

On the irreducibility of Hurwitz spaces of coverings with an arbitrary number of special points

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Abstract. In this paper we study Hurwitz spaces of coverings of Y with an arbitrary number of special points and with monodromy group a Weyl group of type D_d , where Y is a smooth, complex projective curve. We give conditions for which these spaces are irreducible.

1. Introduction

The study of the irreducibility of Hurwitz spaces is a classic topic in algebraic geometry. It is interesting to study Hurwitz spaces of coverings whose monodromy group is all S_d but also whose monodromy group is a Weyl group different from S_d . In fact, coverings with monodromy group a Weyl group appear in the study of spectral curves, integrable systems and Prym-Tyurin varieties (see [6, 15, 16]). Specifically, it is possible to define morphisms from Hurwitz spaces of coverings with monodromy group contained in a Weyl group to Siegel modular varieties which parameterize Abelian varieties. For way of this, some property of these varieties can be studied by using these Hurwitz spaces.

Let Y be a smooth, connected, complex projective curve of genus g . We point out that the irreducibility of Hurwitz spaces of coverings of Y with monodromy group S_d and with an arbitrary number of special points has been studied both when $g = 0$ and when $g > 0$ (see [1, 9, 11, 13–15, 18, 23, 26, 27]). Moreover, for example, Harris, Graber and Starr used the result of [9] in order to prove the existence of sections of one-parameter family of complex rationally connected varieties (see [10]). The irreducibility of Hurwitz spaces of coverings of Y whose monodromy group is a Weyl group different from S_d was studied, for example, in [2, 19–22, 24, 25]. In this paper, we continue such study. Specifically, we consider coverings of Y with an arbitrary number of special points and with monodromy group a Weyl group of type D_d . We give conditions for which the corresponding Hurwitz spaces are irreducible, both when $g = 0$ and when $g > 0$ (see Theorem 4.1). In this way, we extend to coverings with monodromy group a Weyl group of type D_d the results obtained in the case in which the monodromy group is all S_d by Kulikov in [14] and by the author in [26].

Conventions and Notations

Two sequences of coverings, $X_1 \xrightarrow{\pi_1} X'_1 \xrightarrow{f_1} Y$ and $X_2 \xrightarrow{\pi_2} X'_2 \xrightarrow{f_2} Y$, are *equivalent* if there exist two biholomorphic maps $p : X_1 \rightarrow X_2$ and $p' : X'_1 \rightarrow X'_2$ such that $p' \circ \pi_1 = \pi_2 \circ p$ and $f_2 \circ p' = f_1$. We denote

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by $[f \circ \pi]$ the equivalence class containing $f \circ \pi$. Moreover, we denote by t^h the permutation $h^{-1} t h$. Here, $Y^{(n)}$ denote the n -fold symmetric product of Y and Δ is the codimension 1 locus of $Y^{(n)}$ consisting of non simple divisors.

2. Preliminaries

In this section we shortly recall some notions on the Weyl groups of type B_d and D_d . The references for such discussion are [4, 5]. We also invite the reader to look Section 1 of [24]. Moreover, we recall some notions on braid moves. We refer to [3, 7, 11, 12, 17] for the details. Furthermore, in this section, we explain the strategy that we will use in order to prove the main result of this paper.

2.1. Weyl groups of type B_d and D_d

Let $\{\epsilon_1, \dots, \epsilon_d\}$ be the standard base of \mathbb{R}^d and let R be the root system $\{\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j : 1 \leq i, j \leq d\}$. The Weyl group of type B_d , that we denote by $W(B_d)$, is generated by the reflections s_{ϵ_i} , with $1 \leq i \leq d$, and by the reflections $s_{\epsilon_i - \epsilon_j}$, with $1 \leq i < j \leq d$. The Weyl group of type D_d is the subgroup of $W(B_d)$ generated by the reflections with respect to the long roots $\epsilon_1 - \epsilon_i$ and $\epsilon_1 + \epsilon_i$ with $2 \leq i \leq d$. We denote this group with $W(D_d)$.

We recall that $W(B_d)$ is isomorphic to the subgroup of S_{2d} generated by the transpositions $(i - i)$ with $1 \leq i \leq d$ and by the permutations $(i j)(-i - j)$ with $1 \leq i < j \leq d$. Let $(\mathbb{Z}_2)^d$ be the set of the functions from $\{1, \dots, d\}$ into \mathbb{Z}_2 equipped with the sum operation. Let Ψ be the homomorphism from S_d in $Aut((\mathbb{Z}_2)^d)$ which to $t \in S_d$ assigns $\Psi(t)$ where $[\Psi(t)a](j) := a(j^t)$ for each $a \in (\mathbb{Z}_2)^d$. $W(B_d)$ is also isomorphic to the semidirect product, $(\mathbb{Z}_2)^d \times^s S_d$, of $(\mathbb{Z}_2)^d$ and S_d through Ψ (see for example [24], Section 1).

