Filomat 27:8 (2013), 1425–1435
DOI 10.2298/FIL1308425S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Remoteness, proximity and few other distance invariants in graphs

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Abstract. We establish maximal trees and graphs for the difference of average distance and proximity proving thus the corresponding conjecture posed in M. Aouchiche, P. Hansen, Proximity and remoteness in graphs: results and conjectures, Networks 58 (2) (2011) 95102. We also establish maximal trees for the difference of average eccentricity and remoteness and minimal trees for the difference of remoteness and radius proving thus that the corresponding conjectures posed in M. Aouchiche, P. Hansen, Proximity and remoteness in graphs: results and conjectures, Networks 58 (2) (2011) 95102 hold for trees.

1. Introduction

All graphs G in this paper are simple and connected. A vertex set of graph G will be denoted by V, an edge set by E. A number of vertices in G is denoted by n, a number of edges by m. A path on n vertices will be denoted by P_n, while C_n will denote a cycle on n vertices. A *tree* is the graph with no cycles, and a *leaf* in a tree is any vertex of degree 1. We say that a tree G is a *caterpillar tree* if it consists of the path P and the only vertices outside P are leafs neighboring to vertices on P.

The *distance* $d(u,v)$ between two vertices u and v in G is defined as the length of the shortest path connecting vertices u and v. The average distance between all pairs of vertices in G is denoted by $\bar{l}$. The *eccentricity* $e(v)$ of a vertex v in G is the largest distance from v to another vertex of G. The *radius* r of a graph G is defined as the minimum eccentricity in G, while the *diameter* D of G is defined as the maximum eccentricity in G. The average eccentricity of G is denoted by ecc. That is

$$r = \min_{v \in V} e(v),\ D = \max_{v \in V} e(v),\ ecc = \frac{1}{n} \sum_{v \in V} e(v).$$

The *center* of a graph is the vertex v of minimum eccentricity. It is well-known that every tree has either only one center or two centers which are adjacent. The *diametric path* in G is the shortest path from u to v, where $d(u,v)$ is equal to the diameter of G.

The *transmission* of a vertex v in a graph G is the sum of the distances between v and all other vertices of G. The transmission is said to be *normalized* if it is divided by $n-1$. Normalized transmission of a vertex

2010 *Mathematics Subject Classification*. 05C35
Keywords. Remoteness, Proximity, Distance invariants, Extremal trees
Received: 14 June 2012; Accepted: 15 October 2012
Communicated by Dragan Stevanović
The support of the Ministry of Science, Education and Sport of the Republic of Croatia (grant. no. 083-0831510-1511) and of EUROCORES Programme EUROGIGA (project GReGAS) of the European Science Foundation is gratefully acknowledged.
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v will be denoted by $\pi(v)$. The *remoteness* ρ is defined as the maximum normalized transmission, while the *proximity* π is defined as the minimum normalized transmission. That is

$$\pi = \min_{v\in V} \pi(v), \ \rho = \max_{v\in V} \pi(v).$$

In other words, the proximity π is the minimum average distance from a vertex of G to all others, while the remoteness ρ of a graph G is the maximum average distance from a vertex of G to all others. These two invariants were introduced in [1], [2]. A vertex $v \in V$ is *centroidal* if $\pi(v) = \pi(G)$, and the set of all centroidal vertices is the *centroid* of G.

Recently, these concepts and relations between them have been extensively studied (see [1], [2], [3], [4], [11], [12]). For example, in [3] the authors established the Nordhaus–Gaddum theorem for π and ρ. In [4] upper and lower bounds for π and ρ were obtained expressed in number n of vertices in G. Also, relations of both invariants with some other distance invariants (like diameter, radius, average eccentricity, average distance, etc.) were studied. The authors posed several conjectures (one of which was solved in [11]), among which the following.

Conjecture 1.1. *Among all connected graphs G on $n \geq 3$ vertices with average distance $\bar{l}$ and proximity π, the difference $\bar{l} - \pi$ is maximum for a graph G composed of three paths of almost equal lengths with a common end point.*

Conjecture 1.2. *Let G be a connected graph on $n \geq 3$ vertices with remoteness ρ and average eccentricity ecc. Then*

$$ecc - \rho \leq \begin{cases} \frac{3n+1}{4}\frac{n-1}{n} - \frac{n}{2} & \text{if } n \text{ is odd,} \\ \frac{n-1}{4} - \frac{1}{4n-4} & \text{if } n \text{ is even,} \end{cases}$$

with equality if and only if G is a cycle C_n.

Conjecture 1.3. *Let G be a connected graph on $n \geq 3$ vertices with remoteness ρ and radius r. Then*

$$\rho - r \geq \begin{cases} \frac{3-n}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2}{4n-4} - \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

The inequality is best possible as shown by the cycle C_n if n is even and by the graph composed by the cycle C_n together with two crossed edges on four successive vertices of the cycle.

In this paper we prove Conjecture 1.1, and find the extremal trees for $ecc - \rho$ and $\rho - r$ (maximal and minimal trees respectively) showing thus that Conjectures 1.2 and 1.3 hold for trees.

