

A new convex relaxation for quadratically constrained quadratic programming

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Abstract. A new relaxation strategy is presented in this paper to approximately solve the quadratically and linearly constrained quadratic programming. To improve the conservation of traditional semidefinite relaxation (SDR) strategy, we introduce a new linear constraint, which can be derived from the constraints of original problem, to the SDR problem. Furthermore, a randomization method is provided to extract good feasible solution of original problem from optimal solution of relaxed problem. Some numerical examples show that the proposed method can efficiently improve the performance of the traditional SDR strategy.

1. Introduction

Consider the quadratically constrained quadratic programming (QCQP):

$$\begin{aligned} \min_x \quad & x^T P_0 x + 2b_0^T x + d_0 \\ \text{s.t.} \quad & x^T P_i x + 2b_i^T x + d_i \leq 0, \quad i = 1, \dots, l, \end{aligned} \tag{1}$$

where $x, b_i \in \mathbb{R}^n$ and $d_i \in \mathbb{R}$. Note that P_i can be indefinite, so the above problem includes all optimization problems with polynomial objective function and polynomial constraints [1, 2]. If all the matrices P_i are positive semidefinite, then the QCQP problem (1) is convex and can be efficiently solved to the global optimum. However, if one of the P_i is indefinite, then the QCQP is non-convex in general and is computationally difficult to solve. Therefore, it is of practical importance to develop tractable lower bounds on the optimal value and derive good (but not necessarily optimal) feasible solutions to problem (1). An

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important lower bound was given by Shor’s relaxation in [3]:

$$\begin{aligned} & \max_{\gamma_i, t} t & (2) \\ & \text{s.t. } \begin{bmatrix} P_0 & b_0 \\ b_0^T & d_0 - t \end{bmatrix} + \gamma_1 \begin{bmatrix} P_1 & b_1 \\ b_1^T & d_1 - t \end{bmatrix} + \cdots + \gamma_l \begin{bmatrix} P_l & b_l \\ b_l^T & d_l - t \end{bmatrix} \leq 0, \\ & \gamma_i \geq 0, i = 1, \dots, l, \end{aligned}$$

which is a positive semidefinite programming (SDP) with variables t and γ_i . It is shown in [4] that problem (2) is the dual of the following optimization problem:

$$\min_{(x, \Delta) \in \mathcal{A}} \text{Tr}(P_0 \Delta) + 2b_0^T x + d_0, \tag{3}$$

where

$$\begin{aligned} \mathcal{A} &= \{(x, \Delta) \mid \text{Tr}(P_i \Delta) + 2b_i^T x + d_i \leq 0, i = 1, \dots, l, \Delta \geq xx^T\} \\ &= \left\{ (x, \Delta) \mid \text{Tr}(P_i \Delta) + 2b_i^T x + d_i \leq 0, i = 1, \dots, l, \begin{bmatrix} \Delta & x \\ x^T & 1 \end{bmatrix} \geq 0 \right\}. \end{aligned}$$

The problem (3) is traditionally called as the semidefinite relaxation (SDR) of problem (1), and it is easy to verify that both of the two problems yield the same lower bound for (1).

In recent years, the SDP relaxations for combinatorial optimization problems and non-convex QCQPs have attracted much attention due to the remarkable development of interior-point methods for SDP problems [4–7]. In fact, it has been applied to deal with a lot of important engineering problems, which can be cast in the form of a non-convex QCQP or fractional QCQP, in signal processing and communications [8]. Roughly speaking, the SDR is a powerful, computationally efficient approximation technique for a host of very difficult optimization problems [9].

Note that problem (1) is the same as

$$\min_{(x, \Delta) \in \mathcal{O}} \text{Tr}(P_0 \Delta) + 2b_0^T x + d_0,$$

where $\mathcal{O} = \{(x, \Delta) \mid \text{Tr}(P_i \Delta) + 2b_i^T x + d_i \leq 0, i = 1, \dots, l, \Delta = xx^T\}$, we can regard problem (3) as the relaxation of problem (1) by directly replacing non-convex constraint $\Delta = xx^T$ with convex constraint $\Delta \geq xx^T$.

