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Generalized q-integrals via neutrices: Application to the q-beta function

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Abstract. Let $f(x)$ be a continuous function defined on the interval $[0, a]$. In this work, we apply the neutrix limit to generalize the q-integral

$$\int_0^a x^{\alpha-1} \ln^r x f(x) d_q x, \qquad r \in \mathbb{N}$$

for all values of $\alpha \in \mathbb{R}$. We use our results to extend the definition of the q-beta function $B_q(a, b)$ and its derivatives for all values of b and $a \neq 0, -1, -2, \cdots$. Some results for the q-gamma function are derived.

1. Introduction

Neutrices are additive groups of negligible functions that do not contain any constants except zero. Their calculus was developed by van der Corput [1] and Hadamard in connection with asymptotic series and divergent integrals. Recently, the concepts of neutrix and neutrix limit have been used widely in many applications in mathematics, physics and statistics. The technique of neglecting appropriately defined infinite quantities was devised by Hadamard and the resulting finite value extracted from the divergent integral is usually referred to as the Hadamard finite part. Fisher gave general principles for the discarding of unwanted infinite quantities from asymptotic expansions and this has been exploited in the context of distributions, particularly in connection with the composition of distributions, the product and the convolution product of distributions; see [2–4]. Fisher et al. [5–8] used neutrices to define some of the special function (see also [9–11]). Ng and van Dam [12, 13] applied the neutrix calculus, in conjunction with the Hadamard integral, developed by van der Corput, to quantum field theories, in particular, to obtain finite results for the coefficients in the perturbation series. They also applied neutrix calculus to quantum field theory, obtaining finite renormalization in the loop calculations.

In the second half of the twentieth century there was a significant increase of activity in the area of the q-calculus due to applications of the q-calculus in mathematics, statistics and physics. There are many of q-special functions have q-integral representations (q-gamma, q-beta, incomplete q-gamma, q-bessel,...etc). Their q-integral representations have a bounded domain of convergence and they are divergent outside this domain. Better yet, we should look for a mathematical tool that can handle the domain of convergence. Salem [14, 15] has suggested that such a tool is already available in the neutrix calculus developed by van

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der Corput [1]. He applied the neutrix limit to extend the definitions of the q-gamma and the incomplete q-gamma functions and their derivatives to negative integer values. Continuation on the same manner, Ege and Yýldýrým [16] used the neutrix and the neutrix limit to obtain some equalities of the q-gamma function for all real values of x. In this work we apply the neutrix limit to generalize q-integrals

$$\int_0^a x^{\alpha-1} \ln^r x f(x) d_q x \qquad r \in \mathbb{N}_0 \tag{1.1}$$

for all values of α, where $f(x)$ is a continuous function defined on the interval $[0, a]$.

2. Preliminaries and notations

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper. Throughout of this work, q is a positive number, $0 < q < 1$ and the definitions of q-calculus will be taken from the well known books in this field [17, 18].

Definition 2.1. (Notation in q-Calculus): For any complex number a, we define

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \neq 1; \quad [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \quad n \in \mathbb{N} \tag{2.1}$$

with $[0]_q! = 1$ and the q-shifted factorials are defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad n \in \mathbb{N}. \tag{2.2}$$

The limit, $\lim_{n\to\infty}(a;q)_n$, is denoted by $(a;q)_\infty$.

The exponential function e^x has many different q-extensions, one of them is defined as

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{[n]_q!} = (-(1-q)x;q)_\infty. \tag{2.3}$$

Let f be a function defined on a subset of real or complex plane. The q-difference operator is defined by the formula

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0, \quad \text{and} \quad (D_q f)(0) = f'(0) \tag{2.4}$$

provided f is differentiable at origin. The above operator is now sometimes referred to as Euler-Jackson, Jackson q-difference operator or simply the q-derivative.

The q-integration of Jackson is defined for a function f defined on a generic interval $[a, b]$ to be

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x, \tag{2.5}$$

where

$$\int_0^a f(x) d_q x = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n, \tag{2.6}$$

provided the sum converges absolutely.

The q-integrating by parts is given for suitable functions f and g by

$$\int_a^b f(x) D_q g(x) d_q x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) D_q f(x) d_q x. \tag{2.7}$$

We call the function f is q-integrable on the generic interval $[0, a]$ if the series $\sum_{n=0}^{\infty} q^n f(q^n a)$ converges absolutely.

Definition 2.2. (The q-Taylor series): Let $f(x)$ be a continuous function on some interval $[a,b]$ and $c \in (a,b)$. Jackson introduced the following q-counterpart of Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{[x-c]_n}{[n]_q!} D_q^n f(c), \qquad x \in (a,b) \tag{2.8}$$

where $[x-c]_n = \prod_{k=0}^{n-1}(x - cq^k)$ and $[x-c]_0 = 1$.

