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Coincidence and Common Fixed Point Theorems for (ψ, φ) weakly Contractive Mappings in Generalized Metric Spaces

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Abstract. We establish some coincidence and common fixed point theorems for mappings satisfying a generalized (ψ, φ)-weakly contractive condition in complete Hausdorff generalized metric spaces. Our results generalize very recent results of C. Di Bari and P. Vetro [Common fixed points in generalized metric spaces, Appl. Math. Comput. 218 (2012), 7322–7325] and extend and generalize many existing results in the literature.

1. Introduction and preliminaries

It is well known that the contraction mapping principle, formulated and proved in the Ph.D. dissertation of Banach in 1920, which was published in 1922, is one of the most important theorems in classical functional analysis. This contraction mapping principle has been generalized in many directions. Recently, a very interesting generalization was obtained by Branciari in [3] by lessening the structure of a metric space. In fact, Branciari [3] introduced a concept of generalized metric space by replacing the triangle inequality by a more general inequality - by the "rectangular" inequality. So any metric space is a generalized metric space, but the converse is not true (see for example ref. [3]). He proved the Banach's fixed point theorem in such spaces. For more details about fixed-point theory in generalized metric spaces, we refer the reader to [1], [4]-[14].

In this paper, we prove coincidence and common fixed point theorems for two mappings satisfying a generalized (ψ, φ)-weakly contractive condition in complete Hausdorff generalized metric spaces. Presented theorems extend and generalize many existing results in the literature.

2. Definitions and known theorems

Let R^+ denote the set of all positive real numbers and N denote the set of all positive integers.

Definition 2.1. *Let X be a non-empty set and $d : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$ each of them different from x and y, one has*

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(*i*) *$d(x, y) = 0$ if and only if $x = y$,*

(*ii*) *$d(x, y) = d(y, x)$,*

(*iii*) *$d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (rectangular inequality).*

Then (X, d) is called a generalized metric space (or shortly g.m.s.).

Definition 2.2. *Let (X, d) be a g.m.s., $\{x_n\}$ be a sequence in X and $x \in X$. We say that*

(*i*) *A sequence $\{x_n\}$ is convergent to x if and only if $d(x_n, x) \to 0$ as $n \to +\infty$. We denote this by $x_n \to x$.*

(*ii*) *A sequence $\{x_n\}$ is a Cauchy sequence if and only if for each $\epsilon > 0$ there exists a natural number $n(\epsilon)$ such that $d(x_m, x_n) < \epsilon$ for all $n > m \geq n(\epsilon)$.*

(*iii*) *(X, d) is called complete if every g.m.s. Cauchy sequence is convergent in X.*

We denote by Ψ the set of functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(ψ_1) ψ is continuous and nondecreasing,

(ψ_2) $\psi(t) = 0$ if and only if $t = 0$.

We denote by Φ the set of functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(φ_1) φ is lower semi-continuous,

(φ_2) $\varphi(t) = 0$ if and only if $t = 0$.

H. Lakzian and B. Samet in [10], established the following fixed point theorem involving a pair of altering distance functions in a generalized complete metric spaces.

Theorem 2.3. *(Lakzian and Samet [10], Theorem 1). Let (X, d) be a Hausdorff and complete g.m.s. and let $T : X \to X$ be a self- mapping satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y))$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is continuous and $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Definition 2.4. *Let X be a non-empty set and $T, f : X \to X$.*

(*i*) *A point $y \in X$ is called a point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$. The point x is called coincidence point of T and f.*

(*ii*) *The mappings T, f are said to be weakly compatible if they commute at their coincidence point (that is, $Tfx = fTx$ whenever $Tx = fx$).*

Recently C. Di Bari and P. Vetro [8] extended the fixed point Theorem 2.3 of Lakzian and Samet to the following common fixed point theorem for mappings satisfying a (ψ, φ)-weakly contractive condition in generalized metric spaces.

Theorem 2.5. *(Di Bari and Vetro [8], Theorem 1). Let (X, d) be a Hausdorff g.m.s. and let T and f be self-mappings on X such that $TX \subseteq fX$. Assume that (fX, d) is a complete g.m.s. and that the following condition holds:*

$$\psi(d(Tx, Ty)) \leq \psi(d(fx, fy)) - \varphi(d(fx, fy))$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\varphi : [0, +\infty) \to [0, +\infty)$ is lower semi-continuous and $\varphi(t) = 0$ if and only if $t = 0$. Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

In this paper, we prove some coincidence and common fixed point theorems involving (ψ, φ)-weak contractive conditions for two self-mappings on X in complete generalized metric (X, d) spaces by assuming that these are Hausdorff spaces. Our theorems are real generalizations of Theorems 2.3 and 2.5.