All these conjectures were obtained with the use of AutoGraphiX, a conjecture-making system in graph theory (see for example [6] and [7]). Some results on center and centroidal vertices will be used which are already known in literature since those concepts were also quite extensively studied (see for example [5], [8], [9], [10]).

2. Preliminaries

Let us introduce some additional notation for trees and state some auxiliary results known in literature. First, we will often use the notion of the diametric path. So, if a tree G of diameter D has diametric path P, we will suppose that vertices on P are denoted by v_i so that $P = v_0v_1 \ldots v_D$. When deleting edges of P from G, we obtain several connected components which are subtrees rooted in vertices of P. Now, G_i will denote the connected component of tree $G\backslash P$ rooted in vertex v_i of P and V_i will denote the set of vertices of G_i.

Furthermore, for a tree G let $e \in E$ be an edge in G and $u \in V$ a vertex in G. With $G_u(e)$ we will denote the connected component of $G - e$ containing u. We denote $V_u(e) = V(G_u(e))$ and $n_u(e) = |V_u(e)|$. Now the following lemma holds.

Lemma 2.1. *The following statements hold for a tree G:*

1. *a vertex $v \in V(G)$ is a centroidal vertex if and only if for any edge e incident with v it holds that $n_v(e) \geq \frac{n}{2}$,*
2. *G has at most two centroidal vertices,*
3. *if there are two centroidal vertices in G, then they are adjacent,*
4. *G has two centroidal vertices if and only if there is an edge e in G, such that the two components of $G - e$ have the same order. Furthermore, the end vertices of e are the two centroidal vertices of G.*

Proof. See [11]. □

Since we will often use transformation of tree G to G', for the sake of notation simplicity we will write D' for $D(G')$, ρ' for $\rho(G')$, $\pi'(v)$ for $\pi(v)$ in G', etc.

3. Average distance and proximity

To prove Conjecture 1.1 for trees, we will use graph transformations which transform tree to either:
1) path P_n,
2) a tree consisting of four paths of equal length with a common end point,
3) a tree consisting of three paths of almost equal length with a common end point.
So let us first prove that among those graphs the difference $\bar{l} - \pi$ is maximum for the last.

Lemma 3.1. *The difference $\bar{l} - \pi$ is greater for a tree G on $n \geq 4$ vertices consisting of three paths of almost equal length with a common end point than for path P_n.*

Proof. For a path P_n we have $\bar{l}(P_n) = \frac{1+n}{3}$, while $\pi(P_n) = \frac{n^2}{4(n-1)}$ for n even and $\pi(P_n) = \frac{n+1}{4}$ for n odd. Therefore the difference $\bar{l}(P_n) - \pi(P_n)$ equals $\frac{n^2-4}{12(n-1)}$ for n even and $\frac{n+1}{12}$ for n odd. Now, let G be a tree on n vertices consisting of three paths of almost equal length with a common end point. Here we have

$$\bar{l}(G) - \pi(G) = \begin{cases} \frac{7n^2+13n-2}{27n} - \frac{2+n}{6} & \text{for } n = 3k+1 \\ \frac{(7n-8)(1+n)^2}{27n(n-1)} - \frac{n(n+1)}{6(n-1)} & \text{for } n = 3k+2 \\ \frac{7n^2+6n-9}{27(n-1)} - \frac{n(n+1)}{6(n-1)} & \text{for } n = 3k+3 \end{cases}$$

where $k \in \mathbb{N}$. Now, one has to show that the difference $\bar{l}(G) - \pi(G)$ is greater than $\bar{l}(P_n) - \pi(P_n)$ in each of the six possible cases. For example, if n is even and $n = 3k+1$, then

$$\left(\bar{l}(G) - \pi(G)\right) - \left(\bar{l}(P_n) - \pi(P_n)\right) = \frac{(n+2)^3}{108n(n-1)} > 0$$

and the claim holds. In a similar way it can be seen that the claim holds in each of the remaining five cases. □

Lemma 3.2. *The difference $\bar{l} - \pi$ is greater for the tree G on $n \geq 9$ vertices, where $n = 1 \bmod(4)$, consisting of three paths of almost equal length with a common end point than for the tree G' on n vertices consisting of four paths of equal length.*

Proof. First note that we already established the value of $\bar{l}(G) - \pi(G)$ in the proof of Lemma 3.1. Now, let us establish the value of $\bar{l}(G') - \pi(G')$. Note that $\bar{l}(G') = \frac{5n^2+14n-3}{24n}$, while $\pi(G') = \frac{n+3}{8}$. Therefore, $\bar{l}(G') - \pi(G') = \frac{2n^2+5n-3}{24n}$. Now, one has to show that the difference $\bar{l}(G) - \pi(G)$ is greater than $\bar{l}(G') - \pi(G')$ in each of the three possible cases. For example, if $n = 3k+1$ then

$$\left(\bar{l}(G) - \pi(G)\right) - \left(\bar{l}(G') - \pi(G')\right) = \frac{2n^2-13n+11}{216n} > 0.$$

In a similar way it can be seen that the claim holds in each of the remaining two cases. □

Lemma 3.3. *Let G be a tree on $n \geq 6$ vertices with at least four leafs. Then there is a tree G' on n vertices with three leafs for which the difference $\bar{l} - \pi$ is greater or equal than for G.*

Proof. Let u be a centroidal vertex of G, let v be the branching vertex furthest from u. We distinguish two cases.

