer function). Then we have

$$\left\{n \in \mathbb{N} : \left|f_n(y) - f_n(x)\right| < \varepsilon \text{ for all } y \in S(x,\delta)\right\} \supseteq \mathbb{N} \setminus (K \cup \{1, 2, ..., n(\varepsilon)\}),$$

and

$$d_\varphi\left(\left\{n \in \mathbb{N} : \left|f_n(y) - f_n(x)\right| < \varepsilon \text{ for all } y \in S(x,\delta)\right\}\right) \geq d_\varphi(\mathbb{N} \setminus (K \cup \{1, 2, ..., n(\varepsilon)\})) = 1.$$

Therefore, $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}_\varphi$-exhaustive at $x \in \mathbb{R}$.
Now, let us show that $(f_n)_{n\in\mathbb{N}}$ is not exhaustive at $x = 0$. Let $\varepsilon = 2$. For every $\delta > 0$ and every $n \in K$ there exists $y \in S(0,\delta)$, $\left|y\right| < \frac{1}{2}$ such that

$$\left|f_n(y) - f_n(0)\right| = \left|\frac{1}{y} - 0\right| > \varepsilon.$$

Since the set K is infinite, we obtain the required result.

Definition 2.3. Given $D \subseteq X$, let $f, f_n : D \longrightarrow Y$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a filter on $\mathbb{N}$. Let $x \in D$. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{F}$*-α-convergent* to f at the point x if for every sequence $(x_n)_{n\in\mathbb{N}}$ which is $\mathcal{F}$-converging to x, the sequence $(f_n(x_n))_{n\in\mathbb{N}}$ is $\mathcal{F}$-convergent to $f(x)$ (i.e., $\mathcal{F} - \lim f_n(x_n) = f(x)$), and we write $f_n \xrightarrow{\mathcal{F}-\alpha} f$ (at x). If the sequence $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-α-convergent to f at each $x \in D$, then it is said to be $\mathcal{F}$-α-convergent to f on D.

Definition 2.4. Given $D \subseteq X$, let $f, f_n : D \longrightarrow Y$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a filter on $\mathbb{N}$. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{F}$*-pointwise convergent* to f on D if $\mathcal{F} - \lim f_n(x) = f(x)$ for each $x \in D$, i.e.,

$$\{n \in \mathbb{N} : \rho_Y(f_n(x), f(x)) < \varepsilon\} \in \mathcal{F}$$

for every $\varepsilon > 0$. In this case, we write $f_n \xrightarrow{\mathcal{F}-pw} f$ (on D).

Definition 2.5. Given $D \subseteq X$, let $f, f_n : D \longrightarrow Y$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a filter on $\mathbb{N}$. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{F}$*-uniformly convergent* to f on D provided that

$$\{n \in \mathbb{N} : \rho_Y(f_n(x), f(x)) < \varepsilon \text{ for all } x \in D\} \in \mathcal{F}$$

for every $\varepsilon > 0$. In this case, we write $f_n \xrightarrow{\mathcal{F}-u} f$ (on D).

As in the classical analysis, $\mathcal{F}$-uniform convergence implies $\mathcal{F}$-pointwise convergence.

Definition 2.6. Given $D \subseteq X$, let $f, f_n : D \longrightarrow Y$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a filter on $\mathbb{N}$. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be an $\mathcal{F}$*-uniform Cauchy sequence* if for every $\varepsilon > 0$ there exists some $k \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : \rho_Y(f_n(x), f_k(x)) < \varepsilon \text{ for all } x \in D\} \in \mathcal{F}.$$

Theorem 2.7. *Given $D \subseteq X$, let $f, f_n : D \longrightarrow Y$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a free filter on $\mathbb{N}$. Then the following hold:*

(i) If $(f_n)_{n\in\mathbb{N}}$ is exhaustive, then it is $\mathcal{F}$-exhaustive.
(ii) If $(f_n)_{n\in\mathbb{N}}$ is a uniform Cauchy sequence, then it is an $\mathcal{F}$-uniform Cauchy sequence.
(iii) $f_n \xrightarrow{pw} f$ implies $f_n \xrightarrow{\mathcal{F}-pw} f$.
(iv) $f_n \xrightarrow{u} f$ implies $f_n \xrightarrow{\mathcal{F}-u} f$.
(v) $f_n \xrightarrow{\alpha} f$ implies $f_n \xrightarrow{\mathcal{F}-\alpha} f$, where $\mathcal{F}$ is a P-filter.

