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On Certain Classes of Fractional Kinetic Equations

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Abstract. This paper considers certain general forms of fractional kinetic equations and obtains their solutions. The usefulness of the main results are depicted by deriving certain generalized fractional kinetic equations. By adopting important methodology at various stages, we extensively treat solving completely the problems on fractional kinetic equations involving various types of special functions. Relevances and some new consequences of the main results are also pointed out.

1. Introduction and Preliminaries

During last few decades fractional kinetic equations of different forms have been widely used in describing and solving several important problems of physics and astrophysics. The generalized fractional kinetic equations discussed here can be used to investigate a large variety of known fractional kinetic equations. If an arbitrary reaction is characterized by a time dependent quantity $N = N(t)$, then it is possible to calculate the rate $\mathrm{d}N/\mathrm{d}t$ by the mathematical equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -d + p, \tag{1}$$

where d is the destruction rate and p is the production rate of N. In general, the destruction rate d and the production rate p depend on the quantity $N(t)$ itself: $d = d(N)$ or $p = p(N)$, which is a complicated dependence because the destruction or production at time t depends not only on $N(t)$ but also on the past history $N(\eta), \eta < t$, of variable N. Formally, this can be described by the following equation:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -d(N_t) + p(N_t), \tag{2}$$

where the function N_t is defined by $N_t(t^*) = N(t - t^*), t^* > 0$.

Haubold and Mathai [10] studied a special case of this equation, namely,

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = -c_i N_i(t), \tag{3}$$

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with the initial condition that $N_i(t=0) = N_0$ is the number density of species i at time $t = 0$ and constant $c_i > 0$, is known as the standard kinetic equation. The solution of (3) is given by

$$N_i(t) = N_0 e^{-c_i t}. \tag{4}$$

An alternative form of equation (3) upon integration is given by

$$N(t) - N_0 = c_0 D_t^{-1} N(t), \tag{5}$$

where ${}_0D_t^{-1}$ is the standard integral operator. Its fractional generalization considered in [10] can be expressed as

$$N(t) - N_0 = c^{\nu} {}_0D_t^{-\nu} N(t), \tag{6}$$

where ${}_0D_t^{-\nu}$ is the familiar Riemann-Liouville fractional integral operator ([14]; see also [1], [12] and [13])

$${}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-x)^{\nu-1} f(x)\,\mathrm{d}x \ \ (\Re(\nu) > 0). \tag{7}$$

By replacing the Riemann-Liouville fractional integral operator with some other suitable fractional integral operators, we can get many interesting generalizations of the equation (6).

Recently, Chouhan and Saraswat [6] have studied the following equation:

$$N(t) - N_0 f(t) = -c^{\beta} \mathbf{E}^{\gamma}_{\alpha,\beta,\omega;0+} N(t), \tag{8}$$

where $\mathbf{E}^{\gamma}_{\alpha,\beta,\omega;0+} N(t)$ is the integral operator defined by ([16, p.9])

$$\left(\mathbf{E}^{\gamma}_{\alpha,\beta,\omega;a+}\varphi\right)(t) = \int_a^t (t-x)^{\beta-1} E^{\gamma}_{\alpha,\beta}\left(\omega(t-x)^{\alpha}\right)\varphi(x)\,\mathrm{d}x \ \ (\alpha,\beta,\gamma,\omega \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0), \tag{9}$$

where the operator (9) contains the generalized Mittag-Leffler function (see also [8], [12])

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!} \ \ (z,\alpha,\beta,\gamma \in \mathbb{C}, \Re(\alpha) > 0) \tag{10}$$

as its kernel. Formulations of equations similar to (8) were also considered in [11].

Raina in [17] investigated a class of functions defined by

$$\mathcal{F}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \ \ (\rho,\lambda \in \mathbb{C} \ (\Re(\rho) > 0); |x| < R), \tag{11}$$

where the coefficients $\sigma(k)$ is a bounded arbitrary sequence of real (or complex) numbers and R is the set of real numbers. Here and throughout, when $\lambda = 1$ and $\rho = 0$ in (1.11), we shall put (for convenience sake)

$$\mathcal{F}_{0,1}(x) = \mathcal{F}(x) \tag{12}$$

which is used below.