Here, we identify $W(B_d)$ with $(\mathbb{Z}_2)^d \times^s S_d$ and we identify $W(D_d)$ with the subgroup of $(\mathbb{Z}_2)^d \times^s S_d$ generated by the elements $(0; (1, i))$ and $(\bar{1}_{1i}; (1, i))$ with $i = 2, \dots, d$. We use z_{ij} to denote the function of $(\mathbb{Z}_2)^d$ defined as

$$z_{ij}(i) = z_{ij}(j) = z \quad \text{and} \quad z_{ij}(l) = \bar{0} \quad \text{for each } l \neq i, j \quad \text{and } z \in \mathbb{Z}_2.$$

Definition 2.1. Let h be a positive integer. Let $(c; \xi)$ be an element of $W(B_d)$ satisfying the following: ξ is a h -cycle of S_d and c is a function that sends to $\bar{0}$ all the indexes fixed by ξ . We call an such element positive h -cycle if c is either zero or a function which sends to $\bar{1}$ an even number of indexes. We call it negative h -cycle if it is not positive.

We recall that two cycles $(c; \xi)$ and $(c'; \xi')$ in $W(B_d)$ are disjoint if ξ and ξ' are disjoint. Furthermore, all the elements in $W(B_d)$ can be expressed as a product of disjoint positive and negative cycles. The lengths of such disjoint cycles together with their signs determine the signed cycle type of the elements of $W(B_d)$. Two elements of $W(B_d)$ are conjugate if and only if they have the same signed cycle type (see [5]).

2.2. Braid moves

The braid group $\pi_1(Y^{(n)} - \Delta, D)$ is generated by the elementary braids σ_i with $i = 1, \dots, n - 1$ and by the braids ρ_{js}, τ_{js} with $1 \leq j \leq n$ and $1 \leq s \leq g$ (see [3], [7] and [17]).

Definition 2.2. Let G be an arbitrary group. An ordered sequence

$$(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g) := (\underline{t}, \underline{\lambda}, \underline{\mu})$$

of elements in G is a Hurwitz system if $t_i \neq id$ for each $i \in \{1, \dots, n\}$ and $t_1 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g]$. The subgroup of G generated by t_i, λ_s, μ_s with $i = 1, \dots, n$ and $s = 1, \dots, g$ is called the monodromy group of the Hurwitz system. Two Hurwitz systems $(\underline{t}, \underline{\lambda}, \underline{\mu})$ and $(\underline{t}', \underline{\lambda}', \underline{\mu}')$ with elements in G are equivalent if there exists $h \in G$ such that $t'_i = h^{-1} t_i h$, $\lambda'_s = h^{-1} \lambda_s h$ and $\mu'_s = h^{-1} \mu_s h$ for each $i = 1, \dots, n$ and $s = 1, \dots, g$.

Remark 2.3. We notice that when $g = 0$, one has got $t_1 \cdots t_n = id$.

The generators of $\pi_1(Y^{(n)} - \Delta, D)$ acts on Hurwitz systems. One can associate a pair of braid moves to each generator of $\pi_1(Y^{(n)} - \Delta, D)$. We denote by $\sigma'_i, \sigma''_i = (\sigma'_i)^{-1}$ the pair of moves associated to σ_i . We call such moves elementary moves. The moves σ'_i, σ''_i fix all the λ_s , all the μ_s and all the t_h with $h \neq i, i + 1$. They transform (t_i, t_{i+1}) into

$$(t_i t_{i+1} t_i^{-1}, t_i) \quad \text{and} \quad (t_{i+1}, t_{i+1}^{-1} t_i t_{i+1}),$$

respectively (see [11]). We denote by $\rho'_{j_s}, \rho''_{j_s} = (\rho'_{j_s})^{-1}$ and by $\tau'_{j_s}, \tau''_{j_s} = (\tau'_{j_s})^{-1}$ the pair of moves associated to ρ_{j_s} and τ_{j_s} , respectively. The moves $\rho'_{j_s}, \rho''_{j_s}$ fix any element different from t_j and μ_s . Analogously, the moves τ'_{j_s} and τ''_{j_s} modify the element t_j and the element λ_s , only. We notice that the moves $\rho'_{j_s}, \rho''_{j_s}, \tau'_{j_s}$ and τ''_{j_s} transform t_j into an element belonging to the same conjugacy class. Furthermore, we notice that when $\lambda_1 = \dots = \lambda_s = \mu_1 = \dots = \mu_{s-1} = id$, the braid move ρ'_{1_s} transforms

$$\mu_s \quad \text{into} \quad t_1^{-1} \mu_s.$$

Analogously when $\lambda_1 = \dots = \lambda_{s-1} = \mu_1 = \dots = \mu_{s-1} = id$, the braid move τ''_{1_s} transforms

$$\lambda_s \quad \text{into} \quad t_1^{-1} \lambda_s$$

(see [12], Theorem 1.8 and Corollary 1.9).