If we are further given linear constraints on the variable x , such as $0 \leq x \leq e$, where e is the vector with all components being 1, then problem (1) can also be relaxed to a convex problem by utilizing the well-known “ α BB” underestimator [10] and the reformulation-linearization technique (RLT) [11]. The α BB underestimator gives a convex relaxation of the original non-convex problem by replacing all non-convex terms of special structure with customized tight convex lower bounding functions. A generalization of α BB procedure is proposed in [12] to approximately solve a class of non-convex quadratic programs with a non-convex quadratic objective function and convex quadratic constraints. The RLT utilizes the bound constraints on x to derive new convex constraints. However, the RLT doesn’t utilize the available knowledge of matrix Δ from the condition $\Delta = xx^T$, therefore, some methods have been proposed to strengthen the RLT technique. The SDR is combined with the RLT in [13] to derive convex relaxation for the QCQP, and the test problems provided in [13] show that the use of SDP and RLT constraints together can produce bounds that are substantially better than either technique used alone. Kim and Kojima [14] proposed a second-order cone programming relaxation to strengthen the lift-and-project linear programming relaxation by adding convex quadratic valid inequalities derived from the semidefinite condition $\Delta \geq xx^T$. The tighter bounds were presented in [15] by adding linear inequalities implied by $\Delta \geq xx^T$ to the RLT relaxation. The comparison results of convex relaxations for the problem of minimizing a quadratic objective subject to linear and quadratic constraints are proposed in [16].

In this paper, we consider the following quadratic programming with quadratic and linear constraints:

$$\begin{aligned} \min_x \quad & x^T P_0 x + 2b_0^T x + d_0 \\ \text{s.t.} \quad & x^T P_i x + 2b_i^T x + d_i \leq 0, \quad i = 1, \dots, l, \\ & x \geq 0. \end{aligned} \tag{4}$$

Note that the QCQP problem is invariable when we employ an affine transformation to x , the linear constraint $x \geq 0$ is very general, including any case in which the lower bound on x is given. Obviously, problem (4) is intractable in general. However, we can use SDR method to derive the approximate value of problem (4). In order to improve the lower bound, we introduce a convex linear constraint, which is derived from the constraints of original problem, to the SDR of problem (4), then a novel semidefinite relaxation with linear constraint (SDRLC) is provided to approximately solve the original NP-hard problem (4). Of course, the optimal solution of the relaxed problem is not necessary feasible to original problem, so it is a fundamental issue how to convert the globally optimal solution of relaxed problem into a feasible solution of original problem. Recently, the algorithm based on randomization method is being widely employed in science research. The algorithm employs a degree of randomness as part of its logic, and typically uses uniformly random bits as an auxiliary input to guide its behavior, in the hope of achieving good performance in the "average case" over all possible choices of random bits. The randomization method has been proved to be a very efficient way to derive feasible solution of non-convex QCQP problem from its SDR relaxation [9, 17], therefore, it provides us a good strategy to extract an approximate QCQP solution from SDRLC relaxation.

The remainder of this paper is organized as follows. In Section 2, the SDRLC method is presented. The randomized method is adopted to derive a good feasible solution of the QCQP using SDRLC strategy in Section 3. Section 4 provides some numerical examples to compare the presented relaxation strategy with traditional relaxation strategy. A conclusion is given in Section 5.

2. Semidefinite relaxation with linear constraint complement

Note that problem (4) is equivalent to

$$\begin{aligned} \min_{x, \Delta} \quad & \text{Tr}(P_0 \Delta) + 2b_0^T x + d_0 \\ \text{s.t.} \quad & \text{Tr}(P_i \Delta) + 2b_i^T x + d_i \leq 0, \quad i = 1, \dots, l, \\ & \Delta = xx^T, \quad x \geq 0. \end{aligned} \tag{5}$$

Let

$$\begin{aligned} C &= \{(x, \Delta) | \text{Tr}(P_i \Delta) + 2b_i^T x + d_i \leq 0, \quad i = 1, \dots, l, \Delta = xx^T, \quad x \geq 0\}, \\ \widetilde{C} &= \{(x, \Delta) | \text{Tr}(P_i \Delta) + 2b_i^T x + d_i \leq 0, \quad i = 1, \dots, l, \Delta \geq xx^T, \quad x \geq 0\}. \end{aligned}$$