Clearly, many known functions such as the generalized Mittag-Leffler function (10) as mentioned above can be expressed in terms of (11). The well-known Fox-Wright function defined by (see [12], [13])

$$\begin{aligned} {}_p\Psi_q[z] &= {}_p\Psi_q\left[\begin{array}{c} (a_1,\alpha_1),\cdots,(a_p,\alpha_p) \\ \left(b_j,\beta_j\right),\cdots,\left(b_q,\beta_q\right) \end{array};z\right] = {}_p\Psi_q\left[\begin{array}{c} (a_l,\alpha_l)_{1,p} \\ \left(b_j,\beta_j\right)_{1,q} \end{array};z\right] \\ &= \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p}\Gamma(a_l + \alpha_l k)}{\prod_{j=1}^{q}\Gamma\left(b_j + \beta_j k\right)} \frac{z^k}{k!}, \end{aligned} \tag{13}$$

$\left(z, a_l, b_j \in \mathbb{C}, \alpha_l, \beta_j \in \mathbb{R}, l = 1, \cdots, p, j = 1, \cdots, q\right)$

is also a special case of the function (11).

For the purpose of this paper, the fractional integral operator containing (11) as its kernel is defined by

$$\left(\mathcal{J}_{\rho,\lambda,a+;\omega}\varphi\right)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}\left[\omega (x-t)^{\rho}\right] \varphi(t)\, \mathrm{d}t \ \ (x > a), \tag{14}$$

where $a \in \mathbb{R}_+$ $(x > a)$; $\lambda, \rho, \omega \in \mathbb{C}$; $(\Re(\lambda) > 0, \Re(\rho) > 0)$, $\varphi(t)$ is such that the integral on the right side exists.

Apart from the integral operator (9), we shall also use the two known integral operators described below.

i. Srivastava and Tomovski introduced and investigated the integral operator (see [22, p. 202, Eqn. (2.12)])

$$\left(\mathcal{E}^{\omega;\gamma,q}_{a+;\alpha,\beta}\varphi\right)(x) = \int_a^x (x-t)^{\beta-1} E^{\gamma,q}_{\alpha,\beta}\left[\omega (x-t)^{\alpha}\right] \mathrm{d}t \ \ (x > a), \tag{15}$$

$(\gamma, \omega \in \mathbb{C}; \Re(\alpha) > \max\{0, \Re(q) - 1\}; \min\{\Re(\beta), \Re(q)\} > 0)$

where

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{[\gamma]_{qk}}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!} \tag{16}$$

is the generalized Mittag-Leffler function defined by Shukla and Prajapati [21, p. 798, Eqn. (1.4)] in which we have preferred to use the notation (in terms of the quotient of two gamma functions) as

$$[\gamma]_{qk} = \frac{\Gamma(\gamma + qk)}{\Gamma(\gamma)}$$

instead of the Pochhammer symbolic notation $(\gamma)_{qk}$ because q is not a non-negative integer but $q \in \mathbb{C}$ in (16).

ii. As a special case of (14), Raina gives the following integral operator involving the Fox-Wright function (13) (see [17, p. 199, Eqn. (4.1)]):

$$\left(\mathcal{H}^{\lambda,h:(a_p,\alpha_p)}_{\omega,a+:(b_q,\beta_q)}\varphi\right)(x) = \int_a^x (x-t)^{\lambda-1} {}_p\Psi_q\left[\omega (x-t)^{h}\right] \varphi(t)\, \mathrm{d}t, \tag{17}$$

where $\lambda, \omega, h, a_i, b_j \in \mathbb{C}\,(\Re(\lambda) > 0, \Re(h) > 0)$; $\alpha_i, \beta_j \in \mathbb{R}, \forall i = 1, \cdots, p; j = 1, \cdots, q;$

$$\Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1,$$

or

$$\Delta = -1, \left|\omega (x-t)^h\right| < \delta = \prod_{i=1}^{p} |\alpha_i|^{-\alpha_i} \prod_{j=1}^{q} \left|\beta_j\right|^{\beta_j}$$

and if $\left|\omega (x-t)^h\right| = \delta$, then $\Re(\mu) > -\frac{1}{2}$. Note that the integral operator (15) is a special case of (17). In addition, for $\lambda, \omega, h, a_i, b_j \in \mathbb{C}$ $(\Re(\lambda) > 0, \Re(h) > 0)$; $\alpha_i, \beta_j \in \mathbb{R}, i = 1, \cdots, p; j = 1, \cdots, q$ and $\Delta^* = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -\Re(h)$, then $\left(\mathcal{H}^{\lambda,h:(a_p,\alpha_p)}_{\omega,a+:(b_q,\beta_q)}\varphi\right)(x)$ defines a bounded linear operator on $L(a, b)$ $(a < b)$, and satisfying the norm condition that (see [17, p. 200, Theorem 4.])

