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Splitting a Linear System of Operator Equations with Constant Coefficients: A Matrix Polynomial Approach

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Abstract. In this paper, we develop an efficient reduction technique that leads us to study particular classes of linear non-homogenous system of operator equations with constant coefficients. The key concept is assigning a matrix polynomial to the system, and its Smith canonical form provide a reduced system of independent equations. As a consequence of independent structure, each equation of the reduced system contains one unknown.

1. Introduction

Consider a linear system of n operator equations with constant coefficients in unknowns x_i from [11]:

$$\mathscr{L}(\bar{x}) = C\bar{x} + \bar{\varphi}, \tag{1}$$

such that $\sum_{j=1}^{n} c_{i,j}x_j =: \mathscr{L}(x_i) - \varphi_i,\ n > 2$, where $\mathscr{L} : V \to V$ is a linear operator, and V is a vector space over an arbitrary field $\mathbb{F}$. In the system (1), the constant nonzero matrix $C = [c_{i,j}] \in M_{n\times n}(\mathbb{F})$, (the class of $n \times n$ matrices whose elements belong to $\mathbb{F}$), $\bar{\varphi} = [\varphi_1, \cdots, \varphi_n]^T \in V^{n\times 1}$ is the right-hand vector.

System of operator equations represents one of the largest fields within the linear control theory [3, 7, 13] and other applied sciences. The solvability of systems of operator equations has been widely investigated by many authors. Jia et al. [6] have discussed the conditions on operators which would make the system of linear operator equations solvable. The results of [2] deal with the numerical solution of the the system of operator equations via least square methods. A general approach to the case of iterative solutions has also been studied in [17]. Further details about solvability of systems of operator equations may be found in [6, 8, 10, 12, 14].

There are some reduction procedures for transforming a linear system of operator equations to an equivalent solvable form. The strategy which is usually employed is based on the general canonical forms such as Jordan forms [1] and rational canonical forms [11]. A partial reduction procedure based on rational canonical forms has been presented for linear system of operator equations (1) in [11], and recently, the Smith canonical form has been applied for the reduction of the system of integral equations [15, 16].

In this paper, our aim is to extend the applicability of matrix polynomial theory to reduce a system of operator equations. First, an overview on the matrix polynomial concepts and notations is presented in Section 2. Then a matrix polynomial equation equivalent to the linear system of operator equations is introduced in Section 3. Such presentations are useful in applying the Smith canonical form for obtaining a class of independent equations. This concern will be addressed in Section 4, and we show how to use the proposed idea to reduce the system of linear operator equations in special cases. Finally, the conclusion is presented in Section 5.

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2. Matrix polynomials

In this section, some well-known properties of matrix polynomial theory have been collected which will be helpful subsequently. Such matrices arise in the mathematical treatment of various types of linear systems which were studied in, for example [5, 13]. For detailed proofs, the reader may refer to standard references like [4, 5, 9].

Let us recall that a *matrix polynomial* is simply a matrix with polynomial elements which can alternatively be considered as a polynomial with matrix coefficients. We are mainly concerned with the ring of square matrix polynomials $M_{n\times n}(\mathbb{F}[\mathscr{L}])$, (where $\mathbb{F}[\mathscr{L}]$ is the set of all polynomials in $\mathscr{L}$) consists of polynomials with coefficients belonged to $M_{n\times n}(\mathbb{F})$. Note that we also need $GL(n, \mathbb{F}[\mathscr{L}])$, the group of invertible matrix polynomials in $M_{n\times n}(\mathbb{F}[\mathscr{L}])$, more precisely:

Definition 2.1. *A matrix $P \in M_{n\times n}(\mathbb{F}[\mathscr{L}])$ is called unimodular (belongs to $GL(n, \mathbb{F}[\mathscr{L}])$) if and only if $detP$ is a nonzero constant in $\mathbb{F}$.*

It is easily deduced from this property that a matrix polynomial is unimodular if and only if its inverse is a matrix polynomial, which will also be unimodular. Using unimodular matrices, an equivalence relation can be defined on $M_{n\times n}(\mathbb{F}[\mathscr{L}])$, as follows:

Definition 2.2. *Let $P,\ Q \in M_{n\times n}(\mathbb{F}[\mathscr{L}])$. Then P is equivalent to Q over $\mathbb{F}[\mathscr{L}]$, if there exist matrix polynomials $R,\ S \in GL(n, \mathbb{F}[\mathscr{L}])$, such that*

$$R(\mathscr{L})P(\mathscr{L})S(\mathscr{L}) = Q(\mathscr{L}).$$

Definition 2.3. *For every $P \in M_{n\times n}(\mathbb{F})$, the determinantal rank, $\rho(P)$, is defined to be the largest integer r for which there exists a nonzero $r \times r$ minor of P.*

Lemma 2.1. (From [4]) *Let $P,\ Q \in M_{n\times n}(\mathbb{F}[\mathscr{L}])$. If P is equivalent to Q over $\mathbb{F}[\mathscr{L}]$, then $\rho(P) = \rho(Q)$.*

Lemma 2.2. (From [4, 5]) *For every nonzero matrix $P \in M_{n\times n}(\mathbb{F}[\mathscr{L}])$ with $r = \rho(P)$, there exist matrix polynomials $R,\ S \in GL(n, \mathbb{F}[\mathscr{L}])$ such that P is equivalent to a matrix $\Gamma \in M_{n\times n}(\mathbb{F}[\mathscr{L}])$ in the form*

$$R(\mathscr{L})P(\mathscr{L})S(\mathscr{L}) = \Gamma(\mathscr{L}), \tag{2}$$

where

$$\Gamma(\mathscr{L}) = Diag[d_1(\mathscr{L}), \cdots, d_r(\mathscr{L}), 0, \cdots, 0], \tag{3}$$

called the **Smith canonical form** *of $P(\mathscr{L})$, with monic scalar polynomials $d_i(\mathscr{L})$ such that $d_i(\mathscr{L})$ is divisible by $d_{i-1}(\mathscr{L})$.*

Definition 2.4. *The polynomials $d_1(\mathscr{L}), \cdots, d_r(\mathscr{L})$ in the Smith canonical form* (3) *are called the* **invariant factors** *of $P(\mathscr{L})$. Moreover, $R(\mathscr{L})$ and $S(\mathscr{L})$ are named the* **pre-** *and* **post-multipliers**, *respectively.*

Definition 2.5. *The invariant factors of the characteristic matrix of $C \in M_{n\times n}(\mathbb{F})$, (i.e. $\mathscr{L}I - C$) are called* **similarity invariants** *of C.*

3. The Smith canonical form

To proceed in depth, we investigate the system (1) in two special cases $\check{C}, \hat{C} \in M_{n\times n}(\mathbb{F})$ as:

$$\check{C} = \sum_{i,j=1}^{n} c_j \mathcal{E}_{i,j}, \qquad \hat{C} = \sum_{i,j=1}^{n} c_i \mathcal{E}_{i,j}, \tag{4}$$

where $\mathcal{E}_{i,j}$ stands for the matrix with entry (i, j) equal to 1, all other entries equal to zero, and $c_i \neq 0$ for $i = 1, \cdots, n$. First, the matrix polynomials

$$\check{P}(\mathscr{L}) = (\mathscr{L}I - \check{C}), \qquad \hat{P}(\mathscr{L}) = (\mathscr{L}I - \hat{C}), \tag{5}$$

can be assigned to the system (1). In particular, we can impose the additional requirement that $\rho(\mathscr{L}I - \check{C}) = \rho(\mathscr{L}I - \hat{C}) = n$, for the solvability of the system. Let $\Gamma(\mathscr{L})$ be the Smith canonical form of $(\mathscr{L}I - \check{C})$ and $(\mathscr{L}I - \hat{C})$. Since $P(\mathscr{L})$ and $\Gamma(\mathscr{L})$ are equivalent, Lemma 2.1 indicates that $\rho(\Gamma) = n$.

The following lemma is devoted to some preliminary relations needed in the sequel, and its proof is a matter of computation.