Rajkovic et al. [19] proved that if $f(x)$ is a continuous function on $[a,b]$ and $c \in (a,b)$, then there exists $\hat{q} \in (0,1)$ such that for all $q \in (\hat{q},1)$, $\xi \in (a,b)$ can be found between c and x, which satisfies

$$f(x) = \sum_{k=0}^{n-1} \frac{[x-c]_k}{[k]_q!} D_q^k f(c) + \frac{[x-c]_n}{[n]_q!} D_q^k f(\xi) \tag{2.9}$$

Moreover, if $c = 0$, we have for all $q \in (0,1)$ there exists $\xi \in (0,x)$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{x^k}{[k]_q!} D_q^k f(0) + \frac{x^n}{[n]_q!} D_q^k f(\xi) \tag{2.10}$$

Definition 2.3. (Neutrix): A neutrix N is defined as a commutative additive group of functions $f(\xi)$ defined on a domain N' with values in an additive group N'', where further if for some f in N, $f(\xi) = \gamma$ for all $\xi \in N'$, then $\gamma = 0$. The functions in N are called negligible functions.

Definition 2.4. (Neutrix limit): Let N be a set contained in a topological space with a limit point a which does not belong to N. If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant c such that $f(\xi) - c \in N$, then c is called the neutrix limit of f as ξ tends to a and we write $N - lim_{\xi \to a} f(\xi) = c$.

In this paper, we let N be the neutrix having domain $N' = \{\epsilon : 0 < \epsilon < \infty\}$ and range N'' the real numbers, with the negligible functions being finite linear sums of the functions

$$\epsilon^\lambda \ln^{r-1} \epsilon, \qquad \ln^r \epsilon \qquad (\lambda < 0, \quad r \in \mathbb{N}).$$

and all functions $o(\epsilon)$ which converge to zero in the normal sense as ϵ tends to zero [1].

3. Generalized q-integrals

Begin this section by the following lemmas, which help to prove our results.

Lemma 3.1. ([14]) *For all $a, b \neq 0$ and all values of α, we have*

$$\int_a^b x^{\alpha-1} \ln^r x d_q x = \begin{cases} \dfrac{(q-1)(\ln^{r+1} b - \ln^{r+1} a)}{(r+1)\ln q} - \dfrac{\ln^r q}{r+1} \displaystyle\sum_{k=0}^{r-1} \binom{r+1}{k} \ln^{-k} q \int_a^b \frac{\ln^k x}{x} d_q x, \alpha = 0 \\ \dfrac{b^\alpha \ln^r b - a^\alpha \ln^r a}{[\alpha]_q} + \dfrac{q^\alpha \ln^r q}{1-q^\alpha} \displaystyle\sum_{k=0}^{r-1} \binom{r}{k} \ln^{-k} q \int_a^b x^{\alpha-1} \ln^k x d_q x, \alpha \neq 0. \end{cases} \tag{3.1}$$

for $r \in \mathbb{N}$ and when $r = 0$, we have

$$\int_a^b x^{\alpha-1} d_q x = \begin{cases} \dfrac{(q-1)(\ln b - \ln a)}{\ln q}, & \alpha = 0 \\ \dfrac{b^\alpha - a^\alpha}{[\alpha]_q}, & \alpha \neq 0 \end{cases} \tag{3.2}$$

Lemma 3.2. *The neutrix limit as ϵ tends to zero of the q-integral*

$$\int_\epsilon^1 x^{\alpha-1}\ln^r x d_q x \tag{3.3}$$

exists for all values of $\alpha \neq 0$ and $r \in \mathbb{N}$ and

$$N-\lim_{\epsilon\to 0}\int_\epsilon^1 x^{\alpha-1}\ln^r x d_q x = \frac{q^\alpha \ln^r q}{(1-q^\alpha)[\alpha]_q}\sum_{k=1}^{r}\left(\frac{q^\alpha}{1-q^\alpha}\right)^{k-1}\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}i^r \tag{3.4}$$

Furthermore, if $q \to 1$, we get

$$N-\lim_{\epsilon\to 0}\int_\epsilon^1 x^{\alpha-1}\ln^r x dx = \frac{(-1)^r}{\alpha^{r+1}}\sum_{i=1}^{r}(-1)^{r-i}\binom{r}{i}i^r = \frac{(-1)^r r!}{\alpha^{r+1}} \tag{3.5}$$