For any $(x, \Delta) \in C$, we have

$$\text{Tr}(\Delta) = x^T x \leq \left(\max_{x \in C} \|x\|_\infty \right) \sum_{j=1}^n x_j = \left(\max_{x \in C} \|x\|_\infty \right) e^T x.$$

Denote

$$\alpha \triangleq \max_{x \in C} \|x\|_\infty = \max_{x \in C} \{e_1^T x, \dots, e_n^T x\}, \tag{6}$$

where e_j ($j = 1, \dots, n$) be column vectors with all elements being zero except the j -th being 1, then the optimization problem (6) is convex. Thus, from $C \subseteq \widetilde{C}$, problem (5) can be rewritten as

$$\begin{aligned} \min_{x, \Delta} \quad & \text{Tr}(P_0\Delta) + 2b_0^T x + d_0 \\ \text{s.t.} \quad & \text{Tr}(P_i\Delta) + 2b_i^T x + d_i \leq 0, \quad i = 1, \dots, l, \\ & \Delta = xx^T, \quad x \geq 0, \quad \text{Tr}(\Delta) \leq ae^T x. \end{aligned} \tag{7}$$

Since the unique non-convex constraint in (7) is $\Delta = xx^T$, one can directly relax it to the convex constraint $\Delta \geq xx^T$, then the following convex relaxation of problem (7) is derived:

$$\begin{aligned} \min_{x, \Delta} \quad & \text{Tr}(P_0\Delta) + 2b_0^T x + d_0 \\ \text{s.t.} \quad & \text{Tr}(P_i\Delta) + 2b_i^T x + d_i \leq 0, \quad i = 1, \dots, l, \\ & \Delta \geq xx^T, \quad x \geq 0, \quad \text{Tr}(\Delta) \leq ae^T x. \end{aligned} \tag{8}$$

In rest of the paper, we call (8) as the SDRLC problem of (7). Obviously, problem (8) is an SDP, which can be solved by many available SDP solvers.

Remark 1. The linear constraint in problem (4) can be generalized to the case $a \leq x \leq b$, $x \geq a$ or $x \leq b$, which correspond to the constraint $0 \leq x \leq c$, $x \geq 0$ or $x \leq 0$ respectively since the QCQP problem is affine invariable. For the case $x \leq 0$, a similar procedure gives SDRLC of problem (4).

Theorem 1. The optimal value of problem (8) gives a lower bound on the objective value of problem (4), and the lower bound is larger or equal to the one provided by the SDR method.

Proof. Note that the feasible set of problem (7) is a subset of that of problem (8), therefore, the optimal value of problem (4) is not less than that of problem (8). The rest proof follows from the fact that an additional linear constraint is used in the SDRLC strategy compared with the SDR strategy. \square

Remark 2. With an additional linear constrain, the presented method performs better than the SDR methods. However, the parameter α involved in the linear constrain need to be given by solving the optimization problem (6), thus, the computational load of the presented method is higher than that of the SDR method. Fortunately, the computational complexity of problem (6) is not high, therefore, the SDRLC can be a good alternative strategy of the SDR.

Example 1. Consider the following QCQP problem:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \quad i = 1, 2, 3, \\ & x \geq 0, \end{aligned} \tag{9}$$

where,

$$\begin{aligned} f(x) &= -x_1^2 + 5x_2^2 - 20x_1x_2 + 4x_1 + 20x_2, \\ f_1(x) &= 2x_1^2 + 5x_2^2 - 2x_1x_2 + 5x_1 + 4x_2 - 15, \\ f_2(x) &= 2x_1^2 + x_2^2 + 2x_1x_2 - 6x_1 - 4x_2 - 10. \end{aligned}$$

The optimization problem (9) is non-convex owing to the non-convex objective function. Employing different relaxed strategies, the optimal values of problem (9) and corresponding relaxation problems are given as follows:

	original	SDR	SDRLC	SDR + RTL	SDRLC + RTL
objective	-1.1758	-40.4623	-9.1096	-40.4623	-3.2334

As we have expected, the SDRLC approach can provide more accurate approximation of the QCQP problem, and improve the lower bound given by SDR remarkably. At the same time, combined with RLT technique, the proposed method can also provide a tensor approximation of the QCQP problem.