CASE I: $u \neq v$. Let G_v be the subtree of G rooted in v consisting of all vertices w such that path from u to w leads through v. Since v is a branching vertex furthest from u, tree G_v consists of paths with common end v. Let P_1 and P_2 be two such paths. For $i = 1, 2$ let x_i be a vertex in P_i adjacent to v and let y_i be a leaf in P_i. Let G' be the tree obtained from G by deleting edge vx_2 and adding edge x_2y_1. This transformation is illustrated in Figure 1. Note that G' has one leaf less than G. We want to prove that the difference $\bar{l} - \pi$ is greater for G' then for G. For that purpose let us denote $d_1 = d(v, y_1)$ and $d_2 = d(v, y_2)$. Note that

$$\pi(G') \leq \pi'(u) = \pi(u) + \frac{d_1 d_2}{n-1} = \pi(G) + \frac{d_1 d_2}{n-1}$$

and

$$\bar{l}(G') = \bar{l}(G) + \frac{2}{n(n-1)} \cdot d_2(n - d_1 - d_2 - 1)d_1.$$

From here we obtain

$$\bar{l}(G') - \pi(G') \geq \bar{l}(G) - \pi(G) + \frac{d_1 d_2}{n-1}\left(\frac{2(n - d_1 - d_2 - 1)}{n} - 1\right).$$

By Lemma 2.1 we have $n - d_1 - d_2 - 1 \geq \frac{n}{2}$, therefore $\bar{l}(G') - \pi(G') \geq \bar{l}(G) - \pi(G)$.

CASE II: $u = v$. Obviously, v is the only branching vertex in G. Therefore G consists of paths with common end point v. Let P_1 and P_2 be two shortest such path. If $V \backslash (P_1 \cup P_2 \cup \{v\})$ contains at least $\frac{n}{2}$ vertices, then we make the same argument as in case I. Otherwise G is a tree consisting of four paths of equal length with a common end point and the claim follows by Lemma 3.2.

Applying the transformations from cases I and II repeatedly, one obtains the claim. □

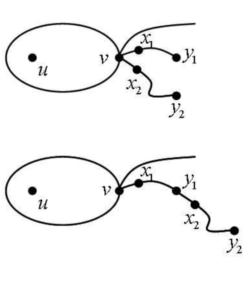

Figure 1: Tree transformation in the proof of Lemma 3.3.

Lemma 3.4. *Among trees with three leafs, the difference $\bar{l} - \pi$ is maximal for the tree G on n vertices consisting of three paths of almost equal length with a common end vertex.*

Proof. Let G be a tree with three leafs. That implies G consists of three paths with a common end vertex. Let u be centroidal vertex of G, let v be the branching vertex furthest from u. If $u \neq v$, then by the same argument as in the case I of the proof of Lemma 3.3 we obtain that the difference $\bar{l} - \pi$ is greater or equal for path P_n than for G. Now the claimed follows from Lemma 3.1. Else if $u = v$, then all three paths graph G consists of have less than $\frac{n}{2}$ vertices. Let v_1 be the leaf furthest from u and v_2 the leaf closest to u. Let G'

be a tree obtained from G by deleting the edge incident to v_1 and adding the edge v_1v_2. We want to prove that the difference $\bar{l} - \pi$ is greater or equal for G' than for G. Let $d_1 = d(u, v_1)$ and $d_2 = d(u, v_2)$. We have

$$\pi'(u) = \pi(u) - \frac{d_1 - d_2 - 1}{n - 1}$$

and

$$\bar{l}(G') = \bar{l}(G') - \frac{2}{n(n-1)}(n - d_1 - d_2 - 1)(d_1 - d_2 - 1).$$

From here we obtain

$$\bar{l}(G') - \pi(G') \geq \bar{l}(G) - \pi(G) + \frac{d_1 - d_2 - 1}{n - 1}\left(1 - \frac{2}{n}(n - d_1 - d_2 - 1)\right).$$

Since all three paths of G have less then $\frac{n}{2}$ vertices, we can conclude that $n - d_1 - d_2 - 1 \leq \frac{n}{2}$ from which follows $\bar{l}(G') - \pi(G') \geq \bar{l}(G) - \pi(G)$. By repeating this tree transformation we obtain the claim. □

We can summarize the results of Lemmas 3.1, 3.2, 3.3 and 3.4 into following theorem.

