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Shellability of complexes of directed trees

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Abstract. The question of shellability of complexes of directed trees was asked by R. Stanley. D. Kozlov showed that the existence of a complete source in a directed graph provides a shelling of its complex of directed trees. We will show that this property gives a shelling that is straightforward in some sense. Among the simplicial polytopes, only the crosspolytopes allow such a shelling. Furthermore, we show that the complex of directed trees of a complete double directed graph is a union of suitable spheres. We prove that the complex of directed trees of a directed graph which is essentially a tree is vertex-decomposable. For these complexes we describe their sets of generating facets.

1. Introduction

A *directed tree* with a *root* r is an acyclic directed graph $T = (V(T), E(T))$ such that for every $x \in V(T)$ there exists a unique path from r to x. A *directed forest* is a family of disjoint directed trees. We say that a vertex y is *below* a vertex x in a directed tree T if there exists a unique path from x to y. In this paper we write $\overrightarrow{xy}$ for a directed edge from x to y.

Definition 1.1. *Let D be a directed graph without loops. The vertices of the* complex of directed trees $\Delta(D)$ *are oriented edges of D. The faces of $\Delta(D)$ are all directed forests that are subgraphs of D.*

The investigation of complexes of directed trees was initiated by D. Kozlov in [10]. The complex of directed trees of a graph G is recognized in [4] as a discrete Morse complex of this graph (the authors treat graph as a 1-dimensional complex). Directed forests of G correspond with Morse matchings on G. Complexes of directed trees are also studied in [7] and [11].

Some important results in combinatorics and discrete geometry are based on investigation of topological properties of appropriate simplicial complexes. One of the key steps in Lovász' proof of Kneser's conjecture is verifying that the neighborhood complex of a Kneser graph $KG_{n,k}$ is $n - 2k - 1$ connected, see [12]. Complexes of all partial matchings in complete bipartite graphs (chessboard complexes) played the fundamental role in the proof of the colourful version of Tverberg's theorem, see [20] and [17]. A detailed treatment of simplicial complexes on graphs and plenty of interesting examples can be found in [9].

A d-dimensional simplicial complex is *pure* if every simplex of dimension less than d is a face of some d-simplex. For further definitions about simplicial complexes and other topological concepts used in this paper we refer the reader to the textbook [16].

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A simplicial complex Δ is *shellable* if it can be built up inductively in a nice way. To be more precise, its facets can be ordered so that each one of them (except for the first one) intersects the union of its predecessors in a non-empty union of maximal proper faces. Very often the following definition of shelling is useful.

Definition 1.2. *A simplicial complex Δ is shellable if Δ is pure and there exists a linear ordering (shelling order) $F_1, F_2, \dots, F_k$ of maximal faces (facets) of Δ such that for all $i < j \leqslant k$, there exists some $l < j$ and a vertex v of F_j, such that*

$$F_i \cap F_j \subseteq F_l \cap F_j = F_j \setminus \{v\}.$$

For a fixed shelling order $F_1, F_2, \dots, F_k$ of Δ, the *restriction* $\mathcal{R}(F_j)$ of the facet F_j is defined by:

$$\mathcal{R}(F_j) = \{v \text{ is a vertex of } F_j : F_j \setminus \{v\} \subset F_i \text{ for some } 1 \leqslant i < j\}.$$

Geometrically, if we build up Δ from its facets according to the shelling order, then $\mathcal{R}(F_j)$ is the unique minimal new face added at the j-th step. The *type* of the facet F_j in the given shelling order is the cardinality of $\mathcal{R}(F_j)$, that is, $type(F_j) = |\mathcal{R}(F_j)|$.

For a d-dimensional simplicial complex Δ we denote the number of i-dimensional faces of Δ by f_i, and call $f(\Delta) = (f_{-1}, f_0, f_1, \dots, f_d)$ the *f-vector*. The empty set is a face of every simplicial complex, so we have that $f_{-1} = 1$. A new invariant, the *h-vector* of Δ is $h(\Delta) = (h_0, h_1, \dots, h_d, h_{d+1})$ defined by the formula

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}.$$

If a simplicial complex Δ is shellable, then $h_k(\Delta) = |\{F \text{ is a facet of } \Delta : type(F) = k\}|$ is an important combinatorial interpretation of $h(\Delta)$. This interpretation of the h-vector was of great significance in the proof of the upper-bound theorem and in the characterization of f-vectors of simplicial polytopes (see chapter 8 in [19]).

If a d-dimensional simplicial complex Δ is shellable, then Δ is homotopy equivalent to a wedge of h_{d+1} spheres of dimension d. A set of maximal simplices from a simplicial complex Δ is a set of *generating simplices* if the removal of their interiors makes Δ contractible.