3. Extraction of approximate solution

In Section 2, we propose a relaxation method to approximately solve quadratic programming (4). If the optimal solution $(\hat{x}, \hat{\Delta})$ of problem (8) satisfies $\hat{\Delta} = \hat{x}\hat{x}^T$, i.e., $\hat{\Delta}$ is rank-one, then \hat{x} must be a feasible (in fact optimal) solution of problem (4). Otherwise, the optimal solution of the relaxed problem is not necessary feasible to the original problem, then some efficient methods must be taken to extract feasible solution of original problem from $(\hat{x}, \hat{\Delta})$. However, it must be emphasized that even the extracted solution is feasible to original problem, it is in general not an optimal solution.

Obviously, if P_i ($i = 1, \dots, l$) are positive semidefinite, then the optimal solution to the SDRLC problem is also feasible to problem (4). However, when there is at least one P_i in the constraints of problem (4) is indefinite, the above fact is not true. So it is necessary to seek for the general method to extract an approximate QCQP solution from the optimal solution of relaxed problem. In [9] and [18], some methods, including rank-one approximation and randomization, have been proposed to derive the feasible solutions of the QCQP problem from SDR solution. As pointed out in [9], randomization is a more efficient but equivalently simple method. In the sequel, we use the randomized method to extract feasible solution of problem (4) from the optimal solution of the relaxed problem (8).

Note that $x^T x \leq ae^T x$ holds for any feasible solution x of problem (4), therefore, problem (4) can be recast as

$$\begin{aligned} \min_x \quad & x^T P_0 x + 2b_0^T x + d_0 \\ \text{s.t.} \quad & x^T P_i x + 2b_i^T x + d_i \leq 0, \quad i = 1, \dots, l, \\ & x^T x \leq ae^T x, \quad x \geq 0. \end{aligned} \tag{10}$$

Let $\zeta \in \mathbb{R}^n$ be a Gaussian random vector with mean x and covariance matrix $\Delta - xx^T$, i.e., $\zeta \sim N(x, \Delta - xx^T)$. Consider the following stochastic optimization problem:

$$\begin{aligned} \min_{x, \Delta} \quad & E(\zeta^T P_0 \zeta + 2b_0^T \zeta + d_0) \\ \text{s.t.} \quad & E(\zeta^T P_i \zeta + 2b_i^T \zeta + d_i) \leq 0, \quad i = 1, \dots, l, \\ & E(\zeta^T \zeta - ae^T \zeta) \leq 0, \\ & E(\zeta) \geq 0. \end{aligned} \tag{11}$$

For the above optimization problem, we choose the parameters x and Δ to minimize the expected value of the quadratic objective while the quadratic and linear constraints are satisfied in expectation. Problem (11) implies that ζ solves problem (10) (i.e., problem (4)) “in expectation”. It is easy to derive that Problem (11) is equivalent to

$$\begin{aligned} \min_{x, \Delta} \quad & E[\text{Tr}(P_0 \zeta \zeta^T) + 2b_0^T \zeta + d_0] \\ \text{s.t.} \quad & E[\text{Tr}(P_i \zeta \zeta^T) + 2b_i^T \zeta + d_i] \leq 0, \quad i = 1, \dots, l, \\ & E[\text{Tr}(\zeta \zeta^T) - ae^T \zeta] \leq 0, \\ & E(\zeta) \geq 0. \end{aligned} \tag{12}$$

Direct computation shows that $E(\zeta \zeta^T) = \Delta$ and $E(\zeta) = x$, together with the fact that $\Delta - xx^T = \text{Var}(\zeta) \geq 0$, therefore, the stochastic optimization problem (12) can be rewritten as the SDRLC problem (8). It provides us a way to derive approximate solutions to problem (4). In fact, after obtaining the optimal solution (x_0, Δ_0) of problem (8), we can pick a random vector ζ from the Gaussian distribution $N(x_0, \Delta_0 - x_0 x_0^T)$, and then

project it to the feasible region of problem (4). The procedure can be performed multiple times and one can pick only the best approximate solution. Because ζ solves problem (4) in expectation, so it suggests that a good approximate solution $\hat{\zeta}$ can be obtained by sampling enough times from the Gaussian distribution $N(x_0, \Delta_0 - x_0 x_0^T)$. Of course, for some special classes of the QCQP problems, the randomization procedure may be simpler, and the projection method can be also different from problem to problem.