$$\left\|\left(\mathcal{H}^{\lambda,h:(a_p,\alpha_p)}_{\omega,a+:(b_q,\beta_q)}\varphi\right)(x)\right\| \le \Omega^* \left\|\varphi\right\|, \varphi \in L(a, b), \tag{18}$$

where the explicit expression of Ω^* is given in [17, p. 200, Eqn. (4.3)]

The purpose of this paper is to investigate new computable extensions of the generalized fractional kinetic equations by involving the fractional integral operator (14) which can effectively and conveniently be used to derive various classes of fractional kinetic equations. We also consider below certain new generalized forms of fractional kinetic equations and point out their consequences and relevances with other known results.

2. Generalized Fractional Kinetic Equations

In this section we will find the solution of the generalized fractional kinetic equation involving the integral operator (14) by applying the Laplace transform technique.

We mention here an important paper of Ťomovski, et al. [23] in which several corrections have been pointed out to some of the earlier works on the subject (see, for instance [19]) of obtaining solutions to certain fractional kinetic equations.

Some useful results related to (11) and (14) will also be mentioned here.

Proposition 2.1. *The Laplace transform of function $\mathcal{F}_{\rho,\lambda}(x)$ is*

$$\mathcal{L}\left[x^{\lambda-1}\mathcal{F}_{\rho,\lambda}(\omega x^{\rho})\right](s) = s^{-\lambda}\mathcal{F}(\omega s^{-\rho}) \quad (\Re(\lambda) > 0, \Re(\rho) > 0, \Re(s) > 0, \omega \in \mathbb{C}), \tag{19}$$

provided that the series on the right-hand side is convergent.

Proof. Using (11), we readily have

$$\begin{aligned}\mathcal{L}\left[x^{\lambda-1}\mathcal{F}_{\rho,\lambda}(\omega x^{\rho})\right](s) &= \mathcal{L}\left[x^{\lambda-1}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}\omega^{k}x^{\rho k}\right] \\ &= \sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}\omega^{k}\mathcal{L}\left[x^{\rho k+\lambda-1}\right](s) \\ &= \sum_{k=0}^{\infty}\sigma(k)\,\omega^{k}s^{-\rho k-\lambda} = s^{-\lambda}\mathcal{F}(\omega s^{-\rho}),\end{aligned} \tag{20}$$

provided that the series denoted by $\mathcal{F}(\omega s^{-\rho})$ is convergent. □

Proposition 2.2. *The Laplace transform of the integral operator $\left(\mathcal{J}_{\rho,\lambda,0+;\omega}\varphi\right)(x)$ is given by*

$$\mathcal{L}\left[\left(\mathcal{J}_{\rho,\lambda,0+;\omega}\varphi\right)(x)\right](s) = s^{-\lambda}\mathcal{F}(\omega s^{-\rho})\,\mathcal{L}[\varphi(t)](s). \tag{21}$$

$$(\Re(\lambda) > 0, \Re(\rho) > 0, \Re(s) > 0, \omega \in \mathbb{C})$$

Proof. By virtue of the convolution theorem of the Laplace transforms, it readily follows that

$$\begin{aligned}\mathcal{L}\left[\left(\mathcal{J}_{\rho,\lambda,0+;\omega}\varphi\right)(x)\right](s) &= \mathcal{L}\left[x^{\lambda-1}\mathcal{F}_{\rho,\lambda}(\omega x^{\rho})\right](s)\,\mathcal{L}[\varphi(t)](s) \\ &= s^{-\lambda}\mathcal{F}(\omega s^{-\rho})\,\mathcal{L}[\varphi(t)](s).\end{aligned} \tag{22}$$