For a given shelling order of a complex Δ we have that $\{F \in \Delta : F \text{ is a facet and } \mathcal{R}(F) = F\}$ is a set of generating facets of Δ. Note that a facet F is in this set if and only if

$$\forall v \in F \text{ there exists a facet } F' \text{ before } F \text{ such that } F \cap F' = F \setminus \{v\}. \tag{1}$$

The concept of shellability for nonpure complexes is introduced in [3]. In the definition of shellability of nonpure complexes we just drop the requirement of purity from Definition 1.2.

For a facet F of a shellable nonpure complex we can define its restriction $\mathcal{R}(F)$ as before. The definitions of f-vector and h-vector of a nonpure complex are extended for double indexed arrays. For a nonpure complex Δ let

$$f_{i,j}(\Delta) = |\{A \in \Delta : |A| = j, i = max\{|T| : A \subseteq T, T \in \Delta\}\}|,$$

$$\text{and } h_{i,j}(\Delta) = \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} f_{i,k}.$$

The above defined arrays are called the *f-triangle* and the *h-triangle* of Δ. If Δ is a shellable complex, we have the following combinatorial interpretation of the h-triangle: $h_{i,j}(\Delta) = |\{F \text{ a facet of } \Delta : |F| = i, |\mathcal{R}(F)| = j\}|$.

If a nonpure simplicial complex Δ is shellable, we know that Δ has a homotopy type of a wedge of spheres, consisting of $h_{j,j}$ copies of the $(j-1)$-spheres (see Theorem 4.1 in [3]). The conditions described in (1) help us to identify a generating set of a nonpure shellable complex.

So, establishing shellability of a simplicial complex is an easy combinatorial way to obtain a lot of information about topology of this complex. In this paper we investigate some specific properties of the shelling of complexes of directed trees. These complexes are more complicated than simplicial polytopes.

However, the only simplicial d-polytope whose boundary admits a shelling satisfying these properties is the crosspolytope. Also, we will use shelling to determine the set of generating simplices of the complex of directed trees for a digraph that is essentially a tree. We want to find a relation between homology of these complexes and some combinatorial invariants for trees.

More information about shellable complexes can be found in [1], [2] and [3].

2. Shelling of complexes of directed trees for graphs with a complete source

A vertex x is a *complete source* of a directed graph D if $\overrightarrow{xy} \in E(D)$ for all $y \in V(D) \setminus \{x\}$. D. Kozlov proved (Theorem 3.1 in [10]) that if a directed graph D has a complete source, then the complex $\Delta(D)$ is shellable. He used a version of shelling described in the following remark.

Remark 2.1. *Let Γ be a simplicial complex. Assume that we can partition the facets of Γ into sets $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m$ in such a way that the following holds:*

$$|\mathcal{F}_0| = 1; \text{ for all } i \leqslant j, \text{ and for different facets } F \in \mathcal{F}_i, F' \in \mathcal{F}_j, \text{ there exist}$$
$$k < j, \text{ a facet } F'' \in \mathcal{F}_k, \text{ and a vertex } v \in F' \text{ such that } F \cap F' \subseteq F'' \cap F' = F' \setminus \{v\}. \tag{2}$$

In that case, any linear order that refines the above partition $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m$ (for $i < j$ we list facets from $\mathcal{F}_i$ before facets from $\mathcal{F}_j$) is a shelling of Γ.
If T is a directed tree and $v \in V(T)$, let $d_T(v)$ denote the out-degree of v, i.e., $d_T(v) = |\{x \in V(T) : \overrightarrow{vx} \in E(T)\}|$.

In the proof of Theorem 3.1 in [10], the facets of $\Delta(D)$ are ordered by their degree sequences lexicographically, i.e., trees T and T' are in the same class if and only if $d_T(v) = d_{T'}(v)$ for all $v \in V(D)$. Substantially, the facets of $\Delta(D)$ are classified by considering the out-degree of the complete source.

Here, we consider a directed graph D with a complete source c and detect some nice properties of a shelling described in the above remark.
If $|V(D)| = n$, for $i = 0, 1, \dots, n-1$, we set $\mathcal{F}_i = \{T \text{ a facet in } \Delta(D) : d_T(c) = n - i - 1\}$. In the same manner as in the proof of Theorem 3.1 in [10] we can verify that the partition $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{n-1}$ fulfills the condition described in Remark 2.1. Namely, if $d_T(c) \geqslant d_{T'}(c)$ and $T \neq T'$, then there exists an edge $\overrightarrow{xy} \in T' \setminus T$ such that $x \neq c$. We define

$$T'' = T' \setminus \{\overrightarrow{xy}\} \cup \{\overrightarrow{cy}\} \text{ if the vertex } c \text{ is not below } y \text{ in } T'; \quad \text{or}$$
$$T'' = T' \setminus \{\overrightarrow{xy}\} \cup \{\overrightarrow{cr}\} \text{ if } c \text{ is below } y \text{ in } T' \text{ and } r \text{ is the root of } T'.$$