Proof. From Lemma 3.1, for all values of $\alpha \neq 0$ and $r \in \mathbb{N}$, we have

$$N-\lim_{\epsilon\to 0}\int_\epsilon^1 x^{\alpha-1}\ln^r x d_q x = \frac{q^\alpha \ln^r q}{1-q^\alpha}\sum_{k=0}^{r-1}\binom{r}{k}\ln^{-k} q N-\lim_{\epsilon\to 0}\int_\epsilon^1 x^{\alpha-1}\ln^k x d_q x$$

Using the mathematical induction would yield

$$\begin{aligned}
N-\lim_{\epsilon\to 0}\int_\epsilon^1 x^{\alpha-1}\ln^r x d_q x &= \frac{q^\alpha \ln^r q}{(1-q^\alpha)[\alpha]_q}\Bigg[1+\frac{q^\alpha}{1-q^\alpha}\sum_{i=1}^{r-1}\binom{r}{i}+\left(\frac{q^\alpha}{1-q^\alpha}\right)^2\sum_{i=2}^{r-1}\binom{r}{i}\sum_{k_1=1}^{i-1}\binom{i}{k_1}\\
&+\cdots+\left(\frac{q^\alpha}{1-q^\alpha}\right)^{r-1}\sum_{i=r-1}^{r-1}\binom{r}{i}\sum_{k_1=1}^{i-1}\binom{i}{k_1}\sum_{k_2=1}^{k_1-1}\binom{k_1}{k_2}\cdots\sum_{k_{r-2}=1}^{k_{r-1}-1}\binom{k_{r-1}}{k_{r-2}}\Bigg]\\
&= \frac{q^\alpha \ln^r q}{(1-q^\alpha)[\alpha]_q}\Bigg[1+\frac{q^\alpha}{1-q^\alpha}\sum_{i=1}^{2}(-1)^{2-i}\binom{2}{i}i^r+\left(\frac{q^\alpha}{1-q^\alpha}\right)^2\sum_{i=1}^{3}(-1)^{3-i}\binom{3}{i}i^r\\
&+\cdots+\left(\frac{q^\alpha}{1-q^\alpha}\right)^{r-1}\sum_{i=1}^{r}(-1)^{r-i}\binom{r}{i}i^r\Bigg]\\
&= \frac{q^\alpha \ln^r q}{(1-q^\alpha)[\alpha]_q}\sum_{k=1}^{r}\left(\frac{q^\alpha}{1-q^\alpha}\right)^{k-1}\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}i^r
\end{aligned}$$

By taking the limit of equation (3.4) as $q \to 1$ would give easily the equation (3.5). □

Lemma 3.3. *The induction yields*

$$N-\lim_{\epsilon\to 0}\int_\epsilon^1 x^{-1}\ln^r x d_q x = 0, \qquad r \in \mathbb{N}_0. \tag{3.6}$$

Theorem 3.4. *Let $f(x)$ be a continuous function on the closed interval $[0,a]$, then the function $x^{\alpha-1}\ln^r x f(x)$ is a q-integrable on $[0,a]$ for $\alpha > 0$ and $r \in \mathbb{N}_0$. Furthermore, for $\alpha > -m, m \in \mathbb{N}$, we have the function*

$$x^{\alpha-1}\ln^r x\left(f(x)-\sum_{n=0}^{m-1}\frac{D_q^n f(0)}{[n]_q!}x^n\right), \quad r \in \mathbb{N}_0 \tag{3.7}$$

is q-integrable on $[0,a]$.

Proof. Since $f(x)$ is a continuous function on the segment $[0,a]$, the extreme value theorem states, it attains its maximum M, i.e. there exists $c \in [0,a]$ such that $f(x) \leq f(c) = M$ for all $x \in [0,a]$. It follows that

$$\int_0^a x^{\alpha-1} \ln^r x f(x) d_q t \leq M \int_0^a x^{\alpha-1} \ln^r x d_q t = (1-q) a^\alpha M \sum_{n=0}^{\infty} q^{\alpha n} (n \ln q + a)^r$$

converges absolutely for $\alpha > 0, r \in \mathbb{N}_0$.