Definition 2.4. Two Hurwitz systems are said braid equivalent if one is obtained from the other by using a finite sequence of braid moves $\sigma'_i, \rho'_{j_s}, \tau'_{j_s}, \sigma''_i, \rho''_{j_s}, \tau''_{j_s}$ where $1 \leq i \leq n - 1, 1 \leq j \leq n$ and $1 \leq s \leq g$. Two ordered sequences of permutations (t_1, \dots, t_l) and (t'_1, \dots, t'_l) are said braid equivalent if (t'_1, \dots, t'_l) is obtained from (t_1, \dots, t_l) by using a finite sequence of braid moves of type σ'_i, σ''_i . We denote the braid equivalence by \sim .

2.3. Hurwitz spaces $H_C^{W(D_d)}(Y)$ and strategy

Let X, X' and Y be smooth, connected, projective complex curves and let g be the genus of Y . Let d, n, k be integers such that $d \geq 3$ and $n > k > 0$. In this paper we are interested in coverings with monodromy group $W(D_d)$ and with an arbitrary number of special points. Such coverings have $2d$ degree and decompose in a sequence of coverings, $X \xrightarrow{\pi} X' \xrightarrow{f} Y$, satisfying the followings:

π is a degree 2 étale covering and f is a degree d coverings, with monodromy group S_d and with n branch points, k of which are simple points and $n - k$ of which are special points.

Let $\underline{e}^1, \dots, \underline{e}^r$ be partitions of d such that $\underline{e}^i = (e^i_1, \dots, e^i_{s_i})$ and $e^i_1 \geq \dots \geq e^i_{s_i}$. Let q_1, \dots, q_r be positive integers such that $q_1 + \dots + q_r = n - k$. Let us denote by $H_C^{W(D_d)}(Y)$ the Hurwitz space of equivalence classes of coverings $f \circ \pi$, defined as above, such that q_i among the special points of f have local monodromy whose cycle type is given by the partition \underline{e}^i , with $i = 1, \dots, r$. Here, $C = (k, q_1 \underline{e}^1, \dots, q_r \underline{e}^r)$.

Our purpose is to study the irreducibility of the space $H_C^{W(D_d)}(Y)$. We notice that such space is smooth. So, if we prove that it is connected then we also prove that it is irreducible. We denote by D and by m , respectively, the branch locus and the monodromy homomorphism of $f \circ \pi$. Let $(\gamma_1, \dots, \gamma_n, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ be a standard generating system for $\pi_1(Y - D, b_0)$. The images via m of $\gamma_1, \dots, \gamma_n, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ determine an equivalence class of Hurwitz systems, $[t_1, \dots, t_n; \underline{\lambda}, \underline{\mu}]$, with monodromy group $W(D_d)$, satisfying the following conditions: k among the t_h are elements of type $(z_{ij}; (i, j))$ and q_i , with $i = 1, \dots, r$, are product of s_i positive disjoint cycles whose lengths are given by the elements of the partition \underline{e}^i . From now on, we will denote by $A_{C,g}$ the set of all equivalence classes of Hurwitz systems as above. We notice that by Riemann's existence theorem, we can identify the set of all the equivalence classes $[f \circ \pi] \in H_C^{W(D_d)}(Y)$ such that $f \circ \pi$ has branch locus D with the set $A_{C,g}$. Let

$$\delta : H_C^{W(D_d)}(Y) \rightarrow Y^{(n)} - \Delta$$

be the map which assigns to each equivalence class $[f \circ \pi]$ the branch locus of $f \circ \pi$. The topology defined on $H_C^{W(D_i)}(Y)$ is such that δ is a topological covering map (see [8]). Therefore the braid group $\pi_1(Y^{(n)} - \Delta, D)$ acts on $A_{C,g}$. The orbits of this action are in one-to-one correspondence with the connected components of $H_C^{W(D_i)}(Y)$. So, in order to prove that $H_C^{W(D_i)}(Y)$ is connected, we will prove that $\pi_1(Y^{(n)} - \Delta, D)$ acts transitively on $A_{C,g}$. We notice that in order to prove this it is sufficient to check that any class in $A_{C,g}$ is braid equivalent to a given normal form.

3. Action of $\pi_1(\mathbb{P}^n - \Delta, D)$ on $A_{C,0}$

In this section, we study the action of the braid group $\pi_1(\mathbb{P}^n - \Delta, D)$ on the set $A_{C,0}$. We give conditions for which this action is transitive.