In both cases simplices T, T', T'' and the vertex $\overrightarrow{xy}$ satisfy the condition described in (2).
Furthermore, for a facet $T \in \Delta(D)$ the unique new face for T in the shelling order defined above is $\mathcal{R}(T) = \{\overrightarrow{xy} \in T : x \neq c\}$. Therefore, the type of T is $type(T) = n - 1 - d_T(c)$, and we obtain that

$$h_i(\Delta(D)) = |\mathcal{F}_i| = |\{T \text{ is a facet of } \Delta(D) : d_T(c) = n - i - 1\}|.$$

Corollary 2.2. *Let G_n be the complete directed graph on n vertices. Then, for $k = 0, 1, \dots, n-1$ we have*

$$h_k(\Delta(G_n)) = \binom{n-1}{k}(n-1)^k.$$

Remark 2.3. *If a directed graph D has a complete source, then the shelling of $\Delta(D)$ is straightforward in the following sense:*

1. *We start the shelling with an appropriate facet F_0 and let $\mathcal{F}_0 = \{F_0\}$.*
2. *When we order all facets from $\mathcal{F}_{i-1}$, let $\mathcal{F}_i$ denote the set of all facets of $\Delta(D) \setminus (\mathcal{F}_0 \cup \cdots \cup \mathcal{F}_{i-1})$ that are neighborly (share a common ridge) to a simplex from $\mathcal{F}_{i-1}$.*
3. *We continue the shelling of $\Delta(D)$ by arranging simplices from $\mathcal{F}_i$ in an arbitrary order.*

(4) In this shelling order, for any facet F we have that type(F) $= i \Leftrightarrow F \in \mathcal{F}_i$.

It may be interesting to find more examples of simplicial complexes that allow a shelling with the properties (1)–(4) from the above remark.

Example 2.4. *Let* D_n *be the directed graph with* $V(D_n) = [n]$ *and* $E(D_n) = \{\overrightarrow{1i} : i \in [n], i \neq 1\} \cup \{\overrightarrow{2j} : j \in [n], j \neq 2\}$. *It is easy to see that* $\Delta(D_n)$ *is combinatorially equivalent to the boundary of the* $(n-1)$*-dimensional crosspolytope.*

Theorem 2.5. *The only simplicial d-dimensional polytope whose boundary admits a shelling as that described in Remark 5 is the crosspolytope.*

Proof. Assume that P is a simplicial d-polytope with a shelling satisfying desired properties. We identify a facet of P with its set of vertices. Let $F_0 = \{v_1, v_2, \ldots, v_d\}$ be the first facet in this shelling.

Let w_i denote the unique new vertex of the facet of P that contains $(d-2)$-dimensional simplex $F_0 \setminus \{v_i\}$. All facets of P whose type is 1 belong to $\mathcal{F}_1$ and therefore have the form $F_0 \setminus \{v_i\} \cup \{w_i\}$. We can conclude that the set of the vertices of P is $V(P) = \{v_1, v_2, \ldots, v_d, w_1, w_2, \ldots, w_d\}$.

There are no general results about the number of simplicial d-polytopes with $2d$ vertices. Grünbaum and Sreedharan [8] proved that there are exactly 37 centrally symmetric 4-dimensional simplicial polytopes with 8 vertices.

For any $S \subseteq [d]$ we consider the $(d-1)$-simplex $F_S = conv\Big(\{v_i : i \notin S\} \cup \{w_j : j \in S\}\Big)$. We do not know that F_S is a facet of P, but we use induction on k to show that

$$\mathcal{F}_k = \{F_S : S \subseteq [d], |S| = k\}. \tag{3}$$

This is true for $k = 0$ and $k = 1$. Assume that the above statement holds for all $t \leqslant k-1$. Let $F \in \mathcal{F}_k$ be a facet (yet not listed) of P that shares a common ridge with a facet $\bar{F}$ from $\mathcal{F}_{k-1}$.

From the inductive hypothesis we have $\bar{F} = F_S$ (for $S \subset [d], |S| = k-1$) and $F = F_S \setminus \{v_i\} \cup \{w_j\}$ for $i, j \notin S$. If $i \neq j$, then the edge $\{v_j w_j\}$ and the $(k-1)$-simplex $\{w_s : s \in S \cup \{j\}\}$ are two different minimal new faces that F contributes in the shelling of P, which is impossible. Therefore, we can conclude that $i = j$, and $F = F_S \setminus \{v_i\} \cup \{w_i\} = F_{S \cup \{i\}}$.
We have that any of the facets that belong to $\mathcal{F}_k$ has the form described in (3).