Proof. Since $\mathcal{F} \supseteq \mathcal{F}_r$, and the concepts defined via the Fréchet filter $\mathcal{F}_r$ and their analogues in the classical analysis are equivalent, the items (i)-(iv) can be proved easily. We only need to prove the item (v).
(v) Let $x \in D$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in D such that $\mathcal{F} - \lim x_n = x$. From Lemma 1.2, since $\mathcal{F}$ is a P-filter, there is a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}$ such that $\lim_{k\to\infty} x_{n_k} = x$. Let us define a sequence $(y_n)_{n\in\mathbb{N}}$ by

$$y_n = \begin{cases} x_{n_k} & ; n = n_k \\ x & ; n \notin K \end{cases}$$

for every $n \in \mathbb{N}$. Then $\lim_{n\to\infty} y_n = x$. Since $f_n \xrightarrow{\alpha} f$ at x, we have $\lim_{n\to\infty} f_n(y_n) = f(x)$.
Let $\varepsilon > 0$. There is an $n(\varepsilon) \in \mathbb{N}$ such that $\rho_Y(f_n(y_n), f(x)) < \varepsilon$ for all $n \geq n(\varepsilon)$. Then

$$K \subseteq \{n \in \mathbb{N} : \rho_Y(f_n(x_n), f(x)) < \varepsilon\} \cup \{1, 2, ..., n(\varepsilon)\},$$

and so $\{n \in \mathbb{N} : \rho_Y(f_n(x_n), f(x)) < \varepsilon\} \in \mathcal{F}$. Consequently, we have $f_n \xrightarrow{\mathcal{F}-\alpha} f$ at $x \in D$. □

Since $\mathbb{N} \in \mathcal{F}$ for any filter $\mathcal{F}$, if the sequence $(f_n)_{n\in\mathbb{N}}$ is equicontinuous then it is $\mathcal{F}$-exhaustive (or exhaustive), where $f_n \in C(X, Y)$ for all $n \in \mathbb{N}$. Moreover, $(f_n)_{n\in\mathbb{N}}$ is equicontinuous if and only if it is $\mathcal{F}_0$-exhaustive, where $\mathcal{F}_0 := \{\mathbb{N}\}$ ($\mathcal{F}_0$ is a trivial filter). Similarly, $(f_n)_{n\in\mathbb{N}}$ is exhaustive if and only if it is $\mathcal{F}_r$-exhaustive.

Now we give an example of a sequence of functions which is statistically α-convergent but not α-convergent.

Example 2.8. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from $[0, 1]$ into $[0, 1]$ defined by

$$f_n(x) = \begin{cases} x^n & \text{if } n \in \left\{1, 4, 9, ..., k^2, ...\right\} \\ \dfrac{x}{n} & \text{if } n \notin \left\{1, 4, 9, ..., k^2, ...\right\} \end{cases}.$$

This sequence is not α-convergent to the function $f = 0$ (that is, $f(x) = 0$ for all $x \in [0,1]$) at $x = 1$. Indeed, the sequence $(x_n)_{n\in\mathbb{N}} = (1 - 1/n)_{n\in\mathbb{N}}$ is convergent to $x = 1$, but the sequence

$$f_n(x_n) = \begin{cases} \left(1 - \frac{1}{n}\right)^n & \text{if } n \in \left\{1, 4, 9, ..., k^2, ...\right\} \\ \frac{n-1}{n^2} & \text{if } n \notin \left\{1, 4, 9, ..., k^2, ...\right\} \end{cases}$$

is not convergent to $0 = f(1)$, because $1/e$ is a limit point of $(f_n(x_n))_{n\in\mathbb{N}}$.
Now we show that $(f_n)_{n\in\mathbb{N}}$ is statistically α-convergent to f on $[0,1]$. Let $y \in [0,1]$ and $(y_n)_{n\in\mathbb{N}}$ be a sequence such that $\mathcal{F}_{st} - \lim y_n = y$. There is a K_1 with $d(K_1) = 1$ such that $\lim_{n\to\infty, n\in K_1} y_n = y$. The set $K_2 = \left\{n \in \mathbb{N} : n \neq k^2 \text{ for each } k \in \mathbb{N}\right\}$ has asymptotic density 1. Let $K = K_1 \cap K_2$. Then $K \in \mathcal{F}_{st}$ and

$$\lim_{\substack{n\to\infty \\ n\in K}} f_n(y_n) = \lim_{\substack{n\to\infty \\ n\in K}} \frac{y_n}{n} \longrightarrow 0 = f(y).$$

Therefore we have $\mathcal{F}_{st} - \lim f_n(y_n) = f(y)$, and so $f_n \overset{\mathcal{F}_{st}-\alpha}{\longrightarrow} f$ at the point $y \in [0,1]$. Consequently, the sequence $(f_n)_{n\in\mathbb{N}}$ is statistically α-convergent to the function f on $[0,1]$.

On ideals, the analogue of the following theorem was given in [6] for sequences of functions defined from metric spaces to lattice groups. We state it without proof.