Lemma 3.1. *Let $i \in \{1, 2, \dots, d - 1\}$. The sequence*

$$((\bar{1}_{1i}; (1, i)), (0; (1, i)), (\star; (1, i)), (\star; (1, i)), (\star; (i, i + 1)), (\star; (i, i + 1)))$$

is braid equivalent to

$$((\star; (i, i + 1)), (\star; (i, i + 1)), (\bar{1}_{1i+1}; (1, i + 1)), (0; (1, i + 1)), (\star; (1, i + 1)), (\star; (1, i + 1))).$$

Proof. Acting by the moves $\sigma''_4, \sigma''_3, \sigma''_2, \sigma'_1$, and if it is necessary with σ'_2 , we obtain the braid equivalent sequence

$$((\star; (i, i + 1)), (\bar{1}_{1i+1}; (1, i + 1)), (0; (1, i + 1)), (\star; (1, i + 1)), (\star; (1, i + 1)), (\star; (i, i + 1))).$$

Now, we use the moves $\sigma'_5, \sigma'_4, \sigma'_3, \sigma'_2$ and so we have got the claim. \square

Lemma 3.2. *The sequence $((\star; (1, i)), (\star; (1, i)), (\star; (i, i+1)), (\star; (i, i+1)))$, with $i \in \{1, 2, \dots, d-1\}$, is braid equivalent to the sequences*

$$((\star; (i, i + 1)), (\star; (i, i + 1)), (\star; (1, i + 1)), (\star; (1, i + 1)))$$

and

$$((\star; (1, i + 1)), (\star; (1, i + 1)), (\star; (i, i + 1)), (\star; (i, i + 1))).$$

Proof. We notice that acting by the braid moves $\sigma''_2, \sigma''_1, \sigma''_3, \sigma''_2$ on the second sequence, we can replace it with the third sequence. So, in order to prove the claim, we must only prove that the first sequence is braid equivalent to the second. We realize this equivalence by using the moves $\sigma''_2, \sigma''_1, \sigma'_3, \sigma'_2$. \square

Proposition 3.3. *The sequence*

$$((\bar{1}_{12}; (1, 2)), (0; (1, 2)), (\star; (1, 2)), (\star; (1, 2)), (\star; (2, 3)), (\star; (2, 3)), \dots, (\star; (d - 1, d)), (\star; (d - 1, d)))$$

is braid equivalent to

$$((\bar{1}_{ij}; (i, j)), (0; (i, j)), (\star; (1, 2)), (\star; (1, 2)), (\star; (2, 3)), (\star; (2, 3)), \dots, (\star; (d - 1, d)), (\star; (d - 1, d)))$$

where i, j are arbitrary indexes of the set $\{1, 2, \dots, d\}$ with $i < j$.

Proof. Let i and j be two indexes of the set $\{1, 2, \dots, d\}$ such that $i < j$. Separately, we analyze the cases $i \neq 1$ and $i = 1$. At first we suppose $i \neq 1$. By using Lemma 3.1, after $i - 2$ steps, we obtain the braid equivalent sequence

$$((\star; (2, 3)), (\star; (2, 3)), \dots, (\star; (i - 1, i)), (\star; (i - 1, i)), (\bar{1}_{1i}; (1, i)), (0; (1, i)), (\star; (1, i)), (\star; (1, i)), (\star; (i, i + 1)), (\star; (i, i + 1)), \dots, (\star; (d - 1, d)), (\star; (d - 1, d))).$$

We notice that using Lemma 3.2, after $j - i$ steps, we have that the sequence

$$((\star; (1, i)), (\star; (1, i)), (\star; (i, i + 1)), (\star; (i, i + 1)), \dots, (\star; (j - 1, j)), (\star; (j - 1, j)))$$

is braid equivalent to

$$((\star; (i, i + 1)), (\star; (i, i + 1)), \dots, (\star; (1, j)), (\star; (1, j)), (\star; (j - 1, j)), (\star; (j - 1, j))).$$

Let $h, h + 1, v, v + 1$ be the places occupied by the elements $(\bar{1}_{1i}; (1, i)), (0; (1, i)), (\star; (1, j)), (\star; (1, j))$, respectively. We act by the moves $\sigma'_{v-1}, \sigma'_v, \dots, \sigma'_{h+2}, \sigma'_{h+3}$ and we move the pair $((\star; (1, j)), (\star; (1, j)))$ at the places $h + 2, h + 3$. Now, we can use the moves $\sigma''_{h+1}, \sigma''_h, \sigma''_h, \sigma''_{h+1}$ in order to obtain the following equivalence

$$((\bar{1}_{1i}; (1, i)), (0; (1, i)), (\star; (1, j)), (\star; (1, j))) \sim ((\bar{1}_{ij}; (i, j)), (0; (i, j)), (\star; (1, j)), (\star; (1, j))).$$

We act with $\sigma''_{h-1}, \sigma''_h, \sigma''_{h-2}, \sigma''_{h-1}, \dots, \sigma''_2, \sigma''_1$ and we move the elements $(\bar{1}_{ij}; (i, j)), (0; (i, j))$ at first and second place. Since the sequence

$$((\star; (1, j)), (\star; (1, j)), (\star; (i, i + 1)), (\star; (i, i + 1)), \dots, (\star; (j - 1, j)), (\star; (j - 1, j)))$$

is braid equivalent to

$$((\star; (i, i + 1)), (\star; (i, i + 1)), \dots, (\star; (j - 1, j)), (\star; (j - 1, j)), (\star; (1, j)), (\star; (1, j))),$$

it is sufficient to use the Lemma 3.2 for $j - 2$ times in order to have the claim.