□

Theorem 2.3. *If $c > 0, \beta > 0, \lambda, \rho, \omega \in \mathbb{C}\,(\Re(\lambda) > 0, \Re(\rho) > 0)$ and $f(t) \in L(0,\infty)$, then the solution of the generalized fractional kinetic equation*

$$N(t) - N_0 f(t) = -c^{\beta}\left(\mathcal{J}_{\rho,\lambda,0+;\omega}N\right)(t) \tag{23}$$

is expressed by

$$N(t) = N_0 \left[f(t) + \sum_{i=1}^{\infty} \left(-c^{\beta}\right)^i \left(\mathcal{J}_{\rho,\lambda i,0+;\omega} f\right)(t) \right], \tag{24}$$

provided that the series on the right-side of (24) *exists, where* $\left(\mathcal{J}_{\rho,\lambda i,0+;\omega} f\right)(t)$ *is the integral operator defined by* (14) *and its kernel function is determined by coefficients* (29) *below.*

Proof. Applying the Laplace transform to equation (23) and using (21) gives

$$\mathcal{L}[N(t)](s) - N_0 \mathcal{L}[f(t)](s) = -c^{\beta} s^{-\lambda} \mathcal{F}(\omega s^{-\rho}) \mathcal{L}[N(t)](s). \tag{25}$$

Solving for $\mathcal{L}[N(t)](s)$, it gives

$$\begin{aligned} \mathcal{L}[N(t)](s) &= N_0 \frac{\mathcal{L}[f(t)](s)}{1 + c^{\beta} s^{-\lambda} \mathcal{F}(\omega s^{-\rho})} \\ &= N_0 \left\{ \sum_{i=0}^{\infty} \left(-c^{\beta}\right)^i \left[s^{-\lambda i} \mathcal{F}^i(\omega s^{-\rho}) \right] \right\} \mathcal{L}[f(t)](s). \\ &\left(\left| c^{\beta} s^{-\lambda} \mathcal{F}(\omega s^{-\rho}) \right| < 1 \right) \end{aligned} \tag{26}$$

In particular, we can directly write $s^{-\lambda i} \mathcal{F}^i(\omega s^{-\rho})$ as the image of the Laplace transform of a known function. To clarify the point, let

$$\mathfrak{F}(\omega s^{-\rho}) = \sum_{k=0}^{\infty} (\omega s^{-\rho})^k = \frac{1}{1 - \omega s^{-\rho}},$$

then (see [12, p. 47, Eqn. (1.9.13)])

$$s^{-\lambda i} \mathfrak{F}^i(\omega s^{-\rho}) = \frac{s^{-\lambda i}}{(1 - \omega s^{-\rho})^i} = \frac{s^{\rho i - \lambda i}}{(s^{\rho} - \omega)^i} = \mathcal{L}\left[x^{\lambda i - 1} E^i_{\rho,\lambda i}(\omega x^{\rho}) \right](s). \tag{27}$$

$$\left(\omega, \alpha \in \mathbb{C}, \Re(s) > 0, \left| \omega s^{-\rho} \right| < 1 \right)$$

where the function $E^i_{\rho,\lambda i}(\omega x^{\rho})(s)$ is given by (10).

We observe that $s^{-\lambda i} \mathcal{F}^i(\omega s^{-\rho})$ $(i = 1, 2, 3, \cdots)$ can be expressed as

$$s^{-\lambda i} \mathcal{F}^i(\omega s^{-\rho}) = s^{-\lambda i} \prod_{j=1}^{i} \sum_{k_j=0}^{\infty} \sigma(k_j) (\omega s^{-\rho})^{k_j} = \sum_{k=0}^{\infty} C(k;i) \omega^k s^{-\lambda i - \rho k}, \tag{28}$$

where the coefficients $C(k;i)$ are given by

$$C(k;i) = \sum_{k_1 + k_2 + \cdots + k_i = k} \prod_{j=1}^{i} \sigma(k_j). \tag{29}$$

Since

$$s^{-(\lambda i + \rho k - 1) - 1} = \frac{\mathcal{L}\left[t^{\lambda i + \rho k - 1} \right](s)}{\Gamma(\lambda i + \rho k)} \quad (\Re(\lambda i + \rho k) > 0), \tag{30}$$

therefore, substituting (28) into (26) and taking into account the formula (30), we obtain that

$$\begin{aligned}\mathcal{L}[N(t)](s) &= N_0\mathcal{L}[f(t)](s) + N_0\sum_{i=1}^{\infty}\left(-c^{\beta}\right)^i\left[s^{-\lambda i}\mathcal{F}^i\left(\omega s^{-\rho}\right)\right]\mathcal{L}[f(t)](s)\\ &= N_0\mathcal{L}[f(t)](s) + N_0\sum_{i=1}^{\infty}\left(-c^{\beta}\right)^i\left\{\sum_{k=0}^{\infty}\frac{C(k;i)\,\omega^k}{\Gamma(\lambda i+\rho k)}\mathcal{L}\left[t^{\lambda i+\rho k-1}\right](s)\,\mathcal{L}[f(t)](s)\right\}.\end{aligned} \tag{31}$$