Theorem 2.9. *Given $D \subseteq X$, let $f, f_n : D \longrightarrow Y$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a free filter on $\mathbb{N}$. Then we have the following:*

(i) If $f_n \overset{\mathcal{F}-\alpha}{\longrightarrow} f$, then $f_n \overset{\mathcal{F}-pw}{\longrightarrow} f$.

(ii) If $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive and $f_n \overset{\mathcal{F}-pw}{\longrightarrow} f$, then $f_n \overset{\mathcal{F}-\alpha}{\longrightarrow} f$.

Now we give two examples of sequences of functions which are $\mathcal{F}$-pointwise convergent but not $\mathcal{F}$-α-convergent.

Example 2.10. Let us consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases},$$

and let $f_n = f$, where $n \in \mathbb{N}$. Then for any free filter $\mathcal{F}$ we have $f_n \overset{\mathcal{F}-pw}{\longrightarrow} f$ at $x = 0$, but $f_n \overset{\mathcal{F}-\alpha}{\not\longrightarrow} f$ at $x = 0$. To see that $(f_n)_{n\in\mathbb{N}}$ is not $\mathcal{F}$-α-convergent, let us consider the sequence $(x_n)_{n\in\mathbb{N}} = (1/n)_{n\in\mathbb{N}}$. Then we have $\mathcal{F} - \lim x_n = 0$, but $f_n(x_n) = 1$ for all $n \in \mathbb{N}$ and $\mathcal{F} - \lim f_n(x_n) = 1$ $(\neq f(0))$. According to Theorem 2.9(ii), $(f_n)_{n\in\mathbb{N}}$ cannot be $\mathcal{F}$-exhaustive at $x = 0$. To see that $(f_n)_{n\in\mathbb{N}}$ is not $\mathcal{F}$-exhaustive, let us take $\varepsilon = 1/2$. For every $\delta > 0$ there exists a $y \in S(0,\delta)$ such that $f(y) - f(0) = 1$, and thus we have

$$\left\{n \in \mathbb{N} : \left|f_n(y) - f_n(0)\right| < 1/2\right\} = \emptyset \notin \mathcal{F}.$$

Example 2.11. Let

$$f_n(x) = \begin{cases} 1, & \text{if } n = k^2 \\ x^n, & \text{if } n \neq k^2 \end{cases}$$

for each $n \in \mathbb{N}$ and

$$f(x) = \begin{cases} 0, & \text{if } x \in [0,1) \\ 1, & \text{if } x = 1 \end{cases}.$$

Then we have $f_n \overset{\mathcal{F}_{st}-pw}{\longrightarrow} f$ (on $[0,1]$), but $(f_n)_{n\in\mathbb{N}}$ is not $\mathcal{F}_{st}$-exhaustive at $x = 1$. Also $(f_n)_{n\in\mathbb{N}}$ is not $\mathcal{F}_{st}$-α-convergent to f at $x = 1$. Indeed, for the sequence $(x_n)_{n\in\mathbb{N}} = \left(1 - \frac{1}{n}\right)_{n\in\mathbb{N}}$ we have $\mathcal{F}_{st} - \lim x_n = 1$, but

$$f_n(x_n) = \begin{cases} 1, & \text{if } n = k^2 \\ \left(1 - \frac{1}{n}\right)^n, & \text{if } n \neq k^2 \end{cases}$$

for each $n \in \mathbb{N}$ and so $\mathcal{F}_{st} - \lim f_n(x_n) = e^{-1} \neq 1 = f(1)$.

The analogue of the following theorem was given in [6] for ideals.

Theorem 2.12. *Given $D \subseteq X$, let $f, f_n : D \longrightarrow Y$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a filter on $\mathbb{N}$. If $f_n \overset{\mathcal{F}-pw}{\longrightarrow} f$ on D and $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive on D, then f is continuous on D.*

In Example 2.10, we have $f_n \overset{\mathcal{F}-pw}{\longrightarrow} f$ at the point $x = 0$, but the function f is not continuous at $x = 0$. Therefore $(f_n)_{n\in\mathbb{N}}$ is not $\mathcal{F}$-exhaustive at $x = 0$.

From Theorems 2.9 and 2.12, we obtain the following corollary.

Corollary 2.13. *If $f_n \overset{\mathcal{F}-\alpha}{\longrightarrow} f$ and $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive, then f is continuous.*

Theorem 2.14. *Given $D \subseteq X$, let $f, f_n : D \longrightarrow Y$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a free filter on $\mathbb{N}$. Then we have the following:*

(i) If $f_n \overset{\mathcal{F}-u}{\longrightarrow} f$ and f is continuous on D, then $f_n \overset{\mathcal{F}-\alpha}{\longrightarrow} f$ on D.
(ii) If D is compact, $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive on D and $f_n \overset{\mathcal{F}-\alpha}{\longrightarrow} f$ on D, then $f_n \overset{\mathcal{F}-u}{\longrightarrow} f$ on D.