3. Main results

In this section, we prove some common fixed point results for two self-mappings satisfying a generalized (ψ, φ)-weakly contractive condition, where a function φ satisfies a less restrictive condition then in the theorems of Lakzian and Samet [10], as well as in theorems of Di Bari, P. Vetro [8].

We denote by Φ^* the set of functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(φ_1) $\liminf_{t \to r+} \varphi(t) > 0$ for all $r > 0$,

(φ_2) $\varphi(t) = 0$ if and only if $t = 0$.

Theorem 3.1. *Let (X, d) be a Hausdorff g.m.s. and let T and f be self-mappings on X such that$TX \subseteq fX$. Assume that (fX, d) is a complete g.m.s. and that the following condition holds:*

$$\begin{aligned} \psi(d(Tx, Ty)) \quad \leq \quad & \psi(\max\{d(fx, fy), \frac{1}{2}[d(fx, Tx) + d(fy, Ty)], d(fy, Tx)\}) \\ & -\varphi(d(fx, fy)) \end{aligned} \tag{1}$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\varphi \in \Phi^*$. Then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Proof. Let $x_0 \in X$. Define sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_n = fx_{n+1} = Tx_n; \quad n \in \{0, 1, 2, \cdots\}. \tag{2}$$

This can be done, since $TX \subseteq fX$. If we assume that $y_n = y_{n-1}$ for some $n \geq 1$, then by (2) we have $y_n = Tx_n = y_{n-1} = fx_n$. Hence $Tx_n = fx_n$. Thus in this case one can directly proved that T and f have a coincidence point x_n in X.

Now we shall suppose that $y_n \neq y_{n-1}$ for all $n \geq 1$. From (1) with $x = x_n$ and $y = x_{n+1}$ we have

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &= \psi(d(Tx_n, Tx_{n+1})) \\ &\leq \psi(\max\{d(fx_n, fx_{n+1}), \frac{1}{2}[d(fx_n, Tx_n) + d(fx_{n+1}, Tx_{n+1})], \\ &\quad d(fx_{n+1}, Tx_n)\}) - \varphi(d(fx_n, fx_{n+1})) \\ &= \psi(\max\{d(y_{n-1}, y_n), \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})], d(y_n, y_n)\}) \\ &\quad -\varphi(d(y_{n-1}, y_n)). \end{aligned}$$

Hence we get

$$\psi(d(y_n, y_{n+1})) \leq \psi(\max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}) - \varphi(d(y_{n-1}, y_n)). \tag{3}$$

From (3), using the monotone property of the function ψ, and as $\varphi(d(y_{n-1}, y_n)) > 0$, we have

$$d(y_n, y_{n+1}) < d(y_{n-1}, y_n) \text{ for all } n \geq 1. \tag{4}$$

From (4) it follows that the sequence of positive reals $\{d(y_n, y_{n+1})\}$ is monotone decreasing and consequently, there exists $q \geq 0$ such that $\lim_{n\to\infty} d(y_n, y_{n+1}) = q$. We shall show that $q = 0$, that is,

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{5}$$

Suppose, to the contrary, that $q > 0$. Letting $n \to \infty$ in (3), and using the continuity of ψ and the property (φ_1) of the function $\varphi \in \Phi^*$, we get

$$\psi(q) \leq \psi(q) - \liminf_{d(y_n, y_{n+1})\to q+} \varphi(q) < \psi(q),$$

a contradiction. Thus we proved (5).

Now we shall show that $y_{n+2} \neq y_n$ for all $n \geq 1$. Suppose, to the contrary, that $y_{n+2} = y_n$. Then, using (4), we have $d(y_n, y_{n+1}) = d(y_{n+2}, y_{n+1}) < d(y_{(n+2)-1}, y_{(n+1)-1}) = d(y_{n+1}, y_n)$, a contradiction. Therefore, $y_{n+2} \neq y_n$ for all $n \geq 1$, which implies that