Now taking the inverse Laplace transform of (31) and applying the convolution theorem, it follows that

$$\begin{aligned}N(t) &= N_0 f(t) + N_0\sum_{i=1}^{\infty}\left(-c^{\beta}\right)^i\left\{\sum_{k=0}^{\infty}\frac{C(k;i)\,\omega^k}{\Gamma(\lambda i+\rho k)}\int_0^t (t-x)^{\lambda i+\rho k-1} f(x)\,\mathrm{d}x\right\}\\ &= N_0 f(t) + N_0\sum_{i=1}^{\infty}\left(-c^{\beta}\right)^i\int_0^t\sum_{k=0}^{\infty}\frac{C(k;i)}{\Gamma(\rho k+\lambda i)}\left[\omega(t-x)^{\rho}\right]^k(t-x)^{\lambda i-1} f(x)\,\mathrm{d}x\\ &= N_0 f(t) + N_0\sum_{i=1}^{\infty}\left(-c^{\beta}\right)^i\int_0^t (t-x)^{\lambda i-1}\mathcal{F}_{\rho,\lambda i}\left[\omega(t-x)^{\rho}\right] f(x)\,\mathrm{d}x\\ &= N_0 f(t) + N_0\sum_{i=1}^{\infty}\left(-c^{\beta}\right)^i\left(\mathcal{J}_{\rho,\lambda i,0+;\omega}f\right)(t)\\ &= N_0\left[f(t) + \sum_{i=1}^{\infty}\left(-c^{\beta}\right)^i\left(\mathcal{J}_{\rho,\lambda i,0+;\omega}f\right)(t)\right].\end{aligned} \tag{32}$$

Note that the interchange of order of integration and summation employed in above equation is permissible because the coefficient $C(k;i)$, as a finite sum of several bounded sequences, is also bounded for all k and the series is absolutely convergent from its definition. This completes the proof. □

As a direct consequence of the above theorem, we mention here a known result. Indeed, if we set $\sigma(k) = (\gamma)_k/k!$, $\rho = \alpha$, $\lambda = \beta$ in Theorem 2.3, we have the following corollary.

Corollary 2.4. *([6, Theorem 3.1]) Corresponding to the fractional kinetic equation* (8)*, there holds its solution given by*

$$N(t) = N_0\left[f(t) + \sum_{i=1}^{\infty}\left(-c^{\beta}\right)^i\mathbf{E}^{\gamma i}_{\alpha,\beta i,\omega;0+}f(t)\right]. \tag{33}$$

Proof. The corollary follows from the summation identity:

$$\sum_{k_1+\cdots+k_i=k}\frac{(\gamma)_{k_1}}{k_1!}\cdots\frac{(\gamma)_{k_i}}{k_i!} = \frac{(\gamma i)_k}{k!}. \tag{34}$$

To prove (34), we need some results about the multivariable Lagrange polynomials. The multivariable Lagrange polynomials are generated by (see [2]; see also [5])

$$\prod_{j=1}^{r}\left\{\left(1-x_j t\right)^{-\alpha_j}\right\} = \sum_{n=0}^{\infty} g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)\,t^n. \tag{35}$$

$$\left(|t| < \min\left\{|x_1|^{-1},\cdots,|x_r|^{-1}\right\}\right)$$

Its explicit representation is given by

$$g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)=\sum_{k_1+\cdots+k_r=n}(\alpha_1)_{k_1}\cdots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}. \tag{36}$$

Clearly, we have

$$\sum_{k_1+\cdots+k_i=k}\frac{(\gamma)_{k_1}}{k_1!}\cdots\frac{(\gamma)_{k_i}}{k_i!}=g_k^{(\gamma,\cdots,\gamma)}(1,\cdots,1). \tag{37}$$

By using ([2, p. 147, Eqn. (36)])

$$g_n^{(\alpha_1,\cdots,\alpha_r)}(x,\cdots,x)=\binom{\alpha_1+\cdots+\alpha_r+n-1}{n}x^n\ \ (n\in\mathbb{N}_0), \tag{38}$$

with $x=1$ and $\alpha_j=\gamma\ (j=1,\cdots,i)$, the result (34) follows. □

If we set $\gamma=0$ in Corollary 2.4, then $\mathbf{E}^{\gamma k}_{\alpha,\beta k,\omega;0+}$ reduces to the Riemann-Liouville fractional operator ${}_0D_t^{-\beta k}$, and we have the following known result.