Proof. **(i)** Let $\varepsilon > 0$ and $x \in D$. Let (x_n) be an arbitrary sequence on D such that $\mathcal{F} - \lim x_n = x$. Since $f_n \overset{\mathcal{F}-u}{\longrightarrow} f$ on D, there is a set $K_1 \in \mathcal{F}$ such that $\rho_Y(f_n(x_n), f(x_n)) < \varepsilon/2$ for every $n \in K_1$. Since f is continuous at x, there is a $\delta > 0$ such that $\rho_Y(f(y), f(x)) < \varepsilon/2$ for every $y \in S(x,\delta)$. Since $\mathcal{F} - \lim x_n = x$, there is $K_2 \in \mathcal{F}$ such that $x_n \in S(x,\delta)$ for every $n \in K_2$. Let $K := K_1 \cap K_2$. Then $K \in \mathcal{F}$ and we have

$$\rho_Y(f_n(x_n), f(x)) \leq \rho_Y(f_n(x_n), f(x_n)) + \rho_Y(f(x_n), f(x)) < \varepsilon$$

for each $n \in K$.
(ii) Assume that $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-α-convergent to f on D. Let $\varepsilon > 0$. From Corollary 2.13, f is continuous on D. Then for every $x \in D$ there is a δ_x such that $\rho_X(x,y) < \delta_x$ implies $\rho_Y(f(x), f(y)) < \varepsilon/3$. Since $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive on D, for every $x \in D$ there exist $\lambda_x \leq \delta_x$ and $K(x) \in \mathcal{F}$ such that $\rho_Y(f_n(y), f_n(x)) < \varepsilon/3$ for all $y \in S(x,\lambda_x)$ and all $n \in K(x)$. Then $\bigcup_{x\in D} S(x,\lambda_x) \supseteq D$. Since D is compact there are finitely many $x^1, x^2, ..., x^m \in D$ such that $D \subseteq \bigcup_{i=1}^m S\left(x^i, \lambda_{x^i}\right)$. Let $K := \bigcap_{i=1}^m K\left(x^i\right) \in \mathcal{F}$. Therefore we have

$$\rho_Y\left(f(y), f\left(x^i\right)\right) < \varepsilon/3$$

and

$$\rho_Y\left(f_n(y), f_n\left(x^i\right)\right) < \varepsilon/3$$

for all $n \in K$ and all $y \in S\left(x^i, \lambda_{x^i}\right)$ (where $i = \{1, 2, ..., m\}$).

From Theorem 2.9(i), we have $f_n \overset{\mathcal{F}-pw}{\longrightarrow} f$ on D. Then for each $i \in \{1, 2, ..., m\}$ we have

$$L_i := \left\{n \in \mathbb{N} : \rho_Y\left(f_n\left(x^i\right), f\left(x^i\right)\right) < \varepsilon/3\right\} \in \mathcal{F}.$$

Let $L = \bigcap_{i=1}^{m} L_i \in \mathcal{F}$ and $M = K \cap L \in \mathcal{F}$.
Let $z \in D$. Then $z \in S\left(x^i, \lambda_{x^i}\right)$ for some $i \in \{1, ..., m\}$ and thus we have

$$\begin{aligned}\rho_Y\left(f_n(z), f(z)\right) &\leq \rho_Y\left(f_n(z), f_n\left(x^i\right)\right) + \rho_Y\left(f_n\left(x^i\right), f\left(x^i\right)\right) + \rho_Y\left(f\left(x^i\right), f(z)\right) \\ &< \varepsilon\end{aligned}$$

for every $n \in M$. Therefore we have

$$M \subseteq \{n \in \mathbb{N} : \rho_Y\left(f_n(z), f(z)\right) < \varepsilon \text{ for all } z \in D\} \in \mathcal{F},$$

and so $f_n \overset{\mathcal{F}-u}{\longrightarrow} f$ on D. □

Theorem 2.15. *Given $D \subseteq X$, let $f, f_n : D \longrightarrow Y$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a free filter. Then the following hold:*

- **(i)** If there is a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}$ such that $(f_{n_k})_{k\in\mathbb{N}}$ is exhaustive at a point $x \in D$, then the sequence $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive at x.
- **(ii)** Let $\mathcal{F}$ be also a P-filter. If $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive at a point $x \in D$, then there is a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}$ such that $(f_{n_k})_{k\in\mathbb{N}}$ is exhaustive at x.