Now, let $i = 1$. We use the Lemma 3.1 for $j - 2$ times, then we move the pair $((\bar{1}_{1j}; (1, j)), (0; (1, j)))$ at first and second place and afterwards we use the Lemma 3.2 for $j - 2$ times. In this way, we have the claim. \square

In what follows, we associate to the partition \underline{e}^i the following element in S_d having cycle type given by \underline{e}^i

$$\varepsilon_i := (1, 2, \dots, e_1^i)(e_1^i + 1, \dots, e_1^i + e_2^i) \cdots \left(\sum_{j=1}^{s_i-1} e_j^i + 1 \dots d \right).$$

Let ε be the following permutation of S_d

$$(\varepsilon_1 \cdots \varepsilon_1 \varepsilon_2 \cdots \varepsilon_2 \cdots \varepsilon_r \cdots \varepsilon_r)^{-1}$$

where ε_i , with $i = 1, \dots, r$, appears q_i times. Let ξ_1, \dots, ξ_q be disjoint cycles of lengths h_1, \dots, h_q , with $h_1 \geq h_2 \geq \dots \geq h_q$, such that $\varepsilon = \xi_1 \cdots \xi_q$. Let $\xi_j = (l_1^j \dots l_{h_j}^j)$ where $l_1^j < l_b^j$ for each $b = 2, \dots, h_j$. In the sequel, we denote by Z_j the sequence of transpositions $((l_1^j, l_2^j), (l_1^j, l_3^j), \dots, (l_1^j, l_{h_j}^j))$ and by Z the concatenation Z_1, Z_2, \dots, Z_q . Moreover, we denote by \tilde{Z}_j the sequence $((0; (l_1^j, l_2^j)), (0; (l_1^j, l_3^j)), \dots, (0; (l_1^j, l_{h_j}^j)))$ and with \tilde{Z} the concatenation $\tilde{Z}_1, \tilde{Z}_2, \dots, \tilde{Z}_q$.

For a convenience of the reader we recall the following results.

Lemma 3.4. ([12], Main Lemma 2.1) Let $(t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$ be a Hurwitz system with values in an arbitrary group G . Suppose that $t_i t_{i+1} = id$. Let H be the subgroup of G generated by $\{t_1, \dots, t_{i-1}, t_{i+2}, \dots, t_n, \lambda_1, \mu_1, \dots, \lambda_g, \mu_g\}$. Then for every $h \in H$ the given Hurwitz system is braid equivalent to

$$(t_1, \dots, t_{i-1}, t_i^h, t_{i+1}^h, t_{i+2}, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g).$$

Proposition 3.5. ([14], Theorem 2.3) Let $[t_1, \dots, t_n]$ be an equivalence class of Hurwitz systems of permutations in S_d , with monodromy group S_d , satisfying the followings: k among the t_j are transpositions and q_i among the t_j are permutations whose cycle type is given by the partition \underline{e}^i of d , for $i = 1, \dots, r$. If $k \geq 3d - 3$, $[t_1, \dots, t_n]$ is braid equivalent to the class $[\tilde{t}_1, \dots, \tilde{t}_n]$ where

$$\tilde{t}_1 = \dots = \tilde{t}_{q_1} = \varepsilon_1, \quad \tilde{t}_{q_j+1} = \dots = \tilde{t}_{q_{j+1}} = \varepsilon_{j+1}$$

with $j = 1, \dots, r - 1$. Moreover the sequence $(\tilde{t}_{n-k+1}, \dots, \tilde{t}_n)$ is equal to

$$(Z, (1, 2), \dots, (1, 2), (2, 3), (2, 3), \dots, (d - 1, d), (d - 1, d))$$

where $(1, 2)$ appears an even number of times.

Now, by using the previous lemmas and the previous propositions, we can prove the following result.

Proposition 3.6. *Let $d \geq 3$ be an odd integer. If $k \geq 3d - 1$ then each class in A_{C_0} is braid equivalent to the class*

$$[(0; \tilde{t}_1), \dots, (0; \tilde{t}_{n-k}), \tilde{Z}, (0; (1, 2)), \dots, (0; (1, 2)), (\bar{1}_{12}; (1, 2)), (0; (1, 2)), (\bar{1}_{12}; (1, 2)), (0; (1, 2)), (0; (2, 3)), (0; (2, 3)), \dots, (0; (d - 1, d)), (0; (d - 1, d))],$$

where $\tilde{t}_1, \dots, \tilde{t}_{n-k}$ are the permutations defined in Proposition 3.5 and $(0; (1, 2))$ appears an even number of times.

