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LIPSCHITZ CONTINUITY OF THE DISTANCE RATIO METRIC ON THE UNIT DISK

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Abstract. For maps of analytic functions $f : D \to D, f(0) = a \in D$, we prove their Lipschitz continuity regarding the distance ratio metric with a Lipschitz constant C depending only on a. Question of the best possible $C = C(a)$ remains open.

1. Introduction

For a subdomain $G \subset \mathbb{R}^n$ and for all $x, y \in G$ the distance ratio metric j_G is defined as

$$j_G(x, y) = \log\left(1 + \frac{|x-y|}{\min\{d(x, \partial G), d(y, \partial G)\}}\right),$$

where $d(x, \partial G)$ denotes the Euclidean distance from x to ∂G. The distance ratio metric was introduced by F.W. Gehring and B.P. Palka [3] and in the above simplified form by M. Vuorinen [9]. As the "first approximation" of the quasihyperbolic metric, it is frequently used in the study of hyperbolic type metrics ([1],[2],[6],[10]) and geometric theory of functions.

For an open continuous mapping $f : G \to G'$ we consider the following condition: there exists a constant $C \geq 1$ such that for all $x, y \in G$ we have

$$j_{G'}(f(x), f(y)) \leq C j_G(x, y),$$

or, equivalently, that the mapping

$$f : (G, j_G) \to (G', j_{G'})$$

between metric spaces is Lipschitz continuous with the Lipschitz constant C.

However, neither the quasihyperbolic metric k_G nor the distance ratio metric j_G are invariant under Möbius transformations. Therefore, it is natural to ask what the Lipschitz constants are for these metrics under conformal mappings or Möbius transformations in higher dimension. F. W. Gehring, B. P. Palka and B. G. Osgood proved that these metrics are not changed by more than a factor 2 under Möbius transformations, see [2], [3]:

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Theorem A *If G and G' are proper subdomains of $\mathbb{R}^n$ and if f is a Möbius transformation of G onto G', then for all $x, y \in G$*

$$m_{G'}(f(x), f(y)) \leq 2m_G(x, y),$$

where $m \in \{j, k\}$.

On the other hand, the next theorem from [7], conjectured in [6], yields a sharp form of Theorem A for Möbius automorphisms of the unit ball.

Theorem B *A Möbius transformation $f : \mathbb{B}^n \to \mathbb{B}^n$, $f(0) = a$, satisfies*

$$j_{\mathbb{B}^n}(f(x), f(y)) \leq (1 + |a|) j_{\mathbb{B}^n}(x, y)$$

for all $x, y \in \mathbb{B}^n$. The constant is best possible.

An interesting problem is to investigate Lipschitz continuity of the distance-ratio metric under some other conformal mappings.
In this paper we obtain results concerning analytic mappings of the open unit disk D into itself. For example, for an analytic f, $f : D \to D$, supposing boundedness of l_1 norm of its Maclorain coefficients, we proved Lipschitz continuity with best possible Lipschitz constant $C = 1$ (Theorem 1, below).

Our second result concerns arbitrary analytic mappings f, normalized by $f(0) = a \in D$. Using heavily the famous Schwarz-Pick lemma we succeed to prove Lipschitz continuity in this case also, with Lipschitz constant C depending only on a. However, question of the best possible $C = C(a)$ remains open.

2. Results

We give firstly a sufficient condition for an analytic mapping to be a contraction, that is to have the Lipschitz constant at most 1.

Theorem 1 *Let $f : D \to D$ be a non-constant mapping given by $f(z) = \sum_{k=0}^{\infty} a_k z^k$, under the condition*

$$\sum_{k=0}^{\infty} |a_k| \leq 1. \tag{$*$}$$

Then for all x, $y \in D$,

$$j_D(f(x), f(y)) \leq j_D(x, y),$$

and this inequality is sharp.

Proof Suppose that $|f(x)| \geq |f(y)|$. Then

$$j_D(x, y) = \log(1 + \frac{|x - y|}{\min\{1 - |x|, 1 - |y|\}})$$

and

$$j_D(f(x), f(y)) = \log(1 + \frac{|f(x) - f(y)|}{1 - |f(x)|}).$$

We have

$$|f(x) - f(y)| = |x - y| |\sum_{k=1}^{\infty} a_k (\sum_{i+j=k-1} x^i y^j)| \leq |x - y| \sum_{k=1}^{\infty} |a_k| (\sum_{i=0}^{k-1} |x|^i)$$

and

$$1 - |f(x)| \geq \sum_{k=1}^{\infty} |a_k| - \sum_{k=1}^{\infty} |a_k||x|^k = (1 - |x|) \sum_{k=1}^{\infty} |a_k|(\sum_{i=0}^{k-1} |x|^i).$$

Hence,

$$j_D(f(x), f(y)) \leq \log(1 + \frac{|x - y|}{1 - |x|}) \leq j_D(x, y).$$

For the sharpness of the inequality let $a_p = 1, a_i = 0, i \neq p$, i.e., $f(z) = z^p$ ($p \in \mathbb{N}$). For s, $t \in (0, 1)$ and $s < t$, we have

$$\frac{j_D(f(t), f(s))}{j_D(t, s)} = \frac{\log \frac{1-s^p}{1-t^p}}{\log \frac{1-s}{1-t}} = \frac{\log \frac{1-s}{1-t} + \log \frac{1+s+\cdots+s^{p-1}}{1+t+\cdots+t^{p-1}}}{\log \frac{1-s}{1-t}}.$$

Letting $t \to 1^-$ we obtain $C = 1$. Therefore this constant is sharp.