4. Some examples

In Section 2, we have proved that the SDRLC method can provide better approximation of the QCQP problem than SDR method. A natural problem is whether or not, by combining a randomization procedure, the SDRLC method can give a better feasible solution of original problem than the SDR method. It is difficult to give an extensive theoretical analysis, however, a lot of numerical examples illustrate that the answer is yes (at least in average). In this section, using randomized method, we provide some examples to compare approximate performance of the relaxation strategies.

Example 2. Consider the following class of QCQP problems:

$$\begin{aligned} \max_x \quad & \|x\|_2^2 \\ \text{s.t.} \quad & x^T \bar{P}_i x \leq 1, \quad i = 1, \dots, l, \end{aligned} \tag{13}$$

where $x \in \mathbb{R}^n$ and $\bar{P}_i \geq 0$ for all $i = 1, \dots, l$. To guarantee that problem (13) makes sense, we require $\sum_{i=1}^l \bar{P}_i > 0$.

The maximization QCQP problem (13), opposed to the downlink transmit beamforming problem in communications, is motivated by the (uplink) receiver intercept beamforming problem in which the base station, equipped with an antenna array, is capable of suppressing signals impinging from irrelevant or hostile emitters, and meanwhile achieving as high a gain as possible for desired signals [8].

The constraints of problem (13) imply that the feasible region of the problem must be bounded. Thus, we can find an appropriate vector a satisfying $x \geq a$ for all x , which is feasible to problem (13). Let $\bar{x} = x - a$, then we can rewrite problem (13) as follows:

$$\begin{aligned} \max_{\bar{x}} \quad & [\bar{x}^T \ 1] \begin{bmatrix} I_n & a \\ a^T & \|a\|^2 \end{bmatrix} \begin{bmatrix} \bar{x} \\ 1 \end{bmatrix} \\ \text{s.t.} \quad & [\bar{x}^T \ 1] \begin{bmatrix} \bar{P}_i & \bar{P}_i a \\ (\bar{P}_i a)^T & a^T \bar{P}_i a \end{bmatrix} \begin{bmatrix} \bar{x} \\ 1 \end{bmatrix} \leq 1, \quad i = 1, \dots, l, \\ & [\bar{x}^T \ 1] \geq 0. \end{aligned}$$

Therefore, without loss of generality, we can recast problem (13) as:

$$\begin{aligned} \max_x \quad & x^T C x \\ \text{s.t.} \quad & x^T P_i x \leq 1, \quad i = 1, \dots, l, \\ & x \geq 0, \end{aligned} \tag{14}$$

where C and P_i are real symmetric matrices. Using the proposed relaxation method, problem (14) can be relaxed to

$$\begin{aligned} \max_{x, \Delta} \quad & \text{Tr}(C\Delta) \\ \text{s.t.} \quad & \text{Tr}(P_i \Delta) \leq 1, \quad i = 1, \dots, l, \\ & \text{Tr}(\Delta) \leq \alpha_1 e^T x, \\ & \Delta \geq x x^T, \quad x \geq 0, \end{aligned} \tag{15}$$

where α_1 is the optimal value of problem (6) by taking

$$\tilde{C} = \{(x, \Delta) | \text{Tr}(P_i \Delta) \leq 1, i = 1, \dots, l, \Delta \geq xx^T, x \geq 0\}.$$

Since the relaxation problem (15) is not tight in general, one can use the randomization approach to extract feasible solution of original problem (14). Let (x_0, Δ_0) be the optimal solution of problem (15), then the extraction procedure can be finished through the following steps:

- (1) Pick random vectors ζ_k ($k = 1, \dots, L$) from Gaussian distribution $N(x_0, \Delta_0 - x_0 x_0^T)$, then introduce new variable $\bar{\zeta}_k$ as follows: for j -th component of $\bar{\zeta}_k$, let

$$\bar{\zeta}_k(j) = \begin{cases} \zeta_k(j), & \text{if } \zeta_k(j) \geq 0; \\ 0, & \text{otherwise,} \end{cases} \quad j = 1, \dots, n, k = 1, \dots, L.$$

- (2) Denote $x(\bar{\zeta}_k) = \bar{\zeta}_k / \max_{1 \leq i \leq l} \sqrt{|\bar{\zeta}_k^T P_i \bar{\zeta}_k|}$.

- (3) Let

$$\hat{x} = \arg \max_{1 \leq k \leq L} x(\bar{\zeta}_k)^T C x(\bar{\zeta}_k)$$

be the approximate solution to problem (14).

Denote the optimal value of the considered maximization QCQP problem by v_{qp} , and the optimal values of corresponding SDR and SDRLC problems by v_{sdr} and v_{slc} respectively, then we have

$$v_{sdp} \geq v_{slc} \geq v_{qp}.$$

Furthermore, denote the objective function of the QCQP problem by $v(x)$, and the extracted approximate solutions to the QCQP problem with SDR and SDRLC method by \hat{x}_{sdr} and \hat{x}_{slc} respectively, then

$$v(\hat{x}_{sdr}) \leq v_{qp} \text{ and } v(\hat{x}_{slc}) \leq v_{qp},$$

therefore,

$$v_{sdr} \geq v_{qp} \geq v(\hat{x}_{sdr}) \text{ and } v_{slc} \geq v_{qp} \geq v(\hat{x}_{slc}).$$

The above inequalities give two interval estimates of v_{qp} corresponding to the SDR and SDRLC relaxation strategies. Since v_{qp} is not available in practice, to compare the two relaxation methods, we consider the lengths $v_{sdr} - v(\hat{x}_{sdr})$ and $v_{slc} - v(\hat{x}_{slc})$ of the relevant interval estimates, named empirical approximate gaps (EAGs). A smaller EAG implies a better relaxation method for the corresponding QCQP problem.

Simulation 1. Consider problem (14) with $n = 4$ and $l = 8$. Let $C = I$ and $P_i = p_i p_i^T$, where each component of p_i ($i = 1, \dots, l$) is independent and identically distributed Gaussian random variable with zero mean and unit variance. Using the randomization procedure with $L = 1000$, the simulation results of SDR and SDRLC methods for problem (14) running 200 independent trials are presented in Table 1.

	mean	min	max	frequency					
				[0, 0.3)	[0.3, 0.6)	[0.6, 1)	[1, 2)	[2, 5)	[5, 17)
SDR	1.8957	0.0068	16.0526	15%	10%	16.5%	23.5%	30%	5%
SDRLC	0.4079	0.0030	1.8167	40.5%	41.5%	13.5%	4.5%	0	0

Table 1: The comparison of the EAGs in the case of positive semidefinite P_i .

As shown in Table 1, for the 200 instances, the average and maximum EAGs of SDR method are 4.6 and 8.8 times larger than those of SDRLC method respectively. Moreover, the percentage of the EAGs less than 0.6 attains 82% using SDRLC method in contrast to 25% using SDR method. Thus, the SDRLC approach provides better approximate solution and interval estimate for problem (14).

Simulation 2. In this simulation, we compare the approximation performance of SDP and SDPLC methods when problem (14) involves some indefinite matrices P_i . We take $l = 8$ and $n = 4$, and the matrices P_i are randomly generated, with 25% indefinite symmetric matrices and 75% rank-one positive semidefinite matrices. With the same procedure as in Simulation 1, the results are given in Table 2.

	mean	min	max	frequency					
				[0, 0.3)	[0.3, 0.6)	[0.6, 1.5)	[1.5, 5)	[5, 20)	[20, 100)
SDR	5.8271	0.0053	99.6494	3%	7%	25.5%	36%	23%	5.5%
SDRLC	1.7565	0.0034	19.8329	15%	30%	30%	17%	8%	0

Table 2: The comparison of the EAGs in the case of 25% indefinite P_i .