Proof. Let $[t] := [t_1, \dots, t_n] \in A_{C_0}$ and let $t_i = (\star; t'_i)$. We notice that the equivalence class $[t'_1, \dots, t'_n]$ satisfies all the hypotheses in Proposition 3.5. So, by Proposition 3.5, $[t]$ is braid equivalent to

$$[(\star; \tilde{t}_1), \dots, (\star; \tilde{t}_{n-k}), (\star; Z), (\star; (12)), \dots, (\star; (d - 1, d))]$$

where, since $k \geq 3d - 1$, the elements of type $(\star; (1, 2))$ are at least four. So, it is no restrictive to suppose that among these elements there are both $(\bar{1}_{12}; (1, 2))$ and $(0; (1, 2))$. In fact, if the elements $(\star; (1, 2))$ are all equal among of them, we can use Lemma 3.4, choosing $h = (\bar{1}_{13}; id)$, in order to replace two of these elements with $(\bar{1}_{12}; (1, 2))$ or $(0; (1, 2))$ depending if \star is equal to 0 or $\bar{1}_{12}$. Then, by Proposition 3.3, we have that $[t]$ is braid equivalent to the class

$$[(a; \tilde{t}_1), \dots, (\star; \tilde{t}_{n-k}), (\star; Z), \dots, (\star; (1, 2)), (\bar{1}_{ij}; (i, j)), (0; (i, j)), (\star; (1, 2)), (\star; (1, 2)), \dots, (\star; (d - 1, d))]$$

where i and j are arbitrary indexes of the set $\{1, 2, \dots, d\}$. Now, we show that this class is braid equivalent to

$$[(0; \tilde{t}_1), \dots, (0; \tilde{t}_{q_1}), (\star; \tilde{t}_{q_1+1}), \dots, (\star; Z), \dots, (\star; (1, 2)), (\bar{1}_{12}; (1, 2)), (0; (1, 2)), (\star; (1, 2)), (\star; (1, 2)), \dots, (\star; (d-1, d))].$$

Separately, we analyze the case in which ε_1 fixes a index $h \in \{1, 2, \dots, d\}$ and the case in which ε_1 does not fix any index of $\{1, 2, \dots, d\}$.

First case: ε_1 fixes $h \in \{1, 2, \dots, d\}$.

Let i_1, i_2, \dots, i_l be the indexes which the function a sends to $\bar{1}$. We suppose that $i_1 < i_2 < \dots < i_{l-1} < i_l$. We choose $\{i, j\}$ equal to $\{i_l, h\}$ and so let $v, v + 1$ be the places occupied by the elements of the pair $((\bar{1}_{i_l h}; (i_l, h)), (0; (i_l, h)))$. By using the moves $\sigma''_{v-1}, \sigma''_v, \dots, \sigma''_1, \sigma''_2$, we bring the elements of this pair to first and second place. In this way, we replace $(a; \varepsilon_1)$ with $(\bar{a}; \varepsilon_1)$ where \bar{a} sends to $\bar{1}$ the indexes $i_1, i_2, \dots, i_{l-1}, i_l - 1$. Here, $i_l - 1$ is the index that precedes i_l in ε_1 . Now, we act by $\sigma''_2, \sigma''_1, \sigma'_3, \sigma'_2, \dots, \sigma'_v, \sigma'_{v-1}$ and so we move the pair $((\bar{1}_{i_l+1 h}; (i_l + 1, h)), (0; (i_l + 1, h)))$ next the element of type $(\star; (1, 2))$. By Proposition 3.3, we can replace this pair with $((\bar{1}_{i_l-1 h}; (i_l - 1, h)), (0; (i_l - 1, h)))$ or $((\bar{1}_{i_l-2 h}; (i_l-2, h)), (0; (i_l-2, h)))$ depending if $(i_l - 1) \neq i_{l-1}$ or $(i_l - 1) = i_{l-1}$. Then, we again reason as above. In this way, after a finite number of steps, we replace $(\bar{a}; \varepsilon_1)$ with $(0; \varepsilon_1)$. We can follow this line for all elements of type $(\star; \varepsilon_1)$ and so we obtain the claim.

Second case: ε_1 does not fix any index of the set $\{1, 2, \dots, d\}$.

Let v_1, \dots, v_{s_1} be disjoint cycles such that $\varepsilon_1 = v_1 \cdots v_{s_1}$. Let i_1, i_2, \dots, i_l be the indexes moved by v_i which a sends to $\bar{1}$ and let $i_1 < i_2 < \dots < i_l$. We can proceed as in the previous case but, this time, we use the pair $((\bar{1}_{i_l i_l-1}; (i_l, i_l - 1)), (0; (i_l - 1, i_l)))$ instead of $((\bar{1}_{i_l h}; (i_l, h)), (0; (i_l, h)))$. In this way, we replace $(a; \varepsilon_1)$ with $(\bar{a}; \varepsilon_1)$ where \bar{a} sends to $\bar{1}$ the indexes $i_1, i_2, \dots, i_{l-1}, i_l - 2$. We again proceed as done in the previous case but now we use the pair $((\bar{1}_{i_s i_s-1}; (i_s, i_s - 1)), (0; (i_s, i_s - 1)))$ where i_s is the greater among of the indexes moved by v_i that \bar{a} sends to $\bar{1}$. Following this line for indexes of any cycle v_i , after a finite number of steps, we replace $(\bar{a}; \varepsilon_1)$ with $(\bar{a}; \varepsilon_1)$ where \bar{a} is a function which sends to $\bar{0}$ either all the indexes moved by v_i or all but two consecutive indexes moved of v_i . Let $h, h + 1$ be the indexes of v_i that \bar{a} sends to $\bar{1}$. At first, we suppose that v_i moves an