Corollary 2.5. *([19, p. 506]) If* $c>0$, $\Re(\beta)>0$ *and* $f(t)\in L(0,\infty)$, *then for the solution of the equation*

$$N(t)-N_0f(t)=-c^{\beta}{}_0D_t^{-\beta}N(t), \tag{39}$$

there exists the formula

$$N(t)=N_0\left[f(t)+\sum_{k=1}^{\infty}\left(-c^{\beta}\right)^k\left({}_0D_t^{-\beta k}f\right)(t)\right]. \tag{40}$$

The cases when the function $f(t)\equiv1$, $f(t)=t^{\mu-1}$ $(\Re(\mu)>0)$ and $f(t)=t^{\mu-1}e^{-at}$ have been studied in [19]. These choices for the function $f(t)$ can be made applicable in our Theorem 2.3 also, but we omit further details in this regard.

The following theorem considers the case when

$$\sigma\left(k_j\right)=\frac{[\gamma]_{qk_j}}{k_j!}$$

in (29), then the coefficients $C(k;i)$ are specified by

$$C(k;i)=\sum_{k_1+\cdots+k_i=k}\frac{[\gamma]_{qk_1}}{k_1!}\cdots\frac{[\gamma]_{qk_i}}{k_i!}. \tag{41}$$

It is clear from Theorem 2.3 that we just need to evaluate and represent the coefficients in (41) in a more compact form as in (34) but it is not easy. However, we adopt a more concise and suitable method used in the proof of the result given below.

Theorem 2.6. *If* $c>0$, $\Re(\lambda)>0$, $\Re(\rho)>\max\{0,\Re(q)-1\}$, $\Re(q)>0$, $\lambda,\omega\in\mathbb{C}$ *and* $f(t)\in L(0,\infty)$, *then for the solution of the equation involving operator* (15)*:*

$$N(t)-N_0f(t)=-c^{\beta}\left(\mathcal{E}^{\omega;\gamma,q}_{0+;\rho,\lambda}N\right)(t) \tag{42}$$

there holds the formula

$$N(t) = N_0 f(t) + N_0 \sum_{i=1}^{\infty} \left(-c^{\beta}\right)^i \underbrace{\int_0^{\infty} \cdots \int_0^{\infty}}_{i\ times} \left(\mathcal{H}^{\lambda i,\rho:-}_{\omega \sum_{j=1}^{i} t_j^q, 0+:(\lambda i,\rho)} f\right)(t)\, \mathrm{d}\mu(t_1, \cdots, t_i), \tag{43}$$

where $\mathrm{d}\mu(t_1, \cdots, t_i)$ *is given below by* (47) *and* $\left(\mathcal{H}^{\lambda i,\rho:-}_{\omega \sum_{j=1}^{i} t_j^q, 0+:(\lambda i,\rho)} f\right)(t)$ *is defined by* (17).

Proof. By applying the representation that

$$[\gamma]_{qk_j} = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} t^{\gamma+qk_j-1} e^{-t} \mathrm{d}t \quad \left(\Re\left(\gamma + qk_j\right) > 0, j = 1, \cdots, i\right) \tag{44}$$

and

$$(t_1 + \cdots + t_i)^k = \sum_{k_1+\cdots+k_i=k} \frac{k!}{k_1! \cdots k_i!} t_1^{k_1} \cdots t_i^{k_i} \tag{45}$$

to (41), and interchanging the order of integration and summation, we obtain

$$\sum_{k_1+\cdots+k_i=k} \frac{[\gamma]_{qk_1}}{k_1!} \cdots \frac{[\gamma]_{qk_i}}{k_i!} = \frac{1}{k!} \int_0^{\infty} \cdots \int_0^{\infty} \left(t_1^q + \cdots + t_i^q\right)^k \mathrm{d}\mu(t_1, \cdots, t_i), \tag{46}$$

where $k \in \mathbb{N}_0$ and

$$\mathrm{d}\mu(t_1, \cdots, t_i) = \prod_{j=1}^{i} \mathrm{d}\mu\left(t_j\right) = \prod_{j=1}^{i} \frac{t_j^{\gamma-1} e^{-t_j}}{\Gamma(\gamma)} \mathrm{d}t, t_j \in [0, \infty), j = 1, \cdots, i. \tag{47}$$