Theorem 3.5. *Among all trees on $n \geq 4$ ($n \neq 5$) vertices with average distance $\bar{l}$ and proximity π, the difference $\bar{l} - \pi$ is maximal for a tree G composed of three paths of almost equal lengths with a common end vertex.*

Therefore, we have proved Conjecture 1.1 for trees on $n \geq 4$ ($n \neq 5$) vertices. Note that for $n = 3$ there is only one tree with n vertices and that is P_3. For $n = 5$ the claim does not hold since for a star S_5 (i.e. a graph consisting of one vertex of degree 4 and 4 vertices of degree 1) holds

$$\bar{l}(S_5) - \pi(S_5) = \frac{16}{10} - \frac{4}{4} = \frac{3}{5},$$

while for a tree G on $n = 5$ vertices composed of three paths of almost equal lengths with a common end vertex holds

$$\bar{l}(G) - \pi(G) = \frac{18}{10} - \frac{5}{4} = \frac{11}{20}.$$

Therefore obviously $\bar{l}(S_5) - \pi(S_5) > \bar{l}(G) - \pi(G)$.

Now we want to prove Conjecture 1.1 for general graphs. If for every graph we find a tree for which difference $\bar{l} - \pi$ is greater or equal, the Conjecture 1.1 for general graphs will follow from Theorem 3.5.

Theorem 3.6. *Among all connected graphs G on $n \geq 4$ ($n \neq 5$) vertices with average distance $\bar{l}$ and proximity π, the difference $\bar{l} - \pi$ is maximal for a graph G composed of three paths of almost equal lengths with a common end vertex.*

Proof. Let G be a connected graph on $n \geq 3$ vertices and let $u \in V(G)$ be a vertex in G such that $\pi(u) = \pi(G)$. Let G' be a breadth-first search tree of G rooted at u. Obviously, $\pi(G) = \pi(u) = \pi'(u) \geq \pi(G')$. As for $\bar{l}$, by deleting edges from G distances between vertices can only increase, therefore $\bar{l}(G) \leq \bar{l}(G')$. Now we have $\bar{l}(G) - \pi(G) \leq \bar{l}(G') - \pi(G')$ and the claim follows from Theorem 3.5. □

4. Average eccentricity and remoteness

Now, let us find maximal trees for $ecc - \rho$, proving thus that the Conjecture 1.2 holds for trees.

Lemma 4.1. *Let G be a tree on n vertices with diameter D and let $P = v_0v_1 \ldots v_D$ be a diametric path in G. If there is $j \leq D/2$ such that the degree of v_k is at most 2 for $k \geq j + 1$, then the difference $ecc - \rho$ is greater or equal for the path P_n than for G.*

Proof. Let w be a leaf in G distinct from v_0 and v_D. Let G' be a tree obtained from G by deleting the edge incident to w and adding the edge $v_D w$. Note that the diameter of G' equals $D+1$. We want to prove that difference $ecc - \rho$ did not decrease by this transformation. First note that eccentricity increased by 1 for at least $n - \frac{D+1}{2}$ vertices. Therefore, $ecc' \geq ecc + \frac{2n-D-1}{2n}$. As for remoteness, first note that $\rho(G) = \pi(v_D)$ and $\rho(G') = \pi'(w)$. Now, let d_w be the distance between vertices w and v_D in G, i.e. $d_w = d(w, v_D)$. Obviously $d_w \geq \frac{D+2}{2}$. Now, we have

$$\pi'(w) = \pi(v_D) + \frac{n - d_w - 1}{n-1} \leq \pi(v_D) + \frac{2n - D - 4}{2(n-1)}.$$

Therefore,

$$ecc' - \rho' \geq ecc - \rho + \frac{2n-D-1}{2n} - \frac{2n-D-4}{2(n-1)} \geq ecc - \rho.$$

We obtain the claim by repeating this transformation. □

Theorem 4.2. *Among trees on $n \geq 3$ vertices, the difference $ecc - \rho$ is maximal for path P_n.*

Proof. Let G be a tree on n vertices and with diameter D. Let $P = v_0 v_1 \ldots v_D$ be a diametric path in G. Let G_i be the tree that is connected component of $G \backslash P$ rooted in v_i and let V_i be the vertex set of G_i. If there is $j \leq D/2$ such that the degree of v_k is at most 2 for $k \geq j+1$, then the claim follows from Lemma 4.1. Else, let v_j and v_k be vertices on P of degree at least 3 such that $j \leq \frac{D}{2} < k$ and $k - j$ is minimum possible. Let w_j be a vertex outside of P adjacent to v_j and let w_k be a vertex outside of P adjacent to v_k. Let G' be a tree obtained from G so that:

1) for every vertex w adjacent to v_j, except $w = w_j$ and $w = v_{j+1}$, edge wv_j is deleted and edge ww_j aded,

2) for every vertex w adjacent to v_k, except $w = w_k$ and $w = v_{k-1}$, edge wv_k is deleted and edge ww_k aded.