In the shelling of the boundary of a polytope, the type of a facet can be interpreted as the number of adjacent facets that preceded it in this order. All facets from $\mathcal{F}_k$ can be listed in an arbitrary order and any of them has the type k. Therefore, we conclude that two facets from $\mathcal{F}_k$ cannot share the same ridge, and we obtain that

$$(d-k+1)|\mathcal{F}_{k-1}| = k|\mathcal{F}_k|.$$

The inductive assumption and the above equations complete the proof of (3). So, we may conclude that P is combinatorially equivalent to the d-dimensional crosspolytope. □

If a directed graph D has a complete source c then the complex $\Delta(D)$ is homotopy equivalent to a wedge of the spheres. In [10], D. Kozlov describes generating facets of $\Delta(D)$ as rooted trees of D having complete source c as a leaf.

Here we study the combinatorics of the spheres in $\Delta(D)$ when D has a complete source. To each tree T that is a generating facet we associate a sphere $S_T \subset \Delta(D)$ that contains T and describe the combinatorial type of S_T.

We consider a directed graph D with n vertices. Assume that c is a complete source of D. Let T be a rooted spanning tree of D with vertex c as a leaf. If $x_1 \to x_2 \to \ldots \to x_k \to c$ is the unique directed path from x_1 (the root of T) to c, let σ_T denote the simplex $\{\overrightarrow{x_1x_2}, \overrightarrow{x_2x_3}, \ldots, \overrightarrow{x_kc}, \overrightarrow{cx_1}\}$. It is obvious that $\sigma_T \notin \Delta(D)$. Also note that $\partial\sigma_T \subset \Delta(D)$.
Let $A_T = \{y_1, y_2, \ldots, y_r\} = V(D) \setminus \{x_1, x_2, \ldots, x_k, c\}$, i.e., A_T contains $r = n-k-1$ vertices that do not belong to the unique directed path from x_1 to c in T. For any $y_i \in A_T$ there exists the unique vertex z_i such that $\overrightarrow{z_iy_i} \in E(T)$. Now, we define

$$S_T = \partial\sigma_T * \{\overrightarrow{z_1y_1}, \overrightarrow{cy_1}\} * \{\overrightarrow{z_2y_2}, \overrightarrow{cy_2}\} * \cdots * \{\overrightarrow{z_ry_r}, \overrightarrow{cy_r}\}. \tag{4}$$

It is not complicated to prove that $S_T \subset \Delta(D)$. The sphere S_T is a $(n-k-1)$-folded bipyramid over the boundary of k-simplex σ_T.

Proposition 2.6. *If a directed graph D has at least two complete sources, then $\Delta(D)$ is the union of the spheres defined in (4).*

Proof. Let us denote two complete sources in D by c and c'. If c is a leaf in T, then we have $T \in S_T$. If c is not a leaf in a tree T, then let $\{u_1, u_2, \ldots, u_k\}$ be the set of all vertices of D such that $\overrightarrow{cu_i} \in E(T)$ for all $i = 1, 2, \ldots, k$.

If the vertex c' is not below c in T, we define

$$T' = T \setminus \{\overrightarrow{cu}_1, \overrightarrow{cu}_2, \ldots, \overrightarrow{cu}_k\} \cup \{\overrightarrow{c'u}_1, \overrightarrow{c'u}_2, \ldots, \overrightarrow{c'u}_k\}.$$

In the case when c' is below c (then we have that $c' = u_i$ or c' is below u_i) and the root of T is r we define

$$T' = T \setminus \{\overrightarrow{cu}_1, \overrightarrow{cu}_2, \ldots, \overrightarrow{cu}_k\} \cup \{\overrightarrow{c'u}_1, \ldots, \overrightarrow{c'u}_{i-1}, \overrightarrow{c'r}, \overrightarrow{c'u}_{i+1}, \ldots, \overrightarrow{c'u}_k\}.$$

In both cases the directed tree T' is a generating facet of $\Delta(D)$. Obviously, the facet T is contained in the sphere $S_{T'}$. □

We conclude now that $\Delta(G_n)$ is the union of the $(n-k-1)$-folded bipyramids over the boundary of a k-simplex. A simple calculation and the well-known formulae for the number of forests with $n-1$ vertices and k trees such that k specified nodes belong to distinct trees (Theorem 3.3 in [14]) give us the number of spheres in $\Delta(G_n)$ of the same combinatorial type.

Corollary 2.7. *For any $n \geqslant 1$ the complex $\Delta(G_n)$ is a union of $(n-1)^{n-1}$ spheres of dimension $n-2$. For $0 < k < n$ there are exactly*

$$\frac{(n-1)!}{(n-k-1)!} k(n-1)^{n-k-2}$$

of these spheres that are $(n-k-1)$-folded bipyramid over the boundary of a $(k-1)$-simplex.