Since $f(x)$ is a continuous function on the segment $[0,a]$, then $f(x)$ has remainder term in q-Taylor formula (2.10). This would yield

$$\int_0^a x^{\alpha-1} \ln^r x \left(f(x) - \sum_{n=0}^{m-1} \frac{D_q^n f(0)}{[n]_q!} x^n \right) = \frac{D_q^m f(\xi)}{[m]_q!} \int_0^a x^{\alpha+m-1} \ln^r x d_q x, \qquad 0 < \xi < a$$
$$= (1-q) a^{\alpha+m} \frac{D_q^m f(\xi)}{[m]_q!} \sum_{n=0}^{\infty} q^{n(\alpha+m)} (n \ln q + a)^r$$

converges absolutely for $\alpha > -m, m \in \mathbb{N}$ and $r \in \mathbb{N}_0$. □

Theorem 3.5. *Let $f(x)$ be a continuous function defined on $[0,a]$. Then the neutrix limit as ϵ tends to zero of the q-integral*

$$\int_\epsilon^a x^{\alpha-1} f(x) d_q x \tag{3.8}$$

exists for $\alpha > -m,\ \ m = 1,2,,...,\ \ \alpha \neq 0,-1,-2,...,-m+1$ and

$$N - \lim_{\epsilon \to 0} \int_\epsilon^a x^{\alpha-1} f(x) d_q x = \int_0^a x^{\alpha-1} \left(f(x) - \sum_{n=0}^{m-1} \frac{D_q^n f(0)}{[n]_q!} x^n \right) d_q x + \sum_{n=0}^{m-1} \frac{D_q^n f(0) a^\alpha}{[\alpha+n]_q [n]_q!}. \tag{3.9}$$

Proof. We have

$$\int_\epsilon^a x^{\alpha-1} f(x) d_q x = \int_\epsilon^a x^{\alpha-1} \left(f(x) - \sum_{n=0}^{m-1} \frac{D_q^n f(0)}{[n]_q!} x^n \right) d_q x + \sum_{n=0}^{m-1} \frac{D_q^n f(0)}{[n]_q!} \int_\epsilon^a x^{\alpha+n-1} d_q x$$
$$= \int_\epsilon^a x^{\alpha-1} \left(f(x) - \sum_{n=0}^{m-1} \frac{D_q^n f(0)}{[n]_q!} x^n \right) d_q x + \sum_{n=0}^{m-1} \frac{D_q^n f(0) a^\alpha}{[\alpha+n]_q [n]_q!}$$
$$- \sum_{n=0}^{m-1} \frac{D_q^n f(0)}{[n]_q!} \frac{\epsilon^{\alpha+n}}{[\alpha+n]_q}, \qquad \alpha \neq 0,-1,-2,\cdots,-m+1$$

Since $f(x)$ is a continuous function on the segment $[0,a]$, Theorem 3.4 tells that the first q-integral on the right hand side converges absolutely as $\epsilon \to 0$ and the last sum consists of terms tend to zero and the rest is (linear sum of $\epsilon^\lambda, \lambda < 0$) negligible function and hence the proof is complete. □

Theorem 3.6. *Let $f(x)$ be a continuous function defined on $[0,a]$. Then the neutrix limit as ϵ tends to zero of the q-integral*

$$\int_\epsilon^a x^{\alpha-1} \ln^r x f(x) d_q x, \qquad r \in \mathbb{N} \tag{3.10}$$

exists for $\alpha > -m,\ \ m = 1,2,,...,\ \ \alpha \neq 0,-1,-2,...,-m+1$

Proof. The q-integral (3.10) can be splitted as

$$\int_\epsilon^a x^{\alpha-1}\ln^r x f(x) d_qx = \int_\epsilon^a x^{\alpha-1}\ln^r x\left(f(x) - \sum_{n=0}^{m-1}\frac{D_q^n f(0)}{[n]_q!}x^n\right)d_qx$$
$$+\sum_{n=0}^{m-1}\frac{D_q^n f(0)}{[n]_q!}\int_1^a x^{\alpha+n-1}\ln^r x d_qx + \sum_{n=0}^{m-1}\frac{D_q^n f(0)}{[n]_q!}\int_\epsilon^1 x^{\alpha+n-1}\ln^r x d_qx$$

It is obvious that the second q-integral converges absolutely for all values of α, r and for $\alpha > -m, m \in \mathbb{N}$ and $r \in \mathbb{N}$ as $\epsilon \to 0$, the first q-integral converges absolutely by using Theorem 3.4. Also the Lemma 3.2 tells that the neutrix limit of the last q-integral exists as $\epsilon \to 0$. This completes the proof. □

Theorem 3.7. *Let $f(x)$ be a continuous function defined on the interval $[0, a]$. Then the function*

$$x^{-m-1}\ln^r x\left(f(x) - \sum_{n=0}^{m}\frac{D_q^n f(0)}{[n]_q!}x^n\right) \tag{3.11}$$

is q-integrable on $[0, a]$ for $m \in \mathbb{N}_0$ and $r \in \mathbb{N}_0$.