$$d(y_n, y_{n+2}) > 0 \text{ for all } n \in \{0, 1, 2, \cdots\}. \tag{6}$$

From (1), for any $n \in N$ we have

$$\begin{aligned}
\psi(d(y_{n+1}, y_{n+3})) &= \psi(d(Tx_n{+}1, Tx_{n+3})) \\
&\leq \psi(\max\{d(fx_{n+1}, fx_{n+3}), \frac{1}{2}[d(fx_{n+1}, Tx_{n+1}) + d(fx_{n+3}, Tx_{n+3})], \\
&\quad d(fx_{n+3}, Tx_{n+1})\}) - \varphi(d(fx_{n+1}, fx_{n+3})) \\
&= \psi(\max\{d(y_n, y_{n+2}), \frac{1}{2}[d(y_n, y_{n+1}) + d(y_{n+2}, y_{n+3})], d(y_{n+2}, y_{n+1})\}) \\
&\quad -\varphi(d(y_n, y_n{+}2)).
\end{aligned}$$

Hence, using (4),

$$\psi(d(y_{n+1}, y_{n+3})) \leq \psi(\max\{d(y_n, y_{n+2}), d(y_n, y_{n+1})\} - \varphi(d(y_n, y_n{+}2)). \tag{7}$$

Thus, from (7), for each $n \in N$, either

$$\psi(d(y_{n+1}, y_{n+3})) \leq \psi(d(y_n, y_{n+2})) - \varphi(d(y_n, y_n{+}2), \tag{8}$$

or

$$\psi(d(y_{n+1}, y_{n+3})) \leq \psi(d(y_n, y_{n+1})) - \varphi(d(y_n, y_n{+}1)). \tag{9}$$

Suppose at first that there is some $n_0 \in N$ such that (8) holds for all $n \geq n_0$. Since from (6) we have $\varphi(d(y_n, y_n{+}2) > 0$, then from (8) we get $\psi(d(y_{n+1}, y_{n+3})) < \psi(d(y_n, y_{n+2}))$. This implies, as ψ is nondecreasing,

$$d(y_{n+1}, y_{n+3}) < d(y_n, y_{n+2}) \quad \text{for all} \quad n \geq n_0.$$

Hence it follows that the sequence $\{d(y_n, y_{n+2})\}$ is monotone decreasing and consequently, there exists $p \geq 0$ such that $d(y_n, y_{n+2}) \to p+$. If we suppose that $p > 0$, then letting $n \to \infty$ in (7) and using the continuity of ψ and the property (φ_1) of φ, we get

$$\psi(p) \leq \psi(p) - \liminf_{d(y_n, y_{n+2})\to p+} \varphi(p) < \psi(p),$$

a contradiction. Thus

$$\lim_{n\to\infty} d(y_n, y_{n+2}) = 0. \tag{10}$$

Suppose now that (9) holds for some infinite subset $\{n_j\}$ of positive integers. Then from (9) we get $\psi(d(y_{n_j+1}, y_{n_j+3})) \leq \psi(d(y_{n_j}, y_{n_j+1}))$ for all $n_j \in N$. Hence, as ψ is nondecreasing,

$$d(y_{n_j+1}, y_{n_j+3}) \leq d(y_{n_j}, y_{n_j+1}) \text{ for all } \; n_j \in N.$$

Letting $j \to \infty$ in the above inequality and using (5) we get

$$\limsup_{j\to+\infty} d(y_{n_j+1}, y_{n_j+3}) \leq \lim_{j\to+\infty} d(y_{n_j}, y_{n_j+1}) = 0.$$

Hence we obtain $\limsup_{k\to\infty} d(y_{n_j+1}, y_{n_j+3}) = 0$. This implies $\lim_{n\to\infty} d(y_n, y_{n+2}) = 0$. Thus we proved that (10) holds.

Now we shall prove that $\{y_n\}$ is a g.m.s. Cauchy sequence. Suppose, to the contrary, that$\{y_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that for each $k \in N$ we can find subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ with $n_k > m_k \geq k$ such that

$$d(y_{m_k}, y_{n_k}) \geq \epsilon. \tag{11}$$

We can choose $n_k > m_k$ in such a way that it is the smallest integer for which (11) holds, that is, such that

$$d(y_{m_k}, y_{n_k-1}) < \epsilon \tag{12}$$

holds. Now, using the rectangular inequality, (12) and (11) we get, as $y_{m_k}, y_{n_k}, y_{n_k-1}, y_{n_k-2}$ are distinct points,

$$\begin{aligned}
\epsilon &\leq d(y_{m_k}, y_{n_k}) \\
&\leq d(y_{m_k}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k-2}) + d(y_{n_k-2}, y_{n_k}) \\
&< \epsilon + d(y_{n_k-2}, y_{n_k-1}) + d(y_{n_k-2}, y_{n_k}).
\end{aligned}$$