This transformation is illustrated in Figure 2. Note that diameter of G' equals $D+2$. We want to prove that $ecc' - \rho' \geq ecc - \rho$. For that purpose, let us denote

$$V'_j = \left\{v \in V_j : d(v, w_j) < d(v, v_j)\right\},$$
$$V'_k = \{v \in V_k : d(v, w_k) < d(v, v_k)\}.$$

Now, let us introduce following partition of set of vertices V

$$X_1 = V_0 \cup \ldots \cup V_{j-1} \cup (V_j \backslash (V'_j \cup \left\{v_j\right\})),$$
$$X_2 = V'_j,$$
$$X_3 = \left\{v_j\right\} \cup V_{j+1} \cup \ldots \cup V_{k-1} \cup \{v_k\},$$
$$X_4 = V'_k,$$
$$X_5 = \left(V_k \backslash (V'_k \cup \{v_k\})\right) \cup V_{k+1} \cup \ldots \cup V_D.$$

Let $x_i = |X_i|$. Now, let us compare $e'(v)$ and $e(v)$ for every vertex $v \in V$. Note that for $v \in X_2 \cup X_3 \cup X_4$ it holds that $e'(v) = e(v) + 1$, while for $v \in X_1 \cup X_5$ it holds that $e'(v) = e(v) + 2$. Therefore,

$$ecc' = ecc + \frac{2x_1 + x_2 + x_3 + x_4 + 2x_5}{n} = ecc + \Delta_1.$$

Now, we want to compare $\pi'(v)$ and $\pi(v)$ for every $v \in V$. We distinguish several cases depending whether $v \in X_1$, $v \in X_2$, $v \in X_3$, $v \in X_4$ or $v \in X_5$. It is sufficient to consider cases $v \in X_1$, $v \in X_2$ and $v \in X_3$, since $v \in X_4$ is analogous to $v \in X_2$ and $v \in X_5$ is analogous to $v \in X_1$.

If $v \in X_1$, then the difference $d'(v,u) - d(v,u)$ equals 0 for $u \in X_1$, equals -1 for $u \in X_2$, equals 1 for $u \in X_3 \cup X_4$ and equals 2 for $u \in X_5$. Therefore,

$$\pi'(v) = \pi(v) + \frac{-x_2 + x_3 + x_4 + 2x_5}{n-1} = \pi(v) + \Delta_2.$$

If $v \in X_2$, then the difference $d'(v,u) - d(v,u)$ equals -1 for $u \in X_1$, equals 0 for $u \in X_2 \cup X_3 \cup X_4$ and equals 1 for $u \in X_5$. Therefore,

$$\pi'(v) = \pi(v) + \frac{-x_1 + x_5}{n-1} = \pi(v) + \Delta_3.$$

If $v \in X_3$, then the difference $d'(v,u) - d(v,u)$ equals 1 for $u \in X_1 \cup X_5$ and equals 0 for $u \in X_2 \cup X_3 \cup X_4$. Therefore,

$$\pi'(v) = \pi(v) + \frac{x_1 + x_5}{n-1} = \pi(v) + \Delta_4.$$

It is easily verified that $\Delta_1 - \Delta_2 \geq 0$, $\Delta_1 - \Delta_3 \geq 0$ and $\Delta_1 - \Delta_4 \geq 0$, so for every $v \in V$ we obtain $ecc' - \pi'(v) \geq ecc - \pi(v)$.

Now, let $u \in V$ be a vertex for which $\pi'(u) = \rho(G')$. We have

$$\begin{aligned} ecc(G') - \rho(G') &= ecc(G') - \pi'(u) \geq ecc(G) - \pi(u) \geq \\ &\geq ecc(G) - \max\{\pi(v) : v \in V\} = ecc(G) - \rho(G). \end{aligned}$$

□

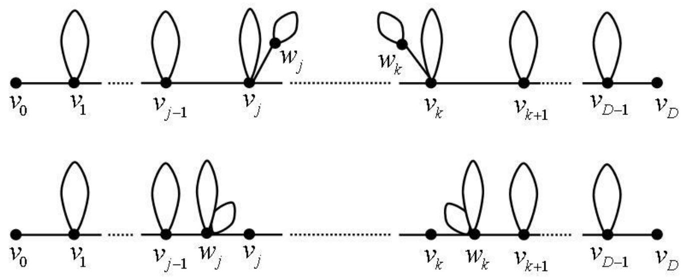

Figure 2: Tree transformation in the proof of Theorem 4.2.

Therefore, we have proved that P_n is the tree which maximizes the difference $ecc - \rho$. Now, from

$$ecc(P_n) - \rho(P_n) = \begin{cases} \frac{n-2}{4} & \text{for even } n, \\ \frac{n}{4} - \frac{2n+1}{4n} & \text{for odd } n. \end{cases}$$

easily follows that Conjecture 1.2 holds for trees.

5. Remoteness and radius

First, we want to find minimal trees for $\rho - r$. For that purpose, the first step is to reduce the problem to caterpillar trees.

Lemma 5.1. *Let G be a tree on n vertices. There is a caterpillar tree G' on n vertices for which the difference $\rho - r$ is less or equal than for G.*

Proof. Let $P = v_0 v_1 \ldots v_D$ be a diametric path in G. Let G_i be the tree that is connected component of $G \backslash P$ rooted in v_i and let V_i be the vertex set of G_i. Let G' be the caterpillar tree obtained from G in a following manner. In a tree G_i let v be the non-leaf vertex furthest from v_i, let $w_1, \ldots, w_k$ be all leafs adjacent to v, and let u be the only remaining vertex adjacent to v. Now, for every $j = 1, \ldots, k$ edge $w_j v$ is deleted and edge $w_j u$ is added. This transformation is illustrated in Figure 3.The procedure is done repeatedly in every G_i