Proof. Since $f(x)$ is a continuous function on the segment $[0, a]$, it satisfies (2.10). It follows that

$$\int_0^a x^{-m-1}\ln^r x\left(f(x) - \sum_{n=0}^{m}\frac{D_q^n f(0)}{[n]_q!}x^n\right) = \frac{D_q^{m+1}f(\xi)}{[m+1]_q!}\int_0^a \ln^r x d_qx, \quad 0 < \xi < a$$
$$= (1-q)a\frac{D_q^{m+1}f(\xi)}{[m+1]_q!}\sum_{n=0}^{\infty} q^n(n\ln q + a)^r$$

converges absolutely for $m \in \mathbb{N}_0$ and $r \in \mathbb{N}_0$. □

Theorem 3.8. *Let $f(x)$ be a continuous function defined on the interval $[0, a]$. Then*

$$N - \lim_{\epsilon\to 0}\int_\epsilon^a x^{-1}\ln^r x f(x) d_qx \tag{3.12}$$

exists for $r \in \mathbb{N}_0$.

Proof. The q-integral (3.12) can be splitted as

$$\int_\epsilon^a x^{-1}\ln^r x f(x) d_qx = \int_\epsilon^a x^{-1}\ln^r x[f(x) - f(0)]d_qx$$
$$+ f(0)\int_1^a x^{-1}\ln^r x d_qx + f(0)\int_\epsilon^1 x^{-1}\ln^r x d_qx$$

The limit of the first q-integral on the right hand side as $\epsilon \to 0$ is being convergent by the previous Theorem. Lemma 3.3 proves that the neutrix limit of the last q-integral exists for $r \in \mathbb{N}_0$. □

Theorem 3.9. *Let $f(x)$ be a continuous function defined on the interval $[0, a]$. Then*

$$N - \lim_{\epsilon\to 0}\int_\epsilon^a x^{-m-1}\ln^r x f(x) d_qx \tag{3.13}$$

exists for $m \in \mathbb{N}$ and $r \in \mathbb{N}_0$.

Proof. The q-integral (3.13) can be splitted as

$$\begin{aligned}\int_{\epsilon}^{a} x^{-m-1}\ln^r x f(x)d_qx &= \int_{\epsilon}^{a} x^{-m-1}\ln^r x\left(f(x)-\sum_{n=0}^{m}\frac{D_q^n f(0)}{[n]_q!}x^n\right)d_qx\\ &+\sum_{n=0}^{m-1}\frac{D_q^n f(0)}{[n]_q!}\int_1^a x^{n-m-1}\ln^r x d_qx+\frac{D_q^m f(0)}{[m]_q!}\int_1^a x^{-1}\ln^r x d_qx\\ &+\sum_{n=0}^{m-1}\frac{D_q^n f(0)}{[n]_q!}\int_{\epsilon}^1 x^{n-m-1}\ln^r x d_qx+\frac{D_q^m f(0)}{[m]_q!}\int_{\epsilon}^1 x^{-1}\ln^r x d_qx\end{aligned}$$

The limit of the first q-integral on the right hand side as $\epsilon \to 0$ is being convergent by Theorem 3.7 for all values of $m \in \mathbb{N}$ and $r \in \mathbb{N}_0$. The second and third integrals converge absolutely for all m, r, using Lemma 3.1. Further, from Lemmas 3.2 and 3.3, we see that the neutrix limit of the q-integral in the last two terms on the right hand side exist. This ends the proof. □

The previous theorems can be outlined as follow:

Theorem 3.10. *Let $f(x)$ be a continuous function defined on the interval $[0, a]$. Then*

$$N-\lim_{\epsilon\to 0}\int_{\epsilon}^{a} x^{\alpha-1}\ln^r x f(x)d_qx \tag{3.14}$$

exists for all real values of α and $r \in \mathbb{N}_0$.

4. On the q-gamma function and its derivatives

The q-gamma function is defined by q-integral representation [17, 18]

$$\Gamma_q(x)=\int_0^{\frac{1}{1-q}} t^{x-1}E_q(-qt)d_qt, \qquad x>0. \tag{4.1}$$

Moreover, it has the recurrence relation

$$\Gamma_q(x+1)=[x]_q\Gamma(x). \tag{4.2}$$

The q-gamma function can be defined for $x < 0$ and $x \neq 0, -1, -2, \cdots$ by using the former recurrence formula. In particular, it follows that if $x > -1$ and $x \neq 0$, then

$$\Gamma_q(x)=[x]_q^{-1}\Gamma_q(x+1)=-[x]_q^{-1}\int_0^{\frac{1}{1-q}} t^x D_q\left(E_q(-t)-1\right)d_qt \tag{4.3}$$

and by q-integrating by parts we have

$$\Gamma_q(x)=\int_0^{\frac{1}{1-q}} t^{x-1}\left(E_q(-qt)-1\right)d_qt+\frac{(1-q)^{1-x}}{1-q^x}. \tag{4.4}$$