Lemma 3.1. *For $i > 1$ and $j > 0$, define: $\lambda_{i,j} = \dfrac{(-1)^j}{j!}\displaystyle\prod_{k=0}^{j-1}(i-k-2)$. It is easy to see that $\lambda_{i,0} = 1$, and $\lambda_{i,j} = 0$, for $j > i - 2$. Moreover, $\lambda_{i,j}$, $i > 2$ satisfies:*

$$\sum_{j=0}^{i-2} \lambda_{i,j} = 0, \tag{6}$$

$$\sum_{k=0}^{i-2} \lambda_{i,k}\lambda_{k+2,j-2} = \delta_{i,j}, \tag{7}$$

$$\sum_{k=1}^{i-2} \lambda_{i,k}\lambda_{k+1,j-2} = \delta_{i,j} - 1, \tag{8}$$

where throughout the rest of this paper, $\delta_{i,j}$ stands for the Kronecker delta.

The main result of the paper (see Theorem 4.1) is based on the Smith factorization of the matrix polynomial. Using Maple® symbolic computation package, we implement the algorithms on a digital computer, automating the factorization process of the Smith canonical form for $(\mathscr{L}I - \check{C})$ and $(\mathscr{L}I - \hat{C})$.

Case 1. In considering the matrix polynomials (5), first we establish the results for $\check{C} \in M_{n\times n}(\mathbb{F})$:

Theorem 3.1. *(I).The pre-multiplier unimodular matrix $\check{R} = \displaystyle\sum_{i=1}^{n} \check{R}_i$, where*

$$\begin{aligned}
\check{R}_1 &= \tfrac{-1}{c_2}\mathcal{E}_{1,1}, \\
\check{R}_2 &= -\mathcal{E}_{2,1} + \mathcal{E}_{2,n}, \\
\check{R}_i &= \sum_{j=n-i+2}^{n} \lambda_{i,n-j}\mathcal{E}_{i,j}, \quad i = 3, \cdots, n-1, \\
\check{R}_n &= \eta_n\mathcal{E}_{n,1} + c_2\mathcal{E}_{n,2} + \sum_{j=3}^{n} \theta_{j,n}\mathcal{E}_{n,j},
\end{aligned} \tag{9}$$

in which:

$$\eta_n = c_1 + 2\Big(\mathscr{L} - \sum_{i=1}^{n} c_i\Big), \qquad \theta_{j,n} = c_j + \lambda_{n,n-j+1}(\mathscr{L} - \sum_{i=1}^{n} c_i), \qquad j = 3, \cdots, n.$$

(II).The Smith matrix of $\check{C}$ is $\Gamma(\mathscr{L}) = \mathcal{E}_{1,1} + \mathscr{L}\displaystyle\sum_{i=2}^{n-1}\mathcal{E}_{i,i} + (\mathscr{L}^2 - \mathscr{L}\sum_{i=1}^{n} c_i)\mathcal{E}_{n,n}$, and $\mathscr{L}^2$ (power of operator $\mathscr{L}$) is defined as usual: $\mathscr{L}^2 = \mathscr{L} o \mathscr{L}$.

(III). The post-multiplier unimodular matrix is: $\check{S} = \sum_{j=2}^{n} \mathcal{E}_{1,j} + \mathcal{E}_{2,1} + \sum_{j=2}^{n} \sigma_{j,n}\mathcal{E}_{2,j} + \sum_{i=3}^{n}\sum_{j=2}^{n} \mu_{i,j,n}\mathcal{E}_{i,j}$,

where:

$$\begin{aligned} &\sigma_{j,n} = \frac{1}{c_2}(\mathscr{L} - c_1 - \sum_{k=3}^{n} \mu_{k,j,n}c_k), && j = 2, \cdots, n, \\ &\mu_{i,j,n} = 1 + \lambda_{n-i+2,j-2}, && i = 3, \cdots, n,\ j = 2, \cdots, n. \end{aligned} \tag{10}$$

Proof. The similarity invariants of $\check{C}$ are exactly the diagonal elements of $\Gamma(\mathscr{L})$, and it will be enough to show that the multipliers satisfy $\Gamma = \check{R}\check{P}\check{S}$, which can be done as follows:

First row: It can be easily checked that $\Gamma_{1,1} = 1$, and for $j = 2, \cdots, n$:

$$\Gamma_{1,j} = \sum_{k=1}^{n}\sum_{i=1}^{n} \check{R}_{1,i}\check{P}_{i,k}\check{S}_{k,j} = \frac{-1}{c_2}\sum_{k=1}^{n} \check{P}_{1,k}\check{S}_{k,j} = \frac{-1}{c_2}\Big[\check{P}_{1,1} + \check{P}_{1,2}\sigma_{j,n} + \sum_{k=3}^{n} \check{P}_{1,k}\mu_{k,j,n}\Big] = -\frac{\mathscr{L} - c_1}{c_2} + \sigma_{j,n} + \frac{1}{c_2}\sum_{k=3}^{n} c_k\mu_{k,j,n},$$

due to the definition of $\sigma_{j,n}$ (10), it is clear that $\Gamma_{1,j} = 0$. Then in the second row, we have:

$$\Gamma_{2,1} = \sum_{k=1}^{n}(\check{R}\check{P})_{2,k}\check{S}_{k,1} = (\check{R}\check{P})_{2,2} = \check{R}_{2,1}\check{P}_{1,2} + \check{R}_{2,n}\check{P}_{n,2} = \check{P}_{n,2} - \check{P}_{1,2},$$

which clearly vanishes since $\check{P}$ satisfies (5). Moreover, for $j = 2, \cdots, n$:

$$\Gamma_{2,j} = \sum_{k=1}^{n}\sum_{i=1}^{n} \check{R}_{2,i}\check{P}_{i,k}\check{S}_{k,j} = \sum_{k=1}^{n}(\check{P}_{n,k} - \check{P}_{1,k})\check{S}_{k,j} = (\check{P}_{n,1} - \check{P}_{1,1})\check{S}_{1,j} + (\check{P}_{n,n} - \check{P}_{1,n})\check{S}_{n,j} = \mathscr{L}(\mu_{n,j,n} - 1).$$

Since $\mu_{n,j,n} = 1 + \delta_{j,2}$, it straightforwardly follows that $\Gamma_{2,j} = \mathscr{L}\delta_{2,j}$. Next, Part I of Theorem shows that the first column $\Gamma_{i,1}$, for $i = 3, \cdots, n-1$, satisfies the following pattern:

$$\Gamma_{i,1} = \sum_{k=1}^{n}(\check{R}\check{P})_{i,k}\check{S}_{k,1} = (\check{R}\check{P})_{i,2}\check{S}_{2,1} = \sum_{k=1}^{n} \check{R}_{i,k}\check{P}_{k,2} = -c_2\sum_{j=n-i+2}^{n} \lambda_{i,n-j}.$$

It follows that $\Gamma_{i,1} = \sum_{k=0}^{i-2} \lambda_{i,k}$, and the result is obvious using (6). Besides, we remark that according to (9), the n-th row have a different pattern, particularly:

$$\Gamma_{n,1} = \sum_{k=1}^{n}(\check{R}\check{P})_{n,k}\check{S}_{k,1} = (\check{R}\check{P})_{n,2}\check{S}_{2,1} = \check{R}_{n,1}\check{P}_{1,2} + \check{R}_{n,2}\check{P}_{2,2} + \sum_{k=3}^{n} \check{R}_{n,k}\check{P}_{k,2} = c_2\Big[\mathscr{L} - \eta_n - c_2 - \sum_{k=3}^{n} \theta_{k,n}\Big].$$

On the other hand, it is straightforward to show that

$$\eta_n + c_2 + \sum_{k=3}^{n} \theta_{k,n} = \mathscr{L}, \tag{11}$$

and taking it into account, we get $\Gamma_{n,1} = 0$. Then for $j = 2, \cdots, n$, we have:

$$\begin{aligned} \Gamma_{n,j} &= \sum_{k=1}^{n}\Big[\check{R}_{n,1}\check{P}_{1,k} + \check{R}_{n,2}\check{P}_{2,k} + \sum_{i=3}^{n} \check{R}_{n,i}\check{P}_{i,k}\Big]\check{S}_{k,j} \\ &= -(\eta_n + c_2 + \sum_{k=3}^{n} \theta_{k,n})\Big[c_1 + c_2\sigma_{j,n} + \sum_{k=3}^{n} c_k\mu_{k,j,n}\Big] + \mathscr{L}\Big[\eta_n + c_2\sigma_{j,n} + \sum_{k=3}^{n} \theta_{k,n}\mu_{k,j,n}\Big]. \end{aligned}$$