For two spanning directed trees T and T' of G_n having 1 as a leaf (generating simplices of $\Delta(G_n)$), from (4) we are able to determine the intersection of the spheres S_T and $S_{T'}$. For example, if $x_1 \to x_2 \to \ldots \to x_k \to 1$ is the common directed path from the root x_1 to 1 in T and T', then $S_T \cap S_{T'}$ is an $n-2$ dimensional disc. If $|E(T) \cap E(T')| = s \geqslant k$, then $S_T \cap S_{T'}$ is an $(n-1-s)$-pyramid over an $(s-k)$-bipyramid over the boundary of a k-simplex.

3. Trees

A subset of the vertex set of a graph is independent if no two vertices in it are adjacent. For a simple graph $G = (V, E)$ the *independence complex* $I(G)$ is the simplicial complex with vertex set V and with faces the independent sets of G. The independence complex has been previously studied in many papers, see for example [6],[13].

Shellability and vertex-decomposability of independence complexes are discussed in[5] and [18]. A complex Δ is *vertex decomposable* if it is a simplex or (recursively) Δ has a shedding vertex, i.e., a vertex v such that both $\Delta \setminus \{v\}$ and $link_\Delta v$ are vertex decomposable. It is well-known that any vertex decomposable complex is shellable too.

A *chordless* cycle of length n in a graph G is a cycle $v_1, v_2, \ldots, v_n, v_1$ in G with no chord, i.e., with no edges except $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.

We will use the following Theorem.

Theorem 3.1 (Theorem 1, [18]). *If G is a graph with no chordless cycles of length other than 3 or 5, then $I(G)$ is vertex decomposable (hence shellable and sequentially Cohen-Macaulay.)*

We follow Kozlov [10] and say that a digraph D is essentially a tree if it becomes an undirected tree when one replaces all directed edges (or pairs of directed edges going in opposite directions) by an edge.

Theorem 3.2. *Let $D = (V(D), E(D))$ be essentially a tree. Then $\Delta(D)$ is vertex decomposable and hence shellable.*

Proof. For a given tree D we define a simple graph G in the following way. For $v \in V(D)$ let $d^-(v) = |\{x \in V(D) : \overrightarrow{xv} \in E(D)\}|$ denote the in-degree of v in D. We replace every $v \in V(D)$ with a complete graph $K_{d^-(v)}$ whose vertices correspond with directed edges having v as sink. Further, if both of directed edges $\overrightarrow{uv}, \overrightarrow{vu}$ are contained in $E(D)$, then the corresponding vertices of $K_{d^-(v)}$ and $K_{d^-(u)}$ are adjacent in G. Formally, we define $V(G) = E(D)$, and edges with the same sink $\overrightarrow{ax}, \overrightarrow{bx}$ are adjacent in G. Also, if $\overrightarrow{ab}, \overrightarrow{ba} \in E(D)$ they are adjacent as vertices of G.

Note that $A \subset V(G)$ is an independent set in G if and only if A is the set of edges of a directed forest in D. Therefore we have that $\Delta(D) = I(G)$. Moreover, the construction of G and the assumption that D is essentially a tree guaranteed that G does not contain a chordless cycle of length other than 3. Now, the statement of our theorem follows from Theorem 3.1. □

Note that these complexes generally are not pure (unlike the case when a digraph has a complete source). We describe a way to find an explicit shelling of $\Delta(D)$. Let D be a directed graph and let $v \in V(D)$ be a leaf in D. In other words there exists the unique vertex $x \in V(D)$ such that $\overrightarrow{vx}$ or $\overrightarrow{xv}$ or both of them are in $E(D)$ and there are no other edges where v is a source or a sink.

Let $D' = D \setminus \{v\}$ and let $\{y_1, y_2, \dots, y_k\} = \{y \in V(D') : \overrightarrow{yx} \in E(D')\}$. Furthermore, let $D_0 = D' \setminus \{\overrightarrow{y_1x}, \overrightarrow{y_2x}, \dots, \overrightarrow{y_kx}\}$ and assume that $\overrightarrow{xy_i} \in E(D)$ for $i = 1, 2, \dots, s$. Now, for $p = 1, 2, \dots, k$ we set $D_p = D_0 \setminus \{\overrightarrow{xy_i}\}$. Note that $D_p = D_0$ for $p > s$.

If a directed graph D is a disjoint union of subgraphs $H_1, H_2, \dots, H_m$ then $\Delta(D) = \Delta(H_1) * \Delta(H_2) * \cdots * \Delta(H_m)$. The join of shellable complexes is also shellable, see [3]. So, we know that the complexes $\Delta(D')$, $\Delta(D_0)$ and $\Delta(D_p)$ are shellable.

Assume that:

(i) $F_1, F_2, \dots, F_t$ is a shelling of $\Delta(D')$;

(ii) $H_1, H_2, \dots, H_r$ is a shelling of $\Delta(D_0)$;

(iii) $G_1^p, G_2^p, \dots, G_{t_p}^p$ is a shelling of $\Delta(D_p)$ (for $p = 1, 2, \dots, k$).

We use the above notation in the next proposition.