odd number of indexes. Then, in order to replace \bar{n} with a function which sends to $\bar{0}$ all the indexes of v_i , it is sufficient to reason as above by using, in the following order, the pairs $((\bar{1}_{hh+2}; (h, h + 2)), (0; (h, h + 2))), ((\bar{1}_{h-1h+3}; (h - 1, h + 3)), (0; (h - 1, h + 3))), \dots, ((\bar{1}_{h-qh+q+2}; (h - q, h + q + 2)), (0; (h - q, h + q + 2)))$. Here, q is such that $(h + q + 2)$ is the index that precedes $h - q$ in ε_1 .

If v_i moves an even number of indexes, we choose one index l moved by a cycle v_e with an odd number of indexes. We notice that a such cycle there exists because d is odd and ε_1 does not fix any index of the set $\{1, 2, \dots, d\}$. Then, we use the pair $((\bar{1}_{h+1l}; (h + 1, l)), (0; (h + 1, l)))$ in order to replace \bar{n} with a function which sends to $\bar{0}$ all the indexes moved by v_i and sends to $\bar{1}$ the indexes $l, l - 1$. Now, we are in the previous situation and then we can proceed as above. Reasoning in this way for all elements of type $(\star; \varepsilon_1)$, we obtain the claim.

We can follow the previous reasoning for any $(\star; \varepsilon_j)$, with $j = 2, \dots, r$, and for any element in the sequence $(\star; Z)$. In this way, we obtain the braid equivalent sequence

$$[(0; \varepsilon_1), \dots, (0; \varepsilon_r), \tilde{Z}, (\star; (1, 2)), \dots, (\star; (1, 2)), (\bar{1}_{12}; (1, 2)), (0; (1, 2)), (\star; (2, 3)), \dots, (\star; (d - 1, d))].$$

We notice that

$$(0; \varepsilon_1) \cdots (0; \varepsilon_r) \tilde{Z} (\star; (1, 2)) \cdots (\star; (2, 3)) \cdots (\star; (d - 1, d)) = (0; id).$$

This implies that the elements of the pair $((\star; (i, i + 1)), (\star; (i, i + 1)))$ are equal among of them for each $i = 2, \dots, d - 1$. Moreover, the elements of type $(\bar{1}_{12}; (1, 2))$ are an even number.

If $(\star; (d - 1, d)) = (\bar{1}_{d-1d}; (d - 1, d))$, we use $d - 3$ times the Lemma 3.2, then we act by the moves $\sigma'_{n-2}, \sigma'_{n-3}, \sigma'_{n-1}, \sigma'_{n-2}, \sigma'_{n-2}, \sigma'_{n-3}, \sigma'_{n-1}, \sigma'_{n-2}$ and, afterwards, we use the Lemma 3.2 other $d - 3$ times. In this way, we replace the pair $((\bar{1}_{d-1d}; (d - 1, d)), (\bar{1}_{d-1d}; (d - 1, d)))$ with $((0; (d - 1, d)), (0; (d - 1, d)))$.

We reason as above for any pair $((\star; (i - 1, i)), (\star; (i - 1, i)))$, with $i = 2, \dots, d - 1$, such that $\star = \bar{1}$. Now, if in the sequence

$$((\star; (1, 2)), \dots, (\star; (1, 2)), (\bar{1}_{12}; (1, 2)), (0; (1, 2)), (0; (2, 3)), \dots, (0; (d - 1, d)))$$

there are more of two $(\bar{1}_{12}; (1, 2))$, we place them as follows

$$((0; (1, 2)), \dots, (0; (1, 2)), (\bar{1}_{12}; (1, 2)), \dots, (\bar{1}_{12}; (1, 2)), (0; (1, 2)), (0; (2, 3)), \dots, (0; (d - 1, d))).$$

Let $j = n - (2d - 4) + 1$ and let h be the place occupied, in the sequence above, by the first elements $(\bar{1}_{12}; (1, 2))$ that we see proceeding from left toward right. We act by the braid moves

$$\sigma''_{j-1}, \sigma''_{j-2}, \sigma''_{j-2}, \sigma''_{j-1}, \sigma''_{j-3}, \sigma''_{j-2}, \sigma''_{j-4}, \sigma''_{j-3}, \dots, \sigma''_{h+1}, \sigma''_{h+2}$$

and so we have that the sequence above is braid equivalent to

$$((0; (1, 2)), \dots, (0; (1, 2)), (\bar{1}_{12}; (1, 2)), (\bar{1}_{13}; (1, 3)), (0; (1, 3)), (0; (1, 2)), \dots, (0; (1, 2)), (\bar{1}_{23}; (2, 3)), (0; (2, 3)), \dots, (0; (d - 1, d)))$$

We obtain the claim by using the following moves

$$\sigma''_{h+2}, \sigma''_{h+1}, \sigma''_{h+3}, \sigma''_{h+2}, \dots, \sigma''_{j-2}, \sigma''_{j-3}, \sigma'_{j-1}, \sigma'_{j-2}, \sigma'_{j-2}, \sigma'_{j-1}, \sigma'_{h+1}, \sigma'_{h+1}, \dots, \sigma'_{j-5}.$$

□

Remark 3.7. We notice that Proposition 3.6 is also true for d even, if for each $i = 1, \dots, r$ one among of the following conditions is satisfied:

- ε_i fixes at least one index of the set $\{1, 2, \dots, d\}$;
- at least one among of the e_j^i , with $j = 1, \dots, s_i$, is an odd integer.