Proof. **(i)** Let a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ be exhaustive at a point $x \in D$ where $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}$. Let $\varepsilon > 0$. Hence there exist a $\delta > 0$ and a $k(\varepsilon) \in \mathbb{N}$ such that

$$\rho_Y\left(f_{n_k}(y), f_{n_k}(x)\right) < \varepsilon$$

for all $y \in S(x, \delta)$ and all $k > k(\varepsilon)$. Then we have

$$K \subseteq \{n \in \mathbb{N} : \rho_Y\left(f_n(y), f_n(x)\right) < \varepsilon\} \cup \left\{n_1, n_2, ..., n_{k(\varepsilon)}\right\}$$

for all $y \in S(x, \delta)$. Since $\mathcal{F}$ is free, we have $\{n \in \mathbb{N} : \rho_Y\left(f_n(y), f_n(x)\right) < \varepsilon\} \in \mathcal{F}$. Consequently, $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive at the point x.
(ii) Let $(f_n)_{n\in\mathbb{N}}$ be $\mathcal{F}$-exhaustive at a point $x \in D$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\{n \in \mathbb{N} : \rho_Y\left(f_n(y), f_n(x)\right) < \varepsilon \text{ for all } y \in S(x, \delta)\} \in \mathcal{F}.$$

There are positive real numbers $\delta_1 > \delta_2 > ... > \delta_t > ...$ such that

$$K_t := \left\{n \in \mathbb{N} : \rho_Y\left(f_n(y), f_n(x)\right) < \frac{1}{t} \text{ for all } y \in S(x, \delta_t)\right\} \in \mathcal{F}$$

for each $t \in \mathbb{N}$. Since $\mathcal{F}$ is a P-filter, there is a set $K \in \mathcal{F}$ such that $|K \setminus K_t| < \infty$ for each $t \in \mathbb{N}$. Let $K := \{n_1 < n_2 < ... < n_k < ...\}$.
Let $\varepsilon > 0$. There is a $t_0 \in \mathbb{N}$ such that $1/t_0 < \varepsilon$, and we have

$$K_t \subseteq \{n \in \mathbb{N} : \rho_Y\left(f_n(y), f_n(x)\right) < \varepsilon \text{ for all } y \in S(x, \delta_t)\}$$

for every $t \geq t_0$. Let us choose a $t^* \in \mathbb{N}$ such that $t^* \geq t_0$. Then we have

$$\left|K \setminus \{n \in \mathbb{N} : \rho_Y\left(f_n(y), f_n(x)\right) < \varepsilon \text{ for all } y \in S(x, \delta_{t^*})\}\right| \leq |K \setminus K_{t^*}| < \infty.$$

Since the difference of two sets on the left side is finite, there is an $n(\varepsilon) \in \mathbb{N}$ such that

$$\rho_Y(f_{n_k}(y), f_{n_k}(x)) < \varepsilon$$

for all $y \in S(x, \delta_{t^*})$ and all $n_k \in K$ with $n_k > n(\varepsilon)$. □

Note 1. The item (i) of the theorem above can also be given on D instead of a single point. In the item (ii), since the set K that we obtained depends on x, it is not easy to find a common set belonging to $\mathcal{F}$ on D.

Note 2. This theorem can be proved similarly for $\mathcal{F}$-pointwise convergence, $\mathcal{F}$-uniform convergence and $\mathcal{F}$-uniform Cauchy condition.

We need to prove the theorem above for $\mathcal{F}$-α-convergence because of additional conditions.

Theorem 2.16. *Given $D \subseteq X$, let $f, f_n : D \longrightarrow Y$ $(n \in \mathbb{N})$ and $\mathcal{F}$ be a free P-filter. Then the following hold:*

(i) If there is a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}$ such that $f_{n_k} \xrightarrow{\alpha} f$ at a point $x \in D$, then $f_n \xrightarrow{\mathcal{F}-\alpha} f$ at x.

(ii) If $f_n \xrightarrow{\mathcal{F}-\alpha} f$ at a point $x \in D$ and $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive at x, then there is a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}$ such that $f_{n_k} \xrightarrow{\alpha} f$ at x.

Proof. **(i)** Let $f_{n_k} \xrightarrow{\alpha} f$ at a point $x \in D$ where $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in D such that $\mathcal{F} - \lim x_n = x$. Since $\mathcal{F}$ is a P-filter, there is a $L \in \mathcal{F}$ such that $\lim_{n\to\infty, n\in L} x_n = x$. Let $M = K \cap L$. Let us define the sequence $(y_{n_k})_{k\in\mathbb{N}}$ by

$$y_{n_k} = \begin{cases} x_{n_k} & ; n_k \in M \\ x & ; n_k \in K \setminus L \end{cases}.$$