Proposition 3.3. *We consider three possible cases.*

(a) If $\overrightarrow{xv} \in E(D)$ and $\overrightarrow{vx} \notin E(D)$, then $F_1 \cup \{\overrightarrow{xv}\}, F_2 \cup \{\overrightarrow{xv}\}, \dots, F_t \cup \{\overrightarrow{xv}\}$ is a shelling of $\Delta(D)$. Also, we have that $h_{i,j}(\Delta(D)) = h_{i-1,j}(\Delta(D'))$.

(b) If $\overrightarrow{xv} \notin E(D)$ and $\overrightarrow{vx} \in E(D)$, then

$$H_1 \cup \{\overrightarrow{vx}\}, \dots, H_r \cup \{\overrightarrow{vx}\}, G_1^1 \cup \{\overrightarrow{y_1x}\}, \dots, G_{t_1}^1 \cup \{\overrightarrow{y_1x}\}, \dots, G_{t_k}^k \cup \{\overrightarrow{y_kx}\}$$

is a shelling of $\Delta(D)$. Furthermore, we have that

$$h_{i,j}(\Delta(D)) = h_{i-1,j}(\Delta(D_0)) + \sum_{p=1}^{k} h_{i-1,j-1}(\Delta(D_p)).$$

(c) If $\overrightarrow{xv}, \overrightarrow{vx} \in E(D)$, then

$$F_1 \cup \{\overrightarrow{xv}\}, F_2 \cup \{\overrightarrow{xv}\}, \dots, F_t \cup \{\overrightarrow{xv}\}, H_1 \cup \{\overrightarrow{vx}\}, H_2 \cup \{\overrightarrow{vx}\}, \dots, H_r \cup \{\overrightarrow{vx}\}$$

is a shelling of D. In that case we have that

$$h_{i,j}(\Delta(D)) = h_{i-1,j}(\Delta(D')) + h_{i-1,j-1}(\Delta(D_0)).$$

Proof.

(a) This is obvious, because $\Delta(D)$ is a cone over $\Delta(D')$ with apex $\overrightarrow{xv}$. Therefore, we have $\mathcal{R}_D(F_i \cup \{\overrightarrow{xv}\}) = \mathcal{R}_{D'}(F_i)$ and $\Delta(D)$ is contractible.

(b) If a facet F of $\Delta(D)$ contains $\overrightarrow{vx}$, then F does not contain any of edges $\{\overrightarrow{y_1x}, \overrightarrow{y_2x}, \dots, \overrightarrow{y_kx}\}$. So, in that case we have that $F = H \cup \{\overrightarrow{vx}\}$, for a facet H of $\Delta(D_0)$. If a facet F' of $\Delta(D)$ does not contain $\overrightarrow{vx}$, then F' must contain exactly one of the edges $\{\overrightarrow{y_1x}, \overrightarrow{y_2x}, \dots, \overrightarrow{y_kx}\}$. Therefore, we have that $F' = G \cup \{\overrightarrow{y_px}\}$ for a facet G of D_p.

The supposed shelling of $\Delta(D_0)$ provides that for $i < j$ and facets H_i, H_j of $\Delta(D_0)$ there exists $k < j$ and $\overrightarrow{wz} \in H_j$ such that

$$(H_i \cup \{\overrightarrow{vx}\}) \cap (H_j \cup \{\overrightarrow{vx}\}) \subseteq (H_k \cup \{\overrightarrow{vx}\}) \cap (H_j \cup \{\overrightarrow{vx}\}) = \left(H_j \cup \{\overrightarrow{vx}\}\right) \setminus \{\overrightarrow{wz}\}.$$

Note that for any p such that $1 \leqslant p \leqslant s$ and a facet G_j^p of $\Delta(D_p)$ there exists a facet H_k of $\Delta(D_0)$ such that $G_j^p \subseteq H_k$. Therefore, for any facet H_i of $\Delta(D_0)$ we have

$$(H_i \cup \{\overrightarrow{vx}\}) \cap (G_j^p \cup \{\overrightarrow{y_px}\}) \subseteq (H_k \cup \{\overrightarrow{vx}\}) \cap (G_j^p \cup \{\overrightarrow{y_px}\}) = G_j^p.$$

Also, for $q \leqslant p$ and a facet G_r^q of $\Delta(D_q)$ we have

$$(G_r^q \cup \{\overrightarrow{y_qx}\}) \cap (G_j^p \cup \{\overrightarrow{y_px}\}) \subseteq (H_k \cup \{\overrightarrow{vx}\}) \cap (G_j^p \cup \{\overrightarrow{y_px}\}) = G_j^p.$$