Apart from that, it can be easily checked that

$$c_1 + c_2\sigma_{j,n} + \sum_{k=3}^{n} c_k\mu_{k,j,n} = \mathscr{L}, \tag{12}$$

and considering $\lambda_{i,0} = 1$, a similar argument shows that

$$\eta_n + c_2\sigma_{j,n} + \sum_{k=3}^{n} \theta_{k,n}\mu_{k,j,n} = \mathscr{L} + (\mathscr{L} - \sum_{j=1}^{n} c_j)\Big[1 + \sum_{k=3}^{n} \lambda_{n-k+2,j-2}\lambda_{n,n-k+1}\Big].$$

Consequently, on account of (8), (11) and (12), we can check that $\Gamma_{n,j} = (\mathscr{L}^2 - \mathscr{L}\sum_{j=1}^{n} c_j)\delta_{n,j}$. We are now in a position to consider $\Gamma_{i,j}$, for $i = 3, \cdots, n-1$ and $j = 2, \cdots, n$, which can be obtained with some tedious manipulation:

$$\Gamma_{i,j} = -(\sum_{k=0}^{i-2} \lambda_{i,k})\Big[c_1 + c_2\sigma_{j,n} + \sum_{k=3}^{n} c_k\mu_{k,j,n}\Big] + \mathscr{L}\Big[\sum_{k=0}^{i-2} \lambda_{i,k}\mu_{n-k,j,n}\Big], \tag{13}$$

considering (12) and (6), shows that $\Gamma_{i,j} = \mathscr{L}\sum_{k=0}^{i-2} \lambda_{i,k}\lambda_{k+2,j-2}$, and to complete the proof, it is enough to use (7) to see that $\Gamma_{i,j} = \mathscr{L}\delta_{i,j}$. □

Case 2. A theorem similar to Theorem 3.1 can be given for the Smith canonical form of $\hat{C} \in M_{n\times n}(\mathbb{F})$, and it can be proved with a similar argument.

Theorem 3.2. *(I).The pre-multiplier unimodular matrix* $\hat{R} = \sum_{i=1}^{n} \hat{R}_i$, *where*

$$\hat{R}_1 = \tfrac{-1}{c_1}\mathcal{E}_{1,1},$$
$$\hat{R}_i = \alpha_{i,n}\mathcal{E}_{i,1} + \sum_{j=n-i+2}^{n} \lambda_{i,n-j}\mathcal{E}_{i,j}, \quad i = 2, \cdots, n-1,$$
$$\hat{R}_n = \gamma_n\mathcal{E}_{n,1} + c_1\mathcal{E}_{n,2} + \sum_{j=3}^{n} \beta_{j,n}\mathcal{E}_{n,j},$$

in which:

$$\alpha_{i,n} = -\sum_{k=0}^{i-2} \lambda_{i,k}\frac{c_{n-k}}{c_1}, \qquad i = 2, \cdots, n-1,$$
$$\gamma_n = c_1 + \Big(1 - \sum_{k=3}^{n} \lambda_{n,n-k+1}\frac{c_k}{c_1}\Big)\Big(\mathscr{L} - \sum_{i=1}^{n} c_i\Big),$$
$$\beta_{j,n} = c_1 + (-1)^n\lambda_{n,j-3}(\mathscr{L} - \sum_{i=1}^{n} c_i), \qquad j = 3, \cdots, n.$$

(II). The similarity invariants of $\hat{C}$ *are* $1, \underbrace{\mathscr{L}, \cdots, \mathscr{L}}_{(n-2)}, \mathscr{L}^2 - \mathscr{L}\sum_{i=1}^{n} c_i.$