Obviously, $y_{n_k} \longrightarrow x$ and so $f_{n_k}(y_{n_k}) \longrightarrow f(x)$. Therefore, for every $\varepsilon > 0$ there is a $k(\varepsilon) \in \mathbb{N}$ such that $\rho_Y(f_{n_k}(y_{n_k}), f(x)) < \varepsilon$ for all $k \geq k(\varepsilon)$. Then

$$M \subseteq \{n \in \mathbb{N} : \rho_Y(f_n(x_n), f(x)) < \varepsilon\} \cup \left\{n_1, n_2, ..., n_{k(\varepsilon)}\right\}.$$

Therefore, for every $\varepsilon > 0$ we have $\{n \in \mathbb{N} : \rho_Y(f_n(x_n), f(x)) < \varepsilon\} \in \mathcal{F}$ since $\mathcal{F}$ is free. Consequently, we have $\mathcal{F} - \lim f_n(x_n) = f(x)$ where $\mathcal{F} - \lim x_n = x$; i.e., $f_n \xrightarrow{\mathcal{F}-\alpha} f$ (at x).

(ii) Let $f_n \xrightarrow{\mathcal{F}-\alpha} f$ and $(f_n)_{n\in\mathbb{N}}$ be $\mathcal{F}$-exhaustive at a point $x \in D$. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in D such that $x_k \to x$. Let $\varepsilon > 0$. From Theorem 2.9(i) we have $f_n \xrightarrow{\mathcal{F}-pw} f$ (at x), and from Note 2, since $\mathcal{F}$ is a P-filter there is a $K_1 \in \mathcal{F}$ such that the subsequence $(f_n)_{n\in K_1}$ is pointwise convergent to f at x. Therefore, there is an $n'(\varepsilon) \in K_1$ such that

$$\rho_Y(f_n(x), f(x)) < \varepsilon/2 \tag{1}$$

for all $n \in K_1$ with $n \geq n'(\varepsilon)$. Since $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-exhaustive at x, from Theorem 2.15(ii) there is a $K_2 \in \mathcal{F}$ such that the subsequence $(f_n)_{n\in K_2}$ is $\mathcal{F}$-exhaustive at x. Therefore, there exist a $\delta > 0$ and an $n^*(\varepsilon) \in K_2$ such that

$$\rho_Y(f_n(y), f_n(x)) < \varepsilon/2 \tag{2}$$

for all $y \in S(x, \delta)$ and all $n \in K_2$ with $n \geq n^*(\varepsilon)$. Since $x_k \to x$, there is a $k_0 \in \mathbb{N}$ such that $x_k \in S(x, \delta)$ for every $k \geq k_0$. Let $K = K_1 \cap K_2 = \{n_1 < n_2 < ... < n_k < ...\}$ and $k_1 = \min\{k \in \mathbb{N} : n_k \geq n'(\varepsilon), n^*(\varepsilon)\}$. From (1) and (2), we have

$$\rho_Y(f_{n_k}(x_k), f(x)) \leq \rho_Y(f_{n_k}(x_k), f_{n_k}(x)) + \rho_Y(f_{n_k}(x), f(x)) < \varepsilon$$

for all $k \geq \max\{k_0, k_1\}$. Then the sequence $(f_{n_k}(x_k))_{k\in\mathbb{N}}$ is convergent to $f(x)$. Consequently, $f_{n_k} \xrightarrow{\alpha} f$ (at x). □

We can also restate Note 1 for this theorem.

The next example shows that Theorem 2.16(ii) does not hold without the condition "$\mathcal{F}$-exhaustiveness".

Example 2.17. Let us consider the sets $J_p = \left\{2^{p-1}(2q-1) : q \in \mathbb{N}\right\}$ for each $p \in \mathbb{N}$. Here, J_p's are pairwise disjoint sets, $d\left(J_p\right) = 1/2^p$ for each $p \in \mathbb{N}$, and $\bigcup_{p=1}^{\infty} J_p = \mathbb{N}$. Let us consider the sequence $(f_n)_{n\in\mathbb{N}}$ defined by

$$f_n(x) = \begin{cases} 1 & ; n \in J_p \text{ and } x = \dfrac{1}{p} \\ 0 & ; \text{otherwise} \end{cases}$$

for each $n \in \mathbb{N}$ in $\mathbb{R}^{\mathbb{R}}$, and the function f defined by $f(x) = 0$ for $x \in \mathbb{R}$.
First, we will show that $f_n \xrightarrow{\mathcal{F}_{st}-\alpha} f$ at $x = 0$. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence such that $\mathcal{F}_{st} - \lim x_n = 0$. Let us define the sets

$$S_p = \left\{n \in \mathbb{N} : x_n = \frac{1}{p}\right\}$$

for each $p \in \mathbb{N}$. Then $d\left(S_p\right)$ must be equal to zero since $\mathcal{F}_{st} - \lim x_n = 0$. Let

$$M_p := J_p \cap S_p$$

for each $p \in \mathbb{N}$. Then we have

$$f_n(x_n) = \begin{cases} 1 & ; n \in \bigcup_{p=1}^{\infty} M_p \\ 0 & ; n \notin \bigcup_{p=1}^{\infty} M_p \end{cases}.$$