Then, substituting (46) into (24), we have

$$\begin{aligned}
\left(\mathcal{J}_{\rho,\lambda i,0+;\omega} f\right)(t) &= \int_0^t (t-x)^{\lambda i-1} f(x) \sum_{k=0}^{\infty} \frac{[\omega (t-x)^{\rho}]^k}{\Gamma(\rho k + \lambda i)\, k!} \int_0^{\infty} \cdots \int_0^{\infty} \left(t_1^q + \cdots + t_i^q\right)^k \mathrm{d}\mu(t_1, \cdots, t_i)\, \mathrm{d}x \\
&= \int_0^t (t-x)^{\lambda i-1} f(x) \int_0^{\infty} \cdots \int_0^{\infty} \sum_{k=0}^{\infty} \frac{[\omega (t-x)^{\rho}]^k}{\Gamma(\rho k + \lambda i)\, k!} \left(t_1^q + \cdots + t_i^q\right)^k \mathrm{d}\mu(t_1, \cdots, t_i)\, \mathrm{d}x \\
&= \int_0^t (t-x)^{\lambda i-1} f(x) \int_0^{\infty} \cdots \int_0^{\infty} {}_0\Psi_1\left[\begin{matrix} - \\ (\lambda i, \rho) \end{matrix} ; \omega \mathbf{t} (t-x)^{\rho}\right] \mathrm{d}\mu(t_1, \cdots, t_i)\, \mathrm{d}x \\
&= \int_0^{\infty} \cdots \int_0^{\infty} \int_0^t (t-x)^{\lambda i-1} f(x)\, {}_0\Psi_1\left[\begin{matrix} - \\ (\lambda i, \rho) \end{matrix} ; \omega \mathbf{t} (t-x)^{\rho}\right] \mathrm{d}x \mathrm{d}\mu(t_1, \cdots, t_i),
\end{aligned} \tag{48}$$

where we write $\mathbf{t} = \sum_{j=1}^{i} t_j^q$.

The inner integral with respect to variable x can be further expressed by using operator (17), that is,

$$\left(\mathcal{H}^{\lambda i,\rho:-}_{\omega \sum_{j=1}^{i} t_j^q, 0+:(\lambda i,\rho)} f\right)(t) = \int_0^t (t-x)^{\lambda i-1} f(x)\, {}_0\Psi_1\left[\begin{matrix} - \\ (\lambda i, \rho) \end{matrix} ; \omega \mathbf{t} (t-x)^{\rho}\right] \mathrm{d}x.$$

Hence,

$$\left(\mathcal{J}_{\rho,\lambda i,0+;\omega}f\right)(t)=\int_{0}^{\infty}\cdots\int_{0}^{\infty}\left(\mathcal{H}^{\lambda i,\rho:-}_{\omega\sum_{j=1}^{i}t_j^{q},0+:(\lambda i,\rho)}f\right)(t)\,\mathrm{d}\mu\left(t_1,\cdots,t_i\right), \tag{49}$$

and (43) follows. □

Remark 2.7. *Since* (9) *is a special case of* (17)*, we have*

$$\left(\mathcal{E}^{\omega;\gamma,q}_{0+;\rho,\lambda}N\right)(t)=\frac{1}{\Gamma(\gamma)}\left(\mathcal{H}^{\lambda,\rho:(\gamma,\rho)}_{\omega,0+:(\lambda,\rho)}N\right)(t), \tag{50}$$

and equation (42) *can also be expressed as*

$$N(t)-N_0f(t)=-\frac{c^{\beta}}{\Gamma(\gamma)}\left(\mathcal{H}^{\lambda,\rho:(\gamma,\rho)}_{\omega,0+:(\lambda,\rho)}N\right)(t). \tag{51}$$