From Table 2, the SDRLC method leads to markedly smaller maximum and average EAGs, and 75% of the AEGs are less than 1.5. In contrast, using the SDR method, only 25.5% of the AEGs are less than 1.5, and 28.5% of the AEGs are larger than 5. Thus, the SDRLC method still has better approximate performance when some of matrices P_i are indefinite. At the same time, one can observe that the average AEGs of both relaxation methods are larger than those given in Simulation 1, which implies that the approximate performances of SDR and SDRLC methods will degrade when the problem (14) involves some indefinite matrices P_i .

Example 3. Consider the following non-convex fractional quadratic optimization problem

$$\begin{aligned}
 \max_x \quad & \frac{x^T R x}{x^T Q x + 1} \\
 \text{s.t.} \quad & x^T P_i x \leq 1, \quad i = 1, \dots, l, \\
 & x \geq 0,
 \end{aligned} \tag{16}$$

where $x \in \mathbb{R}^n$, and R, Q and P_i are positive semidefinite matrices. The problem (16) is NP-hard, and includes problem (14) as a special case. It is an important model in the network beamforming [17, 19].

Using the SDRLC approach, problem (16) is relaxed as

$$\begin{aligned}
 \max_{x, \Delta} \quad & \frac{\text{Tr}(R\Delta)}{\text{Tr}(Q\Delta) + 1} \\
 \text{s.t.} \quad & \text{Tr}(P_i \Delta) \leq 1, \quad i = 1, \dots, l, \\
 & \text{Tr}(\Delta) \leq \alpha_1 e^T x, \\
 & \Delta \geq x x^T, \quad x \geq 0,
 \end{aligned} \tag{17}$$

where α_1 is given as in Example 2. The above problem can be recast as the following epigraph form:

$$\begin{aligned}
 \max_{x, \Delta, t \geq 0} \quad & t \\
 \text{s.t.} \quad & \text{Tr}(R\Delta) \geq t \text{Tr}(Q\Delta) + t, \\
 & \text{Tr}(P_i \Delta) \leq 1, \quad i = 1, \dots, l, \\
 & \text{Tr}(\Delta) \leq \alpha_1 e^T x, \\
 & \Delta \geq x x^T, \quad x \geq 0,
 \end{aligned}$$

which is a quasi-convex problem and can be solved by the bisection method. Denote the optimal solution of problem (17) by (x_1, Δ_1) , then we provide an approximate solution of problem (16) by employing the following Gaussian randomization procedure:

(1) Generate random vectors ζ_k ($k = 1, \dots, L$) from Gaussian distribution $N(x_1, \Delta_1 - x_1 x_1^T)$. Then introduce new variable $\bar{\zeta}_k$ as follows: for j -th component of $\bar{\zeta}_k$, let

$$\bar{\zeta}_k(j) = \begin{cases} \zeta_k(j), & \text{if } \zeta_k(j) \geq 0; \\ 0, & \text{otherwise,} \end{cases} \quad j = 1, \dots, n, k = 1, \dots, L.$$

(2) Denote $x(\bar{\zeta}_k) = \bar{\zeta}_k / \max_{1 \leq i \leq l} \sqrt{|\bar{\zeta}_k^T P_i \bar{\zeta}_k|}$.

(3) Let

$$\hat{x} = \arg \max_{1 \leq k \leq L} \frac{x(\bar{\zeta}_k)^T R x(\bar{\zeta}_k)}{x(\bar{\zeta}_k)^T Q x(\bar{\zeta}_k) + 1}$$

be the approximate solution to problem (16).