More generally, it is easy to prove by mathematical induction that if $x > -n, n \in \mathbb{N}$ and $x \neq 0, -1, -2, \cdots, -n+1$, the q-gamma function has the form

$$\Gamma_q(x)=\int_0^{\frac{1}{1-q}} t^{x-1}\left(E_q(-qt)-\sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}t^i}{[i]_q!}\right)d_qt+(1-q)^{1-x}\sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{(q;q)_i(1-q^{i+x})}. \tag{4.5}$$

It has been shown in [14] that the q-gamma function (4.1) is defined by the neutrix limit

$$\Gamma_q(x) = N - \lim_{\epsilon\to 0}\int_{\epsilon}^{\frac{1}{1-q}} t^{x-1}E_q(-qt)d_qt, \tag{4.6}$$

for $x \neq 0,-1,-2,...,$ and this function is also defined by neutrix limit

$$\Gamma_q(-m) = N - \lim_{\epsilon\to 0}\int_{\epsilon}^{\frac{1}{1-q}} t^{-m-1}E_q(-qt)d_qt = \int_{0}^{\frac{1}{1-q}} t^{-m-1}\left[E_q(-qt) - \sum_{i=0}^{m}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}t^i\right]d_qt$$
$$+ (1-q)^{m+1}\sum_{i=0}^{m-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{(q;q)_i(1-q^{i-m})} + \frac{(-1)^m q^{\frac{m(m+1)}{2}}(1-q)^{m+1}\ln(1-q)}{(q;q)_m\ln q}. \tag{4.7}$$

for $m\in\mathbb{N}$ and when $m=0$ it has the form

$$\Gamma_q(0) = N - \lim_{\epsilon\to 0}\int_{\epsilon}^{\frac{1}{1-q}} t^{-1}E_q(-qt)d_qt = \int_{0}^{\frac{1}{1-q}} t^{-1}\left[E_q(-qt)-1\right]d_qt + \frac{(1-q)\ln(1-q)}{\ln q}. \tag{4.8}$$

It was proved also in [14] the existence of rth derivative of the q-gamma function and defined it for all values of x by the equation

$$\Gamma_q^{(r)}(x) = N - \lim_{\epsilon\to 0}\int_{\epsilon}^{\frac{1}{1-q}} t^{x-1}\ln^r tE_q(-qt)d_qt \qquad r=0,1,2\cdots. \tag{4.9}$$

The author in [14] gave the form of the rth derivatives of the q-gamma function at special cases. We use our results in Lemmas 3.2 and 3.3 to give a general form for rth derivatives of the q-gamma function as follows.

- For $x>-n, n\in\mathbb{N}$ and $r\in\mathbb{N}$

$$\begin{aligned}\Gamma_q^{(r)}(x) &= N - \lim_{\epsilon\to 0}\int_{\epsilon}^{\frac{1}{1-q}} t^{x-1}\ln^r tE_q(-qt)d_qt\\ &= \int_{1}^{\frac{1}{1-q}} t^{x-1}\ln^r tE_q(-qt)d_qt + \int_0^1 t^{x-1}\ln^r t\left(E_q(-qt)d_qt - \sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}t^i\right)\\ &+ \frac{\ln^r q}{1-q}\sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}+i+x}}{[x+i]_q^2[i]_q!}\sum_{k=1}^{r}\left(\frac{q^{x+i}}{1-q^{x+i}}\right)^{k-1}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^r.\end{aligned} \tag{4.10}$$

- For $x=0$ and $r\in\mathbb{N}_0$

$$\begin{aligned}\Gamma_q^{(r)}(0) &= N - \lim_{\epsilon\to 0}\int_{\epsilon}^{\frac{1}{1-q}} t^{-1}\ln^r tE_q(-qt)d_qt\\ &= \int_{1}^{\frac{1}{1-q}} t^{-1}\ln^r tE_q(-qt)d_qt + \int_0^1 t^{-1}\ln^r t\left[E_q(-qt)-1\right]d_qt.\end{aligned} \tag{4.11}$$