We have $\left\{n \in \mathbb{N} : \left|f_n(x_n) - 0\right| \geq \varepsilon\right\} = \emptyset$ for $\varepsilon > 1$. Let $0 < \varepsilon \leq 1$. Then we have

$$\left\{n \in \mathbb{N} : \left|f_n(x_n) - 0\right| \geq \varepsilon\right\} = \bigcup_{p=1}^{\infty} M_p.$$

For $j \in \mathbb{N}$ we have

$$\begin{aligned} d\left(\bigcup_{p=1}^{\infty} M_p\right) &= d(M_1) + d(M_2) + ... + +d\left(M_j\right) + d\left(\bigcup_{p=j+1}^{\infty} M_p\right) \\ &= d\left(\bigcup_{p=j+1}^{\infty} M_p\right) \leq \frac{1}{2^j} \end{aligned}$$

and so we get $d\left(\bigcup_{p=1}^{\infty} M_p\right) = 0$ as $j \to \infty$. Therefore, we get

$$d\left(\left\{n \in \mathbb{N} : \left|f_n(x_n) - 0\right| \geq \varepsilon\right\}\right) = d\left(\bigcup_{p=1}^{\infty} M_p\right) = 0,$$

and so we have $\mathcal{F}_{st} - \lim f_n(x_n) = f(0) = 0$. Consequently, $f_n \xrightarrow{\mathcal{F}_{st}-\alpha} f$ at $x = 0$.
Now, we will show that $f_{n_k} \xrightarrow{\alpha} f$ at $x = 0$ for any set $K = \{n_1 < ... < n_k < ...\} \in \mathcal{F}_{st}$. Let

$$L_p := J_p \cap K$$

for each $p \in \mathbb{N}$. Let us construct the set $\left\{l_1 < l_2 < ... < l_p < ...\right\}$ by $l_1 = \min(L_1)$ and $l_{p+1} = \min\left(L_{p+1} \setminus \left\{1, 2, ..., l_p\right\}\right)$ for each $p \in \mathbb{N}$. Let us define the sequence $(x_k)_{k\in\mathbb{N}}$ by

$$x_k = \begin{cases} \dfrac{1}{p} & ; n_k = l_p, p \in \mathbb{N} \\ 0 & ; \text{otherwise} \end{cases}$$

for every $k \in \mathbb{N}$. Obviously, $x_k \to 0$. But the sequence $(f_{n_k}(x_k))_{k\in\mathbb{N}}$ is not convergent, since

$$f_{n_k}(x_k) = \begin{cases} 1 & ; n_k = l_p, p \in \mathbb{N} \\ 0 & ; \text{otherwise} \end{cases}.$$

Therefore we have $f_{n_k} \overset{\alpha}{\not\longrightarrow} f$ at $x = 0$.
In this example, the sequence $(f_n)_{n\in\mathbb{N}}$ is not $\mathcal{F}_{st}$-exhaustive at $x = 0$. Let $\varepsilon = 1/2$. There is a $p_\delta \in \mathbb{N}$ such that $1/p_\delta < \delta$ for every $\delta > 0$, and we have

$$\left\{n \in \mathbb{N} : \left|f_n(y) - f_n(0)\right| \geq \frac{1}{2} \text{ for } \exists y \in S(0,\delta)\right\} \supseteq \bigcup_{p=p_\delta}^{\infty} J_p.$$

Then we obtain

$$\begin{aligned} d\left(\left\{n \in \mathbb{N} : \left|f_n(y) - f_n(0)\right| \geq \tfrac{1}{2} \text{ for } \exists y \in S(0,\delta)\right\}\right) &\geq d\left(\textstyle\bigcup_{p=p_\delta}^{\infty} J_p\right) \\ &= \tfrac{1}{2^{p_\delta - 1}} > 0. \end{aligned}$$

Whereas, the density on the left should have been zero. Therefore, $(f_n)_{n\in\mathbb{N}}$ is not $\mathcal{F}_{st}$-exhaustive at $x = 0$. □

Now we present a result with respect to the filter $\mathcal{F}_{st}$. The following theorem is an analogue of [1, Theorem 2.3].