Note that the condition ($*$) is sufficient for f to map D into itself but is not necessary at all. Consider for example $f(z) = z + \frac{(1-z)^3}{4}$.

Therefore, the next result is of more general nature.

Theorem 2 *For any analytic function f, $f : D \to D$, $f(0) = a$ and all $z, w \in D$, we have*

$$j_D(f(z), f(w)) \leq C j_D(z, w),$$

where $C = C(a) = \min\{2(1 + |a|), \sqrt{5 + 2|a| + |a|^2}\}$.

Proof

Let $\max\{|z|, |w|\} = r$ and suppose that $|f(z)| \geq |f(w)|$. Then

$$j_D(z, w) = \log(1 + \frac{|z - w|}{1 - r}); \; j_D(f(z), f(w)) = \log(1 + \frac{|f(z) - f(w)|}{1 - |f(z)|}).$$

Our main tool in the proof will be the famous Schwarz-Pick lemma, given by

Theorem C *Let f be an analytic mapping of the unit disk D into itself. Then for any $z_1, z_2 \in D$, we have*

$$\left|\frac{f(z_2) - f(z_1)}{1 - \bar{f}(z_1) f(z_2)}\right| \leq \left|\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right|.$$

Since $f(0) = a$, applying this lemma with $z_1 = 0, z_2 = z$, we obtain an estimation for $|f(z)|, z \in D$, i.e.,

$$|z| \geq |z| \geq \frac{|f(z) - a|}{|1 - \bar{a} f(z)|} \geq \frac{|f(z)| - |a|}{1 - |a||f(z)|},$$

that is,

$$|f(z)| \leq \frac{|z| + |a|}{1 + |a||z|} \leq \frac{r + |a|}{1 + |a|r} \tag{2.1}$$

Further application gives

$$\left|\frac{f(z)-f(w)}{1-\bar{f}(z)f(w)}\right|^{-2}-1\geq\left|\frac{z-w}{1-\bar{z}w}\right|^{-2}-1,$$

that is,

$$\begin{aligned}\left|\frac{f(z)-f(w)}{z-w}\right|^{2}&\leq\frac{(1-|f(z)|^2)(1-|f(w)|^2)}{(1-|z|^2)(1-|w|^2)}\\&\leq\frac{(1-|f(z)|^2)(1-|f(w)|^2)}{(1-r^2)^2},\end{aligned}$$

where we used the well known identity for complex numbers x, y,

$$|1-\bar{x}y|^2-|x-y|^2=(1-|x|^2)(1-|y|^2).$$

Therefore,

$$\frac{|f(z)-f(w)|}{1-|f(z)|}\leq\frac{|z-w|}{1-r}\frac{\sqrt{(1+|f(z)|)(1+|f(w)|)}}{1+r}\sqrt{\frac{1-|f(w)|}{1-|f(z)|}},$$

i.e.,

$$\frac{|f(z)-f(w)|}{1-|f(z)|}\leq\frac{|z-w|}{1-r}\frac{1+|f(z)|}{1+r}\sqrt{1+\frac{|f(z)-f(w)|}{1-|f(z)|}},\tag{2.2}$$

since

$$\frac{1-|f(w)|}{1-|f(z)|}=1+\frac{|f(z)|-|f(w)|}{1-|f(z)|}\leq1+\frac{|f(z)-f(w)|}{1-|f(z)|}.$$

Also, by (2.1) we get $\frac{1+|f(z)|}{1+r}\leq 1+|a|:=2c$ and, denoting $\frac{|z-w|}{1-r}:=X\in[0,+\infty)$, the inequality (2.2) gives

$$1+\frac{|f(z)-f(w)|}{1-|f(z)|}\leq(cX+\sqrt{1+c^2X^2})^2.$$

Therefore

$$\frac{j_D(f(z),f(w))}{j_D(z,w)}\leq\frac{2\log(cX+\sqrt{1+c^2X^2})}{\log(1+X)}\leq\frac{2\log(1+2cX)}{\log(1+X)}\leq 4c=2(1+|a|):=C_1,$$

by the Bernoulli inequality. The second constant $C_2=\sqrt{5+2|a|+|a|^2}$ is obtained from the inequality

$$\log(cX+\sqrt{1+c^2X^2})\leq\sqrt{1+c^2}\log(1+X),$$

which can be proved in the standard way. Namely, denote

$$g(X):=\log(cX+\sqrt{1+c^2X^2})-\sqrt{1+c^2}\log(1+X),\ X\in[0,\infty).$$

Then

$$g'(X)=\frac{c}{\sqrt{1+c^2X^2}}-\frac{\sqrt{1+c^2}}{1+X}=\frac{-(cX-1)^2}{(1+X)\sqrt{1+c^2X^2}(c(1+X)+\sqrt{(1+c^2)(1+c^2X^2)})}.$$

Hence the function g is non-increasing and $g(X) \leq g(0) = 0$.

Finally, it follows that $C = \min\{C_1, C_2\}$.

Although the constant C can be improved a little bit by the above method, an intriguing question is to determine the best possible constant $C^* = C^*(a)$ such that the inequality

$$\frac{j_D(f(z), f(w))}{j_D(z, w)} \leq C^*(a)$$

holds for any analytic mapping $f,\ f : D \to D,\ f(0) = a$ and all $z, w \in D$.

Our result and Theorem B give

$$1 + |a| \leq C^*(a) \leq \min\{2(1 + |a|),\ \sqrt{5 + 2|a| + |a|^2}\}.$$

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