$(2 \leq i \leq D-2)$ until the caterpillar tree G' is obtained. Note that G' has the same diameter (and therefore radius) as G. What remains to be proved is that remoteness in G' is less or equal than in G. It is sufficient to prove that the described transformation does not increase remoteness. Obviously, $\pi'(u) \leq \pi(u)$ for every $u \in V \backslash \{v\}$. Number $\pi'(v)$ can be greater than $\pi(v)$, but note that $\pi'(v) = \pi'(w_i) \leq \pi(w_i) \leq \rho$. Therefore $\rho' \leq \rho$. □

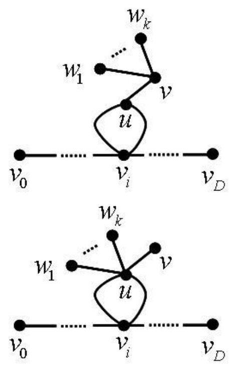

Figure 3: Tree transformation in the proof of Lemma 5.1.

Now that we reduced the problem to the caterpillar trees, let us prove some auxiliary results for such trees. First note that because of Lemma 2.1, a leaf in a tree can not be centroidal vertex. Therefore, in a caterpillar tree a centroidal vertex must be on diametric path P.

Lemma 5.2. *Let $G \neq P_n$ be a caterpillar tree on n vertices with diameter D, remoteness ρ and only one centroidal vertex. Let $P = v_0 v_1 \ldots v_D$ be the diametric path in G such that $v_j \in P$ is the only centroidal vertex in G and every of the vertices $v_{j+1}, \ldots, v_D$ is of the degree at most 2. Then there is a caterpillar tree G' on n vertices of the diameter $D+1$ and the remoteness at most $\rho + \frac{1}{2}$.*

Proof. If v_j is of degree 2, then by Lemma 2.1 follows that $j \leq \frac{D}{2}$, so $\rho = \pi(v_D)$. Let w be any leaf in G distinct from v_0 and v_D. Let G' be a graph obtained from G by first deleting edge incident to w, then deleting edge $v_{j-1}v_j$ and adding path $v_{j-1}wv_j$ instead. This transformation is illustrated in Figure 4. Note that the diameter of G' is $D+1$, while the remoteness is still obtained for v_D. Note that the distance from v_D has increased by 1 for at most $\frac{n}{2} - 1$ vertices. Therefore, $\pi'(v_D) \leq \pi(v_D) + \frac{n-2}{2(n-1)}$ from which follows $\rho' \leq \rho + \frac{1}{2}$ and the claim is proved in this case.

If the degree of v_j is greater than 2, then v_j must have at least one neighbor that is a leaf. Let us denote that leaf neighboring to v_j by w. Let $V_L = V_1 \cup \ldots \cup V_{j-1}$ and $V_R = V_{j+1} \cup \ldots \cup V_D$. Since v_j is a centroidal vertex, from Lemma 2.1 follows that V_L and V_R have at most $\frac{n}{2}$ vertices. If any of them had exactly $\frac{n}{2}$ vertices, then G would have two centroidal vertices by Lemma 2.1, which would be contradiction with v_j being the only centroidal vertex. Therefore, we conclude $|V_L| \leq \frac{n-1}{2}$ and $|V_R| \leq \frac{n-1}{2}$. Now it is possible to divide the set of vertices $V_j \backslash \left\{v_j\right\}$ into two subsets V_j' and V_j'' such that $\left|V_L \cup V_j'\right| \leq \frac{n-1}{2}$ and $\left|V_R \cup V_j''\right| \leq \frac{n-1}{2}$. Let G' be a graph obtained from G by first deleting the edge incident to w, then deleting the edge $v_j v_{j+1}$ and adding a path $v_j w v_{j+1}$ instead, and finally for every vertex $v \in V_j''$ the edge vv_j is deleted and the edge vw added. This transformation is illustrated in Figure 4. Note that the diameter of G' is $D+1$. Now, if $v \in V_L \cup V_j' \cup \left\{v_j, w\right\}$ the distance $d(v,u)$ has increased by 1 only if $u \in V_R \cup V_j''$, therefore $\pi'(v) \leq \pi(v) + \frac{1}{2}$. If $v \in V_R \cup V_j''$ the distance $d(v,u)$ has increased by 1 only if $u \in V_L \cup V_j'$, therefore $\pi'(v) \leq \pi(v) + \frac{1}{2}$. We conclude $\rho' \leq \rho + \frac{1}{2}$, and the claim is proved in this case too. □

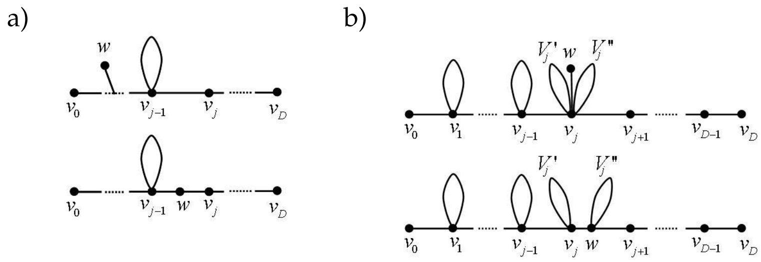

Figure 4: Tree transformations in the proof of Lemma 5.2: a) v_j is of degree 2, b) v_j is of degree at least 3.