- For $x=-n, n\in\mathbb{N}$ and $r\in\mathbb{N}$

$$\begin{aligned}\Gamma_q^{(r)}(-n) &= N - \lim_{\epsilon\to 0}\int_{\epsilon}^{\frac{1}{1-q}} t^{-n-1}\ln^r tE_q(-qt)d_qt\\ &= \int_{1}^{\frac{1}{1-q}} t^{-n-1}\ln^r tE_q(-qt)d_qt + \int_0^1 t^{-n-1}\ln^r t\left(E_q(-qt)d_qt - \sum_{i=0}^{n}\frac{(-1)^i q^{\frac{i(i+1)}{2}}}{[i]_q!}t^i\right)\\ &+ \frac{\ln^r q}{1-q}\sum_{i=0}^{n-1}\frac{(-1)^i q^{\frac{i(i+1)}{2}+i-n}}{[i-n]_q^2[i]_q!}\sum_{k=1}^{r}\left(\frac{q^{i-n}}{1-q^{i-n}}\right)^{k-1}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^r.\end{aligned} \tag{4.12}$$

5. The q-beta function and its derivatives

The q-beta function is defined by q-integral representation [17, 18]

$$B_q(a,b) = \int_0^1 x^{b-1} \frac{(qx;q)_\infty}{(q^a x;q)_\infty} d_q x, \quad b > 0, a \neq 0, -1, -2, ... \tag{5.1}$$

The q-binomial theorem would yield the function

$$f(x) = \frac{(qx;q)_\infty}{(q^a x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(q^{1-a};q)_n q^{na}}{(q;q)_n} x^n, \qquad |x| < q^{-a} \tag{5.2}$$

is a continuous function on the interval [0,1] so we can apply our results in section three to define the q-beta function and its derivatives for all values of b as follow:

Definition 5.1. The q-analogue of the beta function is defined for all values of b and $a \neq 0, -1, -2, \cdots$ by

$$B_q(a,b) = N - \lim_{\epsilon \to 0} \int_\epsilon^1 x^{b-1} \frac{(qx;q)_\infty}{(q^a x;q)_\infty} d_q x. \tag{5.3}$$

If we want to make the details of this definition more precise, by using Lemmas 3.2 and 3.3, we redefine as follow:

- For $b > -m, m \in \mathbb{N}; b \neq 0, -1, -2, \cdots, -m+1$ and $a \neq\in \mathbb{N}_0$

$$\begin{aligned} B_q(a,b) &= N - \lim_{\epsilon \to 0} \int_\epsilon^1 x^{b-1} \frac{(qx;q)_\infty}{(q^a x;q)_\infty} d_q x \\ &= \int_0^1 x^{b-1} \left(\frac{(qx;q)_\infty}{(q^a x;q)_\infty} - \sum_{n=0}^{m-1} \frac{(q^{1-a};q)_n q^{na}}{(q;q)_n} x^n \right) d_q x + \sum_{n=0}^{m-1} \frac{(q^{1-a};q)_n q^{na}}{[n+b]_q (q;q)_n}. \end{aligned} \tag{5.4}$$

- For $b = 0$ and $a \neq 0, -1, -2, \cdots$

$$B_q(a,0) = N - \lim_{\epsilon \to 0} \int_\epsilon^1 x^{-1} \frac{(qx;q)_\infty}{(q^a x;q)_\infty} d_q x = \int_0^1 x^{-1} \left(\frac{(qx;q)_\infty}{(q^a x;q)_\infty} - 1 \right) d_q x. \tag{5.5}$$

- For $b = -m, m \in \mathbb{N}$ and $a \neq 0, -1, -2, \cdots$

$$\begin{aligned} B_q(a,-m) &= N - \lim_{\epsilon \to 0} \int_\epsilon^1 x^{-m-1} \frac{(qx;q)_\infty}{(q^a x;q)_\infty} d_q x \\ &= \int_0^1 x^{-m-1} \left(\frac{(qx;q)_\infty}{(q^a x;q)_\infty} - \sum_{n=0}^{m} \frac{(q^{1-a};q)_n q^{na}}{(q;q)_n} x^n \right) d_q x + \sum_{n=0}^{m-1} \frac{(q^{1-a};q)_n q^{na}}{[n-m]_q (q;q)_n}. \end{aligned} \tag{5.6}$$

Definition 5.2. The *rth* partial derivative with respect to b of the q-beta function is defined for all values of b and $a \neq 0, -1, -2, \cdots$ as

$$\frac{\partial^r}{\partial b^r} \left(B_q(a,b) \right) = N - \lim_{\epsilon \to 0} \int_\epsilon^1 x^{b-1} \ln^r x \frac{(qx;q)_\infty}{(q^a x;q)_\infty} d_q x, \qquad r \in \mathbb{N}_0. \tag{5.7}$$

Also, the details of this definition can be organized as follow:

- For $b > -m, m \in \mathbb{N}; b \neq 0, -1, -2, \cdots, -m+1; a \neq 0, -1, -2, \cdots$ and $r \in \mathbb{N}$

$$\begin{aligned}\frac{\partial^r}{\partial b^r}\left(B_q(a,b)\right) &= N-\lim_{\epsilon\to 0}\int_\epsilon^1 x^{b-1}\ln^r x\frac{(qx;q)_\infty}{(q^ax;q)_\infty}d_qx\\ &= \int_0^1 x^{b-1}\ln^r x\left(\frac{(qx;q)_\infty}{(q^ax;q)_\infty}-\sum_{n=0}^{m-1}\frac{(q^{1-a};q)_nq^{na}}{(q;q)_n}x^n\right)d_qx\\ &+\frac{\ln^r q}{1-q}\sum_{n=0}^{m-1}\frac{(q^{1-a};q)_nq^{(a+1)n+b}}{[n+b]_q^2(q;q)_n}\sum_{k=1}^{r}\left(\frac{q^{n+b}}{1-q^{n+b}}\right)^{k-1}\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}i^r.\end{aligned}\tag{5.8}$$

- For $b = 0; a \neq 0, -1, -2, \cdots$ and $r \in \mathbb{N}_0$

$$\frac{\partial^r}{\partial b^r}\left(B_q(a,b)\right)|_{b=0} = N-\lim_{\epsilon\to 0}\int_\epsilon^1 x^{-1}\ln^r x\frac{(qx;q)_\infty}{(q^ax;q)_\infty}d_qx = \int_0^1 x^{-1}\ln^r x\left(\frac{(qx;q)_\infty}{(q^ax;q)_\infty}-1\right)d_qx.\tag{5.9}$$

- For $b = -m, m \in \mathbb{N}; a \neq 0, -1, -2, \cdots$ and $r \in \mathbb{N}$

$$\begin{aligned}\frac{\partial^r}{\partial b^r}\left(B_q(a,b)\right)|_{b=-m} &= N-\lim_{\epsilon\to 0}\int_\epsilon^1 x^{-m-1}\ln^r x\frac{(qx;q)_\infty}{(q^ax;q)_\infty}d_qx\\ &= \int_0^1 x^{-m-1}\ln^r x\left(\frac{(qx;q)_\infty}{(q^ax;q)_\infty}-\sum_{n=0}^{m}\frac{(q^{1-a};q)_nq^{na}}{(q;q)_n}x^n\right)d_qx\\ &+\frac{\ln^r q}{1-q}\sum_{n=0}^{m-1}\frac{(q^{1-a};q)_nq^{(a+1)n-m}}{[n-m]_q^2(q;q)_n}\sum_{k=1}^{r}\left(\frac{q^{n-m}}{1-q^{n-m}}\right)^{k-1}\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}i^r.\end{aligned}\tag{5.10}$$

Remark 5.3. It is known that the neutrix limit, if it exists, is unique and it is precisely the same as the ordinary limit, if it exists. We can verify this by using the equation (5.8) as follows

$$\begin{aligned}B_q(a,b) &= \int_0^1 x^{b-1}\left(\frac{(qx;q)_\infty}{(q^ax;q)_\infty}-\sum_{n=0}^{m-1}\frac{(q^{1-a};q)_nq^{na}}{(q;q)_n}x^n\right)d_qx+\sum_{n=0}^{m-1}\frac{(q^{1-a};q)_nq^{na}}{[n+b]_q(q;q)_n}\\ &= \sum_{n=0}^{\infty}\frac{(q^{1-a};q)_nq^{na}}{[n+b]_q(q;q)_n} = \frac{1}{[b]_q}\sum_{n=0}^{\infty}\frac{(q^{1-a};q)_n(q^b;q)_nq^{na}}{(q;q)_n(q^{b+1};q)_n}\\ &= {}_2\phi_1(q^{1-a},q^b;q^{b+1};q,q^a)\end{aligned}$$

where ${}_2\phi_1$ is the basic Gauss hypergeometric function which it has the well known identity

$${}_2\phi_1(a,b;c;q,a^{-1}b^{-1}c) = \frac{(a^{-1}c;q)_\infty(b^{-1}c;q)_\infty}{(c;q)_\infty(a^{-1}b^{-1}c;q)_\infty}\qquad\left|\frac{c}{ab}\right|<1.$$

Using this identity would yield the well known definition of the q-beta function [16]

$$B_q(a,b) = \frac{(1-q)(q;q)_\infty(q^{a+b};q)_\infty}{(q^a;q)_\infty(q^b;q)_\infty} = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)},\quad a,b\neq 0,-1,-2,\cdots.\tag{5.11}$$

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