(III). The post-multiplier unimodular matrix is: $\hat{S} = \sum_{j=2}^{n} \mathcal{E}_{1,j} + \mathcal{E}_{2,1} + \sum_{j=2}^{n} \rho_{j,n}\mathcal{E}_{2,j} + \sum_{i=3}^{n}\sum_{j=2}^{n} \nu_{i,j,n}\mathcal{E}_{i,j},$

in which:

$$\begin{aligned} \rho_{j,n} &= \frac{1}{c_1}\Big(\mathscr{L} + c_2 + \lambda_{n,j-1}c_1 - \sum_{i=1}^{n} c_i\Big), \qquad & j = 2,\cdots,n,\\ \nu_{i,j,n} &= \lambda_{n+2-i,j-2} + \frac{c_i}{c_1}, & i = 3,\cdots,n,\ j = 2,\cdots,n. \end{aligned}$$

4. The main results

We develop structural formulas for reducing the system (1) to a system of independent equations. The following lemma initiates our strategy of the reduction process based on the Smith canonical form:

Lemma 4.1. (From [5]) *Let* $\Gamma(\mathscr{L})$ *be the Smith normal form of* $P(\mathscr{L}) = \mathscr{L}I - C$. *Then the system* $P(\mathscr{L})\bar{x} = \bar{\varphi}$, *can be reduced to the system*

$$\Gamma(\mathscr{L})\bar{y} = \bar{\psi}, \tag{14}$$

where $\bar{\psi} = R(\mathscr{L})\bar{\varphi}$ *is its free column and* $\bar{x} = S(\mathscr{L})\bar{y}$ *is a column of the unknowns.*

Now, we state the main theorem dealing with the reduction of the system (1). It is worthy to mention that since the major objective of this paper is the study of corresponding independent system which arises in (14), we only need the Smith matrix and the pre-multipliers.

Theorem 4.1. *Suppose the assumption of Lemma 4.1 for* $P(\mathscr{L}) = (\mathscr{L}I - C)$. *The reduced system of* (1) *in two special cases of* (4) *is:*

$$\begin{aligned} y_1 &= \psi_1, \\ \mathscr{L}y_i &= \psi_i, \qquad i = 2,\cdots,n-1, \\ (\mathscr{L}^2 - \mathscr{L}\sum_{i=1}^{n} c_i)y_n &= \psi_n, \end{aligned} \tag{15}$$

where the column $\bar{\psi} = [\psi_1,\cdots,\psi_n]^T$ *is determined by* $\bar{\psi} = R(\mathscr{L})\bar{\varphi}$.

Proof. To make the proof easier to follow, we proceed in three steps:

First step: The required step is assigning the matrix polynomial (5) to the system (1) in special cases of (4). So, the system can be written as $P(\mathscr{L})\bar{x} = \bar{\varphi}$.

Second step: Following Lemma 4.1, we aim to find the reduced system of (1). The left-hand side of the equation (14) shows the necessity of the Smith matrix which is given in Part II of Theorems 3.1 and 3.2. In fact, the matrix polynomial $\Gamma(\mathscr{L})$ establishes the operators who act on the vector $\bar{y}$. Making the right-hand side of (15) needs $\bar{\psi}$ to be calculated.

Third step: Take into account that $\bar{\psi} = R(\mathscr{L})\bar{\varphi}$, and consider the matrix polynomials $\check{R}$ and $\hat{R}$ presented in Part I of Theorems 3.1 and 3.2. Now, direct computation leads to the following $\check{\bar{\psi}}$ and $\hat{\bar{\psi}}$ vectors corresponding to the referred special cases:

$$\begin{cases} \check{\psi}_1 = \frac{-1}{c_2}\varphi_1, \\ \check{\psi}_2 = -\varphi_1 + \varphi_n, \\ \check{\psi}_i = \displaystyle\sum_{j=n-i+2}^{n} \lambda_{i,n-j}\varphi_j, \qquad i = 3,\cdots,n-1, \\ \check{\psi}_n = \eta_n\varphi_1 + c_2\varphi_2 + \displaystyle\sum_{j=3}^{n} \theta_{j,n}\varphi_j, \end{cases}$$

and

$$\begin{cases} \hat{\psi}_1 = \frac{-1}{c_1}\varphi_1, \\ \hat{\psi}_i = \alpha_{i,n}\varphi_1 + \sum_{j=n-i+2}^{n} \lambda_{i,n-j}\varphi_j, \qquad i = 2, \cdots, n-1, \\ \hat{\psi}_n = \gamma_n\varphi_1 + c_1\varphi_2 + \sum_{j=3}^{n} \beta_{j,n}\varphi_j. \end{cases}$$