Lemma 5.3. *Let $G \neq P_n$ be a caterpillar tree on n vertices with diameter D, remoteness ρ and exactly two centroidal vertices. Let $P = v_0 v_1 \dots v_D$ be a diametric path in G such that $v_j, v_{j+1} \in P$ are centroidal vertices and every of the vertices $v_{j+1}, \dots, v_D$ is of degree at most 2. Then there is a caterpillar tree G' on n vertices of the diameter $D + 1$ and the remoteness at most $\rho + \frac{1}{2}$.*

Proof. Since v_{j+1} is centroidal vertex, from Lemma 2.1 follows that $j \leq \frac{D}{2}$, so $\rho = \pi(v_D)$. Let w be any leaf in G distinct from v_0 and v_D. Let G' be a graph obtained from G by first deleting the edge incident to w, then deleting the edge $v_j v_{j+1}$ and adding the path $v_j w v_{j+1}$ instead. The diameter of G' is $D + 1$ and the remoteness is still obtained for v_D. Note that distances from v_D increased by 1 for at most $\frac{n}{2} - 1$ vertices, so $\pi'(v_D) \leq \pi(v_D) + \frac{n-2}{2(n-1)}$. Therefore, $\rho' \leq \rho + \frac{1}{2}$. □

Lemma 5.4. *Let $G \neq P_n$ be a caterpillar tree on n vertices with diameter D, remoteness ρ and exactly two centroidal vertices of different degrees. Let $P = v_0 v_1 \dots v_D$ be a diametric path in G such that $v_j, v_{j+1} \in P$ are centroidal vertices and every of the vertices $v_0, \dots, v_{j-1}, v_{j+2}, \dots, v_D$ is of degree at most 2. Then there is a caterpillar tree G' on n vertices of the diameter $D + 1$ and the remoteness at most $\rho + \frac{1}{2}$.*

Proof. Let $d_1 = d(v_0, v_j)$ and $d_2 = d(v_{j+1}, v_D)$. Without loss of generality we may assume that $d_1 \leq d_2$. Since the degrees of v_j and v_{j+1} differ, from Lemma 2.1 we conclude $d_1 \neq d_2$. Therefore, $d_1 < d_2$. From this follows $j + 1 \leq \frac{D}{2}$, so $\rho = \pi(v_D)$. Let G' be a graph obtained from G so that for every leaf w incident to v_j (distinct from v_0) we delete the edge wv_j and add the edge wv_{j+1}. The diameter of G' is still D, while the remoteness ρ' is less or equal than ρ. Note that G' has diametric path $P = v_0 v_1 \dots v_D$ and only one centroidal vertex which is v_{j+1}. All other vertices on P are of degree at most 2. Therefore, we can apply Lemma 5.2 on G' and the claim follows. □

Lemma 5.5. *Let $G \neq P_n$ be a caterpillar tree on n vertices with diameter D, remoteness ρ and exactly two centroidal vertices of equal degrees. Let $P = v_0 v_1 \dots v_D$ be a diametric path in G such that $v_j, v_{j+1} \in P$ are centroidal vertices and every of the vertices $v_0, \dots, v_{j-1}, v_{j+2}, \dots, v_D$ is of degree at most 2. Then the difference $\rho - r$ is less or equal for path P_n than for G.*

Proof. Let $d_1 = d(v_0, v_j)$ and $d_2 = d(v_{j+1}, v_D)$. Since v_j and v_{j+1} have equal degrees, and every of the vertices $v_0, \dots, v_{j-1}, v_{j+2}, \dots, v_D$ is of degree at most 2, we conclude that $d_1 = d_2$. Now, we will transform the tree twice which is illustrated in Figure 5. First, since G is not a path, both v_j and v_{j+1} must have a pendent leaf. Denote those leafs with w_1 and w_2 respectively. Let G' be a graph obtained from G by first deleting edges incident to w_1 and w_2, then deleting edge $v_j v_{j+1}$ and adding path $v_j w_1 w_2 v_{j+1}$ instead. Note that $D' = D + 2$. Therefore, $r' = r + 1$. Note that remoteness in both G and G' is obtained for v_0 and v_D. Since distances from v_0 have increased by 2 for at most $\frac{n}{2} - 1$ vertices, we conclude $\pi'(v_0) \leq \pi(v_0) + \frac{2(n-2)}{2(n-1)}$ from which follows $\rho' \leq \rho + 1$. Thus we obtain $\rho' - r' \leq \rho - r$. If G' is a path, then the claim is proved. Else, we transform G' so that for every leaf w in G' incident to v_j edge wv_j is deleted and edge ww_1 is added. Also, for every leaf w in G' incident to v_{j+1} edge wv_{j+1} is deleted and edge ww_2 is added. Note that this transformation changes neither radius neither remoteness. Thus we obtain the tree on which we can repeat the whole procedure. After repeating procedure finite number of times we obtain the path P_n and the claim is proved. □

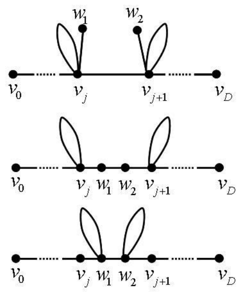

Figure 5: Tree transformations in the proof of Lemma 5.5.