So, we obtain that the order defined in (b) is a shelling order for $\Delta(D)$. In this order we have that the restriction of the facets of $\Delta(D)$ is

$$\mathcal{R}_D(H_i \cup \{\overrightarrow{vx}\}) = \mathcal{R}_{D_0}(H_i) \text{ and } \mathcal{R}_D(G_i^p \cup \{\overrightarrow{y_px}\}) = \mathcal{R}_{D_p}(G_i^p) \cup \{\overrightarrow{y_px}\}.$$

(c) In this case a facet of $\Delta(D)$ has the form

$$\{\overrightarrow{xv}\} \cup F, \text{ for a facet } F \text{ of } \Delta(D') \text{ or } \{\overrightarrow{vx}\} \cup H, \text{ for a facet } H \text{ of } \Delta(D_0).$$

Again, for a facet H_j of $\Delta(D_0)$ there exists a facet F_i of $\Delta(D')$ such that $H_j \subseteq F_i$. In the similar manner as in (b) we can prove that the considered order is a shelling order. Further, the restriction in this order is

$$\mathcal{R}_D(F_i \cup \{\overrightarrow{xv}\}) = \mathcal{R}_{D'}(F_i) \text{ and } \mathcal{R}_D(H_i \cup \{\overrightarrow{vx}\}) = \mathcal{R}_{D_0}(H_i) \cup \{\overrightarrow{vx}\}.$$

□

The complexity of the above described algorithm for shelling is $O(2^n)$. The worst case example for our algorithm is $\Delta(P_n)$, where P_n is a double directed path with n vertices.

The trivial, brute-force algorithm that simply generates all permutations of t facets and checks whether that ordering is a shelling takes $O(t! \cdot t)$, see [15]. The number of facets of $\Delta(P_n)$ exponentially depends on the number of vertices.

Remark 3.4. *Now, we can identify a set of generating facets of $\Delta(D)$. We use the same notation as above. If $\overrightarrow{xv} \in E(D)$ and $\overrightarrow{vx} \notin E(D)$, then $\Delta(D)$ is contractible. If $\overrightarrow{xv} \notin E(D)$ and $\overrightarrow{vx} \in E(D)$, let $\mathcal{G}_p$ denote a set of generating faces of $\Delta(D_p)$ for $p = 1, 2, \dots, k$. Then, a generating set of facets of $\Delta(D)$ is*

$$\bigcup_{p=1}^{k} \{G \cup \{\overrightarrow{y_px}\} : G \in \mathcal{G}_p\}.$$

If $\overrightarrow{xv}, \overrightarrow{vx} \in E(D)$, then a set of generating facets of $\Delta(D)$ is $\{H \cup \{\overrightarrow{vx}\} : H$ is a generating facet of $\Delta(D_0)\}$.

A *directed acyclic graph* is a directed graph without directed cycles. By successive applications of Proposition 3.3 and Remark 3.4 we obtain the following result of A. Engström.

Theorem 3.5 (Theorem 2.12, [7]). *If G is a directed acyclic graph, then $\Delta(G)$ is homotopy equivalent to a wedge of $\prod_{v\in V(G)\setminus R}(d^-(v)-1)$ spheres of dimension $|V(G)|-|R|-1$, where R is the set of vertices without edges directed to them.*

A *double directed tree D* is obtained by replacing every edge of a tree by a pair of directed edges going in opposite directions. Now, we investigate homotopy type of $\Delta(D)$, where D is a double directed tree.

Definition 3.6. *A tree T with $2n$ vertices (n leaves and n non-leaves) such that every non-leaf is adjacent to exactly one leaf we call* basic tree. *Also, we say that a tree with exactly two vertices is a basic tree. We say that the edge connecting a non-leaf and a leaf is peripheral.*

We can produce a basic tree if we start with an arbitrary tree T' and add a leaf to each node of T'. We use description of generating facets from Remark 3.4 to obtain the following proposition.

Proposition 3.7. *Let D be a double directed tree with $2n$ vertices obtained from a basic tree T by replacing every edge of T by a pair of directed edges going in opposite directions. Then we have that $\Delta(D)\simeq \mathbb{S}^{n-1}$.*

Proof. Assume that $v_1,v_2,\dots,v_n$ are leaves of T. We label the rest of the vertices of T with $u_1,u_2,\dots,u_n$ so that $v_iu_i\in E(T)$ for all $i=1,2,\dots,n$. By applying Remark 3.4 successively we obtain that the set of peripheral edges $\{\overrightarrow{v_iu_i} : i=1,2,\dots,n\}$ is the unique generating facet of $\Delta(D)$. □

We denote the unique generating facet for a basic tree T by G_T, that is, $G_T=\{\overrightarrow{v_1u_1},\overrightarrow{v_2u_2},\dots,\overrightarrow{v_nu_n}\}$.

Let D be a double directed tree obtained from a tree T. We describe a bijection between generating simplices of $\Delta(D)$ and decompositions of T into basic trees.