Simulation 3. Consider an instance of problem (16) by taking P_i , R and Q as follows:

$$P_1 = \begin{bmatrix} 0.1425 & -0.2353 & -0.4085 & -0.2949 \\ -0.2353 & 0.3884 & 0.6745 & 0.4869 \\ -0.4085 & 0.6745 & 1.1714 & 0.8455 \\ -0.2949 & 0.4869 & 0.8455 & 0.6102 \end{bmatrix}'$$

$$P_2 = \begin{bmatrix} 0.0875 & -0.2364 & 0.0389 & -0.1683 \\ -0.2364 & 0.6385 & -0.1051 & 0.4546 \\ 0.0389 & -0.1051 & 0.0173 & -0.0748 \\ -0.1683 & 0.4546 & -0.0748 & 0.3237 \end{bmatrix}'$$

$$P_3 = \begin{bmatrix} 2.1760 & -1.3879 & -0.5751 & 1.2121 \\ -1.3879 & 0.8853 & 0.3668 & -0.7731 \\ -0.5751 & 0.3668 & 0.1520 & -0.3204 \\ 1.2121 & -0.7731 & -0.3204 & 0.6752 \end{bmatrix}'$$

$$P_4 = \begin{bmatrix} 0.0548 & 0.2322 & -0.0206 & 0.0622 \\ 0.2322 & 0.9842 & -0.0873 & 0.2635 \\ -0.0206 & -0.0873 & 0.0077 & -0.0234 \\ 0.0622 & 0.2635 & -0.0234 & 0.0705 \end{bmatrix}'$$

$$R = \begin{bmatrix} 0.1232 & 0.2604 & 0.3334 & 0.1820 \\ 0.2604 & 0.5506 & 0.7049 & 0.3849 \\ 0.3334 & 0.7049 & 0.9023 & 0.4927 \\ 0.1820 & 0.3849 & 0.4927 & 0.2690 \end{bmatrix}'$$

$$Q = \begin{bmatrix} 2.0837 & -1.4547 & 0.6404 & 1.4238 \\ -1.4547 & 1.0156 & -0.4471 & -0.9940 \\ 0.6404 & -0.4471 & 0.1968 & 0.4376 \\ 1.4238 & -0.9940 & 0.4376 & 0.9729 \end{bmatrix}'$$

The global optimal solution and objective value of problem (16), and the extracted approximate solutions from the optimal solutions of SDRLC and SDR with randomization methods for different sample size L are listed in Table 3.

We can observe in Table 3 that, to obtain feasible solution with the same objective value, the SDRLC method needs far less sample size than the SDR method. For example, with randomization, the SDRLC method only needs 500 samples to derive feasible solution with objective 2.0648, which is close enough to the optimal value 2.0660 of problem (16), however, the sample size using SDR method is larger than 500000 to obtain the same approximate effect. Therefore, the proposed method can provide better feasible solution.

5. Conclusion

A new relaxation method is presented to approximately solve non-convex QCQP problem, and randomization is introduced to extract good feasible solution of original QCQP problem from the optimal solution

optimal	solution		objective	
	(0.6237, 0.9153, 0.6144, 0)		2.0660	
L	SDR	SDRLC	SDR	SDRLC
500	(0.6539, 0.5409, 0.8406, 0)	(0.6180, 0.9164, 0.6114, 0)	1.2804	2.0648
1000	(0.6441, 0.6157, 0.7941, 0)	(0.6178, 0.9165, 0.6115, 0)	1.4367	2.0654
5000	(0.4605, 0.7026, 0.6800, 0)	(0.6240, 0.9152, 0.6146, 0)	1.6556	2.0659
10000	(0.6858, 0.9011, 0.6196, 0)	(0.6232, 0.9154, 0.6142, 0)	1.9950	2.0659
50000	(0.6827, 0.9015, 0.6155, 0)	(0.6273, 0.9147, 0.6161, 0)	1.9864	2.0659
100000	(0.6731, 0.8797, 0.6522, 0)	(0.6215, 0.9158, 0.6134, 0)	1.9958	2.0660
500000	(0.6212, 0.9145, 0.6140, 0)	(0.6249, 0.9151, 0.6150, 0)	2.0636	2.0660

Table 3: The comparison of the approximation performances of SDRLC and SDR approaches.

of the SDRLC problem. The new relaxation strategy provides a tensor solution than the traditional SDR strategy theoretically. Some examples are given to compare the approximate performances of the SDR and SDRLC approaches.

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