Letting $k \to +\infty$ in the above inequality, using (5) and (10), we obtain

$$d(y_{m_k}, y_{n_k}) \to \epsilon + . \tag{13}$$

From (1) with $x = x_{n_k}$ and $y = x_{m_k}$, using that $n_k > m_k$ and (4), we get

$$\begin{aligned}
\psi(d(y_{m_k+1}, y_{m_k+1})) &= \psi(d(Tx_{m_k+1}, Tx_{n_k+1})) \\
&\leq \psi(\max\{d(fx_{m_k+1}, fx_{n_k+1}), d(fx_{n_k+1}, Tx_{m_k+1}), \\
&\quad \frac{d(fx_{m_k+1}, Tx_{m_k+1}) + d(fx_{n_k+1}, Tx_{n_k+1})}{2}\}) \\
&\quad -\varphi(d(fx_{m_k+1}, fx_{n_k+1})). \\
&= \psi(\max\{d(y_{m_k}, y_{n_k}), d(y_{m_k}, y_{m_k+1}), d(y_{n_k}, y_{m_k+1})\}) \\
&\quad -\varphi(d(y_{m_k}, y_{n_k})).
\end{aligned}$$

Hence, using that $n_k > m_k$, (4) and (4), we get

$$\begin{aligned}
\psi(d(y_{m_k+1}, y_{m_k+1})) &\leq \psi(\max\{d(y_{m_k}, y_{n_k}), d(y_{m_k}, y_{m_k+1}), d(y_{m_k+1}, y_{n_k})\}) \\
&\quad -\varphi(d(y_{m_k}, y_{n_k})).
\end{aligned} \tag{14}$$

From the rectangular inequality, we have

$$\begin{aligned}
& d(y_{m_k}, y_{n_k}) - d(y_{m_k}, y_{m_k+1}) - d(y_{n_k+1}, y_{n_k}) \\
\leq\; & d(y_{m_k+1}, y_{n_k+1}) \\
\leq\; & d(y_{m_k+1}, y_{m_k}) + d(y_{m_k}, y_{n_k}) + d(y_{n_k}, y_{n_k+1}),
\end{aligned}$$

Letting $k \to +\infty$, using (13) and (5), we obtain

$$d(y_{m_k+1}, y_{n_k+1}) \to \epsilon. \tag{15}$$

Similarly, from the rectangular inequality,

$$\begin{aligned} & d(y_{m_k}, y_{n_k}) - d(y_{m_k-1}, y_{m_k}) - d(y_{n_k-1}, y_{n_k}) \\ \leq \; & d(y_{m_k-1}, y_{n_k-1}) \\ \leq \; & d(y_{m_k-1}, y_{m_k}) + d(y_{m_k}, y_{n_k}) + d(y_{n_k-1}, y_{n_k}). \end{aligned}$$

Letting $k \to +\infty$ we obtain

$$d(y_{m_k-1}, y_{n_k-1}) \to \epsilon. \tag{16}$$

Again, from the rectangular inequality,

$$d(y_{m_k-1}, y_{n_k-1}) - d(y_{m_k-1}, y_{m_k+1}) - d(y_{n_k-1}, y_{n_k}) \leq d(y_{m_k+1}, y_{n_k}),$$

and

$$\leq d(y_{m_k+1}, y_{n_k}) \leq d(y_{m_k+1}, y_{m_k-1}) + d(y_{m_k-1}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}).$$

Letting $k \to +\infty$, using (16), (10) and (5), we obtain

$$d(y_{m_k+1}, y_{n_k}) \to \epsilon. \tag{17}$$

Now, letting $k \to +\infty$ in (14), using (15), (13), (5), (17) and the continuity of ψ and the property (φ_1) of $\varphi \in \Phi^*$, we obtain

$$\psi(\epsilon) \leq \psi(\epsilon) - \lim_{d(y_{m_k}, y_{n_k}) \to \epsilon+} \varphi(d(y_{m_k}, y_{n_k})) < \psi(\epsilon),$$

which is a contradiction with $\epsilon > 0$. Hence, $\{y_n\}$ is a Cauchy sequence. Since (fX, d) is g.m.s. complete, there exists $z \in fX$ such that $\lim_{n\to+\infty} y_n = z$. Let $w \in X$ be such that $fw = z$. Then