4. Main Result

In this short section, we state and prove the main result of this paper. Such result is a direct consequence of Proposition 3.6 and Remark 3.7.

Theorem 4.1. *Let $k \geq 3d - 1$. If d is odd or if d is even and the conditions of Remark 3.7 are satisfied, the Hurwitz space $H_C^{W(D_d)}(Y)$ is irreducible.*

Proof. The irreducibility of $H_C^{W(D_d)}(\mathbb{P}^1)$, under the hypothesis $k \geq 3d - 1$, follows by Proposition 3.6 and Remark 3.7, immediately. Thus, we suppose $g \geq 1$. We have got the claim if we prove that each equivalence class in $A_{C,g}$ is braid equivalente to a class of the form $[\bar{t}; (0; id), (0; id), \dots, (0; id), (0; id)]$. In fact, the class $[\bar{t}]$ belongs to $A_{C,0}$ and so the theorem follows by Proposition 3.6 and Remark 3.7.

Let $[\bar{t}; \underline{\lambda}, \underline{\mu}] \in A_{C,g}$ with $t_i = (*; t'_i)$, $\lambda_k = (*; \lambda'_k)$ and $\mu_k = (*; \mu'_k)$. We notice that $[t'_1, \dots, t'_n; \lambda'_1, \mu'_1, \dots, \lambda'_g, \mu'_g]$ is the equivalence class of Hurwitz systems associated to a degree $d \geq 3$ covering of Y , with monodromy group S_d , with k simple points and with $n - k$ special points, q_i among of which have local monodromies with cycle type given by the partition e^i of d . Since, under the hypothesis $k \geq 3d - 1$, the Hurwitz space parameterizing coverings as above is irreducible (see [26], Theorem 2), the equivalence class $[\bar{t}; \underline{\lambda}, \underline{\mu}]$ is braid equivalent to a class of the form

$$[\bar{t}_1, \dots, \bar{t}_n; (\star; id), (\star; id), \dots, (\star; id), (\star; id)].$$

Furthermore, following the proof of Proposition 3.6, we have that the above class is braid equivalent to

$$[\dots, (\bar{1}_{ij}; (i, j)), (0; (i, j)), \dots; (a_1; id), (b_1; id), \dots, (a_g; id), (b_g; id)]$$

where i, j are arbitrary indexes of $\{1, \dots, d\}$. Now, if $a_s = 0$ and $b_v = 0$ for each $1 \leq s, v \leq g$, the claim follows by Proposition 3.6 and Remark 3.7. So, we suppose $a_s \neq 0$ and $a_l = b_l = 0$, for each $l \leq s - 1$. We recall that all the functions a_v and b_v , with $v = 1, \dots, g$, different from zero send to $\bar{1}$ an even number of indexes (see Definition 2.1). So, let e, h be two among of the indexes that a_s sends to $\bar{1}$. We choose $\{i, j\}$ equal to $\{e, h\}$ and we bring, acting by moves of type σ''_* , to first and second place the elements of the pair $((\bar{1}_{eh}; (e, h)), (0; (e, h)))$. Now, we use the moves $\tau''_{1s}, \sigma''_1, \tau''_{1s}$ and we replace $(a_1; id)$ with $(\bar{1}_{eh}; id) (a_1; id)$ where $\bar{1}_{eh} + a_1$ is a function that sends e, h to $\bar{0}$.

Again, following the proof of Proposition 3.6, we obtain that our class is braid equivalent to a class of the form

$$[\dots, (\bar{1}_{ij}; (i, j)), (0; (i, j)), \dots; (0; id), \dots, (0; id), (a'_s; id), \dots, (a'_g; id), (b'_g; id)]$$

where a'_s is a function that sends to $\bar{1}$ the same number of indexes sent to $\bar{1}$ by $\bar{1}_{eh} + a_1$. Now, we can proceed as above. In this way, after a finite number of steps, we replace $(a_s; id)$ with $(0; id)$.

We notice that we reason in the same way, if $b_s \neq 0$, $a_l = b_l = 0$, for each $l \leq s - 1$, and $a_s = 0$ but we apply the braid move $\rho'_{1s}, \sigma''_1, \rho'_{1s}$ to transform $(b_s; id)$ into $(0; id)$. Following this line, for all the functions a_v and b_v different from zero, we obtain the claim. \square

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