Actually, the vector $\bar{\psi}$ completes the reduced form of the system in (14). □

Note that the system (15) consists of a class of independent equations, and from an algorithmic point of view, it is desirable to search for methods to solve independent equations of type (15) with one unknown in each equation, rather than systems like (1).

The rest of the section deals with particular instances where the system (1) can be explicitly reduced to yield interesting class of independent equations. The obtained results are illustrated for the special case of $n = 3$. This example likely plays the role of clarifying our presented matrix polynomial based method.

Suppose the linear system of operator equations (1) for $n = 3$, and its assigned matrix polynomial $P(\mathscr{L}) = \mathscr{L}I - C$. The Smith form of $\check{P}(\mathscr{L})$ and $\hat{P}(\mathscr{L})$, is as follows:

$$\Gamma(\mathscr{L}) = Diag[1, \mathscr{L}, \mathscr{L}^2 - \mathscr{L}\sum_{j=1}^{3} c_j], \tag{16}$$

and $\check{R}$ and $\hat{R}$ corresponding to $\check{C}$ and $\hat{C}$, respectively:

$$\check{R}(\mathscr{L}) = \begin{pmatrix} \frac{-1}{c_2} & 0 & 0 \\ -1 & 0 & 1 \\ c_1 + 2(\mathscr{L} - \sum_{j=1}^{3} c_j) & c_2 & c_3 - (\mathscr{L} - \sum_{j=1}^{3} c_j) \end{pmatrix},$$

and

$$\hat{R}(\mathscr{L}) = \begin{pmatrix} \frac{-1}{c_1} & 0 & 0 \\ \frac{-c_3}{c_1} & 0 & 1 \\ c_1 + (\mathscr{L} - \sum_{j=1}^{3} c_j)(1 + \frac{c_3}{c_1}) & c_1 & c_1 - (\mathscr{L} - \sum_{j=1}^{3} c_j) \end{pmatrix}.$$

Following the proposed method leads to the computation of the reduced system (14). Since Γ in (16) is same for $\check{C}$ and $\hat{C}$, the left-hand side of the reduced system remains as (15) for $n = 3$, and the right-hand side vector $\bar{\psi}$ differs in two cases, as follows:

$$\check{\psi} = \Big[\frac{-1}{c_2}\varphi_1, -\varphi_1 + \varphi_3, \Big(c_1 + 2(\mathscr{L} - \sum_{j=1}^{3} c_j)\Big)\varphi_1 + c_2\varphi_2 + \Big(c_3 - (\mathscr{L} - \sum_{j=1}^{3} c_j)\Big)\varphi_3\Big]^T,$$

$$\hat{\psi} = \Big[\frac{-1}{c_1}\varphi_1, \frac{-c_3}{c_1}\varphi_1 + \varphi_3, \Big(c_1 + (\mathscr{L} - \sum_{j=1}^{3} c_j)(1 + \frac{c_3}{c_1})\Big)\varphi_1 + c_1\varphi_2 + \Big(c_1 - (\mathscr{L} - \sum_{j=1}^{3} c_j)\Big)\varphi_3\Big]^T.$$

Actually, this reduced system consists of three independent equations, supposed preferable to the main system. It is worthy to mention that these results can be easily deduced from Theorem 4.1 for $n = 3$.

5. Conclusion

In this paper, the properties of matrix polynomial theory for the linear system of operator equations have been applied. The interesting idea of this work is assigning a matrix polynomial to the systems of operator equations which appear in almost every branch of science and engineering. Moreover, Smith factorization of the referred matrix polynomial transforms the system to a class of independent equations. The proposed approach could be of interest in the theory of general systems of operator equations such as systems of differential and integral equations which will be the focus of our future work.

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