$$\lim_{n\to+\infty} y_n = fw. \tag{18}$$

We shall prove that

$$fw = Tw. \tag{19}$$

Suppose, to the contrary, that $d(fw, Tw) > 0$. Then $\varphi(d(fw, Tw)) > 0$, which implies that $\varphi(\max\{d(fw, Tw), d(y_{n-1}, fw), d(y_{n-1}, y$
0. Now, applying the inequality (1) with $x = x_n$ and $y = w$, we obtain

$$\begin{aligned} \psi(d(Tx_n, Tw)) \;\; \leq \;\; & \psi(\max\{d(fx_n, fw), \frac{1}{2}[d(fx_n, Tx_n) + d(fw, Tw)], d(fw, Tx_n)\}) \\ & -\varphi(d(fx_n, fw)) \\ \leq \;\; & \psi(\max\{d(y_{n-1}, fw), \frac{1}{2}[d(y_{n-1}, y_n) + d(fw, Tw)], d(fw, y_n)\}). \end{aligned}$$

Hence, using that ψ is nondecreasing, we have

$$d(Tx_n, Tw) \leq \max\{d(y_{n-1}, fw), \frac{1}{2}[d(y_{n-1}, y_n) + d(fw, Tw)], d(fw, y_n)\}.$$

Letting $n \to +\infty$ in the above inequality, using (18) and (5), we get

$$\limsup_{n\to\infty} d(Tx_n, Tw) \leq \frac{1}{2} d(fw, Tw)). \tag{20}$$

From the rectangular inequality,

$$d(fw, Tw)) \leq d(fw, y_{n-1}) + d(y_{n-1}, y_n) + d(y_n, Tw).$$

Letting $n \to +\infty$ in the above inequality, using (18), (20) and (5), we get

$$d(fw, Tw)) \leq \limsup_{n\to\infty} d(y_n, Tw) \leq \frac{1}{2} d(fw, Tw).$$

Hence $d(fw, Tw)) = 0$, which implies $fw = Tw$. Thus we proved that $z = fw = Tw$ and so z is a point of coincidence of T and f.

Now we show that z is a unique point of coincidence. Let z_1 be another point of coincidence in X, that is, let $z_1 = fv = Tv$. Suppose that $z_1 \neq z$. Then $fv \neq fw$ and so $\varphi(d(fv, fw))) > 0$. From (1) we have

$$\begin{aligned} \psi(d(Tv, Tw)) &\leq \psi(\max\{d(fv, fw), \frac{1}{2}[d(fv, Tv) + d(fw, Tw)], d(fw, Tv)\}) \\ &\quad -\varphi(d(fv, fw))) \\ &\leq \psi(\max\{d(fv, fw), 0, d(fw, Tv)\}) - \varphi(d(fv, fw))) \\ &< \psi(d(Tv, Tw)), \end{aligned}$$

a contradiction. Therefore, $z_1 = z$. Thus we proved that T and f have a unique point of coincidence.

If T and f are weakly compatible, then from $fw = Tw = z$ we have $Tfw = fTw$, that is, $Tz = fz$. Let $v = Tz = fz$. Since z is a unique point of coincidence of T and f, then $v = z$. Therefore, we have $z = fz = Tz$. Thus we proved that z is the unique common fixed point of T and f. This complete the proof.

Remark 3.2. *Theorem 3.1 is a generalization of Theorem 2.5 of C. Di Bari and P. Vetro [8]..*

From Theorem 3.1, if we choose $f = I_X$ the identity mapping on X, we obtain the following fixed point result.

Corollary 3.3. *Let (X, d) be a Hausdorff complete g.m.s. space and let T be a self-mapping on X such that the following condition holds:*

$$\psi(d(Tx, Ty)) \leq \psi(\max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], d(y, Tx)\}) - \varphi(d(x, y)) \tag{21}$$

for all $x, y \in X$, where $\psi \in \Psi$ and $\varphi \in \Phi$. Then T has a unique fixed point.