Let $v_1,v_2,\dots,v_n$ be a fixed linear order of $V(D)$ and choose the first leaf $v\in V(D)$ in this order. Assume that $N(v)=\{x\}$ and $N(x)=\{v,y_1,\dots,y_k\}$. From Remark 3.4 we know that all generating facets of $\Delta(D)$ have to contain the edge $\overrightarrow{vx}$. Next, we are looking for generating facets of complex $\Delta(D_0)$ where $D_0=(D\setminus\{v\})\setminus\{\overrightarrow{y_1x},\dots,\overrightarrow{y_kx}\}$. From (b) of Proposition 3.3 we have that a generating facet of $\Delta(D_0)$ must contain edges $\overrightarrow{z_1y_1},\overrightarrow{z_2y_2},\dots,\overrightarrow{z_ky_k}$ where $z_i\in N(y_i)$ and $z_i\neq x$. If $d_T(y_i)=2$ for all $i=1,2,\dots,k$, we consider a subtree of T spanned by $\{v,x,y_1,\dots,y_k,z_1,\dots,z_k\}$. In the case when $N_T(y_i)=\{x,z_i,u_1,\dots,u_r\}$, a generating facet of $\Delta(D)$ that contains $\{\overrightarrow{vx},\overrightarrow{z_1y_1},\dots,\overrightarrow{z_ky_k}\}$ also must contain edges $\overrightarrow{w_ju_j}$ for $j=1,2,\dots,r$.

By repeating this procedure we obtain a subtree B_1 of T such that

(1) B_1 is a basic tree and $v\in V(B_1)$,
(2) for any $x\in V(B_1)$ that is not a leaf in B_1 we have that $d_{B_1}(x)=d_T(x)$,
(3) $|V(B_1)|>2$ whenever $|V(T)|>2$.

Note that there can be more possibilities for a basic tree B_1, see Figure 1. If we can not find a subtree B_1 that satisfies the above conditions, then we obtain that $\Delta(D)$ is contractible. After we choose a basic tree B_1 that satisfies (1)–(3) we proceed in the same way with $T'=T\setminus\{x\in V(B_1): d_{B_1}(x)=d_T(x)\}$. Note that T' is a forest or a tree.

Let v' be the first leaf of T' and let T_1 be the maximal tree of T' that contains v'. Now, we are looking for B_2, a subtree of T_1 that satisfies (1)–(3). If we can decompose T into $B_1,B_2,\dots,B_m$ we say that $(B_1,B_2,\dots,B_m)$ is an ordered decomposition of T into m basic trees.

An ordered decomposition $(B_1,B_2,\dots,B_m)$ of T that satisfies (1)–(3) produces a generating facet $G_{B_1}\cup G_{B_2}\cup\cdots\cup G_{B_m}$ of $\Delta(D)$.

Theorem 3.8. *Let D be a double directed tree with n vertices. Let μ_m denote the number of ordered decompositions of D into m basic trees. Then we have that*

$$\Delta(D)\simeq\bigvee_m\left(\bigvee^{\mu_m}\mathbb{S}^{\frac{n+m-3}{2}}\right).$$

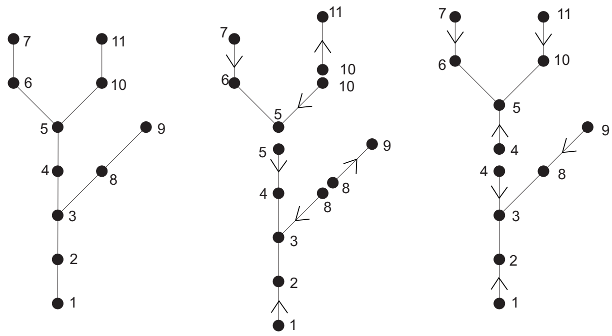

Figure 1: *A tree and its decompositions into basic trees. Oriented edges represent generating facets of* $\Delta(D)$. Note that $\Delta(D) \simeq \mathbb{S}^5 \vee \mathbb{S}^6$

Proof. We described above a bijection between generating sets of $\Delta(D)$ and ordered decompositions of D that satisfy (1)–(3). Consider such an ordered partition $(B_1, B_2, \ldots, B_m)$ with m basic trees. If a basic tree B_i contains $2s_i$ vertices (and $2s_i - 1$ edges) it contains s_i edges of a generating set of $\Delta(D)$. Then we have $2s_1 - 1 + 2s_2 - 1 + \cdots + 2s_m - 1 = n - 1$, and this decomposition corresponds with

$$s_1 + s_2 + \cdots + s_m - 1 = \frac{n + m - 1}{2} - 1$$

dimensional generating facet of $\Delta(D)$. □

Corollary 3.9. *For a double directed tree D with n vertices the nontrivial homology of* $\Delta(D)$ *can occur in dimensions between* $\left\lceil \frac{n-2}{2} \right\rceil$ *and* $n - \lfloor \frac{n}{3} \rfloor - 2$.

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