Now that we have established auxiliary results for caterpillar trees, we can find minimal trees for $\rho - r$ among caterpillar trees.

Lemma 5.6. *Let G be a caterpillar tree on n vertices. If n is odd, then the difference $\rho - r$ is less or equal for path P_n then for G. If n is even, then the difference $\rho - r$ is less or equal for path P_{n-1} with a leaf appended to a central vertex than for G.*

Proof. Let D be the diameter in G and let $P = v_0v_1 \dots v_D$ be the diametric path in G. Suppose $D \leq n-3$. That means G has at least two leafs outside P. Let $v_j \in P$ be a centroidal vertex in G. If there are two vertices v_k and v_l on P ($k < j < l$) with a pendent leaf on them (distinct from v_0 and v_D), then the caterpillar tree G' obtained from G by deleting a leaf from v_j and v_k and adding a leaf on v_{j+1} and v_{k-1} has the same radius and the remoteness which is less or equal than in G. By repeating this procedure, we obtain a caterpillar tree G' of the same diameter as G with diametric path $P = v_0v_1 \dots v_D$ such that:

1. G' has exactly one centroidal vertex $v_j \in P$ and every of the vertices $v_{j+1}, \dots, v_D$ is of degree at most 2,
2. G' has two centroidal vertices $v_j, v_{j+1} \in P$ and every of the vertices $v_{j+1}, \dots, v_D$ is of degree at most 2,
3. G' has two centroidal vertices $v_j, v_{j+1} \in P$ and every of the vertices $v_0, \dots, v_{j-1}, v_{j+2}, \dots, v_D$ is of degree at most 2.

Therefore, on the obtained graph G' one of the Lemmas 5.2, 5.3, 5.4 or 5.5 can be applied. If Lemma 5.5 is applied, the claim is proved. Else if Lemma 5.2, 5.3 or 5.4 is applied, we obtain graph G' of diameter $D+1$ and remoteness $\rho + \frac{1}{2}$. Since for $D+1$ it holds that $D+1 \leq n-2$, we can apply the whole procedure with $G = G'$ (as the second step) and thus obtain a caterpillar tree G' of diameter $D+2$ and remoteness $\rho' \leq \rho + 1$. Since for thus obtained G' it holds that $D' = D+2$, we conclude $r' = r+1$. Therefore, $\rho' - r' \leq \rho - r$.

Repeating this double step, we obtain a caterpillar tree G' of diameter $D' = n-2$ or $D' = n-1$ for which the difference $\rho - r$ is less or equal than for G. Now we distinguish several cases with respect to D' and parity of n. Suppose first $D' = n-1$. Then $G' = P_n$. If n is odd then the claim is proved. If n is even it is easily verified that the difference $\rho - r$ is less for path P_{n-1} with a leaf appended to a central vertex than for $G' = P_n$ and the claim is proved in this case too. Suppose now that $D' = n-2$. That means G' is a path P_{n-1} with a leaf appended to one vertex of P_{n-1}. If n is odd, then deleting the only leaf in G' to extend it to P_n increases radius by 1 and remoteness by less than 1, so the claim holds. If n is even, then deleting the leaf in G' outside P_{n-1} and appending it to central vertex of P_{n-1} preserves the radius and decreases the remoteness. Therefore, the claim holds in this case too. □

We can summarize the results of these lemmas in the following theorem which gives minimal trees for $\rho - r$.

Theorem 5.7. *Let G be a tree on n vertices. If n is odd, then the difference $\rho - r$ is less or equal for path P_n then for G. If n is even, then the difference $\rho - r$ is less or equal for path P_{n-1} with a leaf appended to a central vertex than for G.*

Proof. Follows from Lemmas 5.1 and 5.6. □

For a path P_n on odd number of vertices n it holds that $\rho - r = \frac{1}{2}$ which, together with Theorem 5.7, obviously implies that trees on odd number of vertices satisfy Conjecture 1.3. Now, let us consider graph G on even number of vertices n consisting of a path P_{n-1} with a leaf appended to a central vertex. For G it holds that $\rho - r = \frac{n}{2(n-1)}$ which implies that trees on even number of vertices satisfy Conjecture 1.3 too.

6. Conclusion

We have established that the maximal tree for $\bar{l} - \pi$ is a tree composed of three paths of almost equal lengths with a common end point. Thus, we proved that Conjecture 1.1 posed in [4] for general graph holds for trees. Using reduction of a graph to a corresponding subtree, this result enabled us to prove Conjecture 1.1 for general graphs too. Furthermore, we established that the maximal tree for $ecc - \rho$ is the path P_n and that the minimal tree for $\rho - r$ is the path P_n in case of odd n and the path P_{n-1} with a leaf appended to a central vertex in case of even n. Since for these extremal trees Conjectures 1.2 and 1.3 posed in [4] hold, it follows that those conjectures hold for trees.

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