Theorem 2.18. *Given $D \subseteq X$, let $f, f_n : D \longrightarrow Y$ $(n \in \mathbb{N})$. Then the sequence $(f_n)_{n\in\mathbb{N}}$ is statistically α-convergent to the function f if, and only if, both (f_{2n}) and (f_{2n-1}) are statistically α-convergent to f.*

Proof. First, suppose that $f_n \overset{\mathcal{F}_{st}-\alpha}{\longrightarrow} f$ at $x \in D$. Let (y_n) and (z_n) be arbitrary sequences in D such that $\mathcal{F}_{st} - \lim y_n = x$ and $\mathcal{F}_{st} - \lim z_n = x$. Let us define a sequence (x_n) such that $x_{2n} = y_n$ and $x_{2n-1} = z_n$ for every $n \in \mathbb{N}$. It was shown in [1] that the sequence (x_n) is statistically convergent to x.
Since $f_n \overset{\mathcal{F}_{st}-\alpha}{\longrightarrow} f$ at $x \in D$, we have $\mathcal{F}_{st} - \lim f_n(x_n) = f(x)$, i.e., $K(\varepsilon) := \{n \in \mathbb{N} : \rho_Y(f_n(x_n), f(x)) < \varepsilon\} \in \mathcal{F}_{st}$ for every $\varepsilon > 0$. Define the sets

$$L(\varepsilon) := \{n \in \mathbb{N} : \rho_Y(f_n(x_n), f(x)) < \varepsilon \text{ and } n = 2k,\ k \in \mathbb{N}\} = \{2l_1, 2l_2, ...\}$$

and

$$\begin{aligned} M(\varepsilon) &:= \{n \in \mathbb{N} : \rho_Y(f_n(x_n), f(x)) < \varepsilon \text{ and } n = 2k-1,\ k \in \mathbb{N}\} \\ &= \{2m_1 - 1, 2m_2 - 1, ...\}. \end{aligned}$$

Since $d(K(\varepsilon)) = 1$, we have $d(L(\varepsilon)) = d(M(\varepsilon)) = 1/2$.
For the subsequence (f_{2n}), we get $L'(\varepsilon) := \{n \in \mathbb{N} : \rho_Y(f_{2n}(x_{2n}), f(x)) < \varepsilon\} = \{l_1, l_2, ...\}$. Since $d(L(\varepsilon)) = 1/2$ and $d(L'(\varepsilon)) = 2d(L(\varepsilon))$, we have $d(L'(\varepsilon)) = 1$. So we have $\mathcal{F}_{st} - \lim f_{2n}(y_n) = f(x)$ and $f_{2n} \overset{\mathcal{F}_{st}-\alpha}{\longrightarrow} f$ at x. Similarly, for the subsequence (f_{2n-1}), we get $M'(\varepsilon) := \{n \in \mathbb{N} : \rho_Y(f_{2n-1}(x_{2n-1}), f(x)) < \varepsilon\} = \{m_1, m_2, ...\}$. Since $d(M(\varepsilon)) = 1/2$ and $d(M'(\varepsilon)) = 2d(M(\varepsilon))$, we have $d(M'(\varepsilon)) = 1$. So we have $\mathcal{F}_{st} - \lim f_{2n-1}(z_n) = f(x)$ and $f_{2n-1} \overset{\mathcal{F}_{st}-\alpha}{\longrightarrow} f$ at x.

Now suppose that $f_{2n} \overset{\mathcal{F}_{st}-\alpha}{\longrightarrow} f$ and $f_{2n-1} \overset{\mathcal{F}_{st}-\alpha}{\longrightarrow} f$ at $x \in D$. Let $\mathcal{F}_{st} - \lim x_n = x$. From [1], we have $\mathcal{F}_{st} - \lim x_{2n} = x$ and $\mathcal{F}_{st} - \lim x_{2n-1} = x$. Then, for every $\varepsilon > 0$, we have $L'(\varepsilon), M'(\varepsilon) \in \mathcal{F}_{st}$. Then $d(L(\varepsilon)) = \frac{1}{2}d(L'(\varepsilon)) = \frac{1}{2}$ and $d(M(\varepsilon)) = \frac{1}{2}d(M'(\varepsilon)) = \frac{1}{2}$. Thus we get

$$K(\varepsilon) = \{n \in \mathbb{N} : \rho_Y(f_n(x_n), f(x)) < \varepsilon\} = L(\varepsilon) \cup M(\varepsilon)$$

and

$$d(K(\varepsilon)) = d(L(\varepsilon)) + d(M(\varepsilon)) = 1,$$

since $L(\varepsilon) \cap M(\varepsilon) = \emptyset$. Consequently, we have $\{n \in \mathbb{N} : \rho_Y(f_n(x_n), f(x)) < \varepsilon\} \in \mathcal{F}_{st}$ for every $\varepsilon > 0$. This means that $f_n \xrightarrow{\mathcal{F}_{st}-\alpha} f$ at $x \in D$. □

This theorem can also be proved for $\mathcal{F}$-pointwise convergence, $\mathcal{F}$-uniform convergence, $\mathcal{F}$-exhaustiveness and $\mathcal{F}$-uniform Cauchy condition.

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