Remark 3.4. *Corollary 3.3 is a generalization of Theorem 2.3 of Lakzian and Samet [10].*

Remark 3.5. *Now, we give a simple example that support the result of our Theorem 3.1.*

Example. Let $X = \{0, 1, 2, 3\}$. Define $d : X \times X \to R$ as follows:

$d(x, y) = d(y, x)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $y = x$. Further, let

$d(0, 3) = d(2, 3) = 1;\ \ d(0, 2) = d(1, 3) = 2;$

$d(0,1) = 4; \;\; d(1,2) = 5.$

Then it is easy to show that (X,d) is a complete generalized metric space, but (X,d) is not a metric space because the triangle inequality does not hold for all $x,y,z \in X$:

$$5 = d(0,1) > d(0,2) + d(2,1) = 2 + 1 = 3.$$

Now define a mappings $T, f : X \to X$ as follows:

$$\begin{aligned}
&Tx = 0, \text{ if } x \in \{0,1,2\}\\
&Tx = 2, \text{ if } x = 3;\\
&f(0) = 0, \;\; f(1) = 2, \;\; f(2) = 3, \;\; f(3) = 1.
\end{aligned}$$

Then, T and f satisfy (1) with $\psi(t) = 2t$ and $\varphi(t) = t/2$. Indeed, $d(Tx,Ty) > 0$ only if $x \in \{0,1,2\}$ and $y = 3$. We have

$$\begin{aligned}
&4 = \psi(d(T(0),T(3))) \le 2 \cdot d(f(0),f(3)) - \frac{d(f(0),f(3))}{3} = 10 - \frac{4}{2} = 8;\\
&4 = \psi(d(T(1),T(3))) \le 2 \cdot d(f(1),f(3)) - \frac{d(f(1),f(3))}{2} = 10 - \frac{5}{2} = 7,5;\\
&4 = \psi(d(T(2),T(3))) \le 2 \cdot [\frac{d(f(2),T(2)) + d(f(3),T(3))}{2}] - \frac{d(f(2),f(3))}{3} = [1+5] - \frac{3}{2}.
\end{aligned}$$

Therefore, T and f satisfy the inequality (1).Clearly, $T(X) \subset f(X)$ and T and f are weakly compatible. So we can apple our Theorem 3.1 and T and f have a unique fixed point $z = 0$.

Remark 3.6. *Note that for the above example there is not $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(d(T(x),T(y))) \le \psi(d(f(x),f(y))) - \varphi(d(f(x),f(y)))$$

for $x = 2$ and $y = 3$ we have $d(T(x),T(y)) = 2$ and $d(f(x),f(y)) = 2$. Thus,

$$4 = \psi(2) > \psi(2) - \varphi(2) = 4 - 1 = 3.$$

Therefore, Theorem 2.5 of C. Di Bari and P. Vetro [8], as well as Theorem 2.3 of Lakzian and Samet [10], can not be applied in this example.

From Theorem 3.1, we can derive many interesting fixed point results in generalized metric spaces involving contractive conditions of integrable type. Denote by L the set of functions $\phi : [0,+\infty) \to [0,+\infty)$ which are Lebesgue integrable on each compact subset of $[0,+\infty)$ such that for every $\epsilon > 0$, we have

$$\int_0^{\epsilon} \phi(s)ds > 0.$$

Since the function $\psi : [0,+\infty) \to [0,+\infty)$ defined by $\psi(t) = \int_0^t \phi(s)ds$ belongs to Ψ, we obtain the following theorem.

Theorem 3.7. *Let (X,d) be a Hausdorff g.m.s. and let T and f be self-mappings on X such that $TX \subseteq fX$. Assume that (fX,d) is a complete g.m.s. and that the following condition holds:*

$$\int_0^{d(Tx,Ty)} \phi(s)ds \le \int_0^{\max\{d(fx,fy),\frac{1}{2}[d(fx,Tx)+d(fy,Ty)],d(fy,Tx)\}} \phi(s)ds - \varphi(d(fx,fy))$$

for all $x,y \in X$, where $\phi \in L$ and $\varphi \in \Phi$. If T and f are weakly compatible, then T and f have a unique fixed point.

Taking $\varphi(t) = (1-k)\int_0^s \phi(s)ds$ for $k \in [0,1)$ in Theorem 3, we obtain the following result.

Theorem 3.8. *Let (X,d) be a Hausdorff g.m.s. and let T and f be self-mappings on X such that $TX \subseteq fX$. Assume that (fX,d) is a complete g.m.s. and that the following condition holds:*

$$\int_0^{d(Tx,Ty)} \phi(s)ds = \lambda \int_0^{\max\{d(fx,fy),\frac{1}{2}[d(fx,Tx)+d(fy,Ty)],d(fy,Tx)\}\}} \phi(s)ds$$

for all $x,y \in X$, where $\lambda \in [0,1)$. If T and f are weakly compatible, then T and f have a unique fixed point.

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