In recent years the fractional kinetic equations involving specific $f(t)$ have generated extensive interest among scientists and mathematicians. A large number of equations and their solutions have been found and studied with the help of special functions (see, for example [3], [6], [7], [9], and [20]). Here we consider the equation (23) when $f(t)$ is expressed by

$$f(t)=t^{\alpha-1}\mathcal{F}_{\rho,\alpha}\left(\omega t^{\rho}\right).$$

For convenience and clarity sake, we use the superscripts to indicate the particular coefficient of the kernel function. For example,

$$\mathcal{F}^{\sigma_1}_{\rho,\lambda}\left(\omega t^{\rho}\right)=\sum_{n=0}^{\infty}\frac{\sigma_1(n)}{\Gamma(\rho n+\lambda)}\left[\omega t^{\rho}\right]^{n}, \tag{52}$$

and correspondingly

$$\left(\mathcal{J}^{\sigma_1}_{\rho,\lambda,a+;\omega}\varphi\right)(x)=\int_{a}^{x}(x-t)^{\lambda-1}\,\mathcal{F}^{\sigma_1}_{\rho,\lambda}\left[\omega(x-t)^{\rho}\right]\varphi(t)\,\mathrm{d}t\ \ (x>a). \tag{53}$$

Theorem 2.8. *If* $c>0,\beta>0,\ \lambda,\alpha,\rho,\omega_1,\omega_2\in\mathbb{C}$ $(\Re(\alpha)>0,\ \Re(\lambda)>0,\ \Re(\rho)>0)$ *and* $f(t)\in L(0,\infty)$*, then the solution of the generalized fractional kinetic equation*

$$N(t)-N_0t^{\alpha-1}\mathcal{F}^{\sigma_2}_{\rho,\alpha}\left(\omega_2t^{\rho}\right)=-c^{\beta}\left(\mathcal{J}^{\sigma_1}_{\rho,\lambda;0+;\omega_1}N\right)(t) \tag{54}$$

is expressed by

$$N(t)=N_0t^{\alpha-1}\left[\mathcal{F}^{\sigma_2}_{\rho,\alpha}\left(\omega_2t^{\rho}\right)+\sum_{i=1}^{\infty}(-1)^{i}c^{\beta i}t^{\lambda i}\mathcal{F}^{\Omega(m;\sigma_3,\sigma_2;\omega_1,\omega_2)}_{\rho,\lambda i+\alpha}\left(\omega_1t^{\rho}\right)\right], \tag{55}$$

where function $\mathcal{F}^{\Omega(m;\sigma_3,\sigma_2;\omega_1,\omega_2)}_{\rho,\lambda i+\alpha}\left(\omega_1t^{\rho}\right)$ *is given by* (56) *below.*

Proof. The solution of (54) can be obtained by using (24) with

$$f(t)=t^{\alpha-1}\mathcal{F}^{\sigma_2}_{\rho,\alpha}\left(\omega_2t^{\rho}\right).$$

We have (on using (11) and (14))

$$\mathcal{J}^{\sigma_3}_{\rho,\lambda i,0+;\omega_1}\left[x^{\alpha-1}\mathcal{F}^{\sigma_2}_{\rho,\alpha}(\omega_2 x^\rho)\right](t) = \int_0^t (t-x)^{\lambda i-1}\mathcal{F}^{\sigma_3}_{\rho,\lambda i}\left[\omega_1(t-x)^\rho\right]x^{\alpha-1}\mathcal{F}^{\sigma_2}_{\rho,\alpha}(\omega_2 x^\rho)\,\mathrm{d}x$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\sigma_3(m)\,\sigma_2(n)\,\omega_1^m\omega_2^n}{\Gamma(\rho m+\lambda i)\,\Gamma(\rho n+\alpha)}\int_0^t x^{\alpha+\rho n-1}(t-x)^{\lambda i+\rho m-1}\,\mathrm{d}x$$
$$= t^{\alpha-1}t^{\lambda i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\sigma_3(m)\,\sigma_2(n)}{\Gamma(\rho(m+n)+\lambda i+\alpha)}\omega_1^m\omega_2^n t^{\rho(m+n)}$$
$$= t^{\alpha-1}t^{\lambda i}\sum_{m=0}^{\infty}\frac{\Omega(m;\sigma_3,\sigma_2;\omega_1,\omega_2)}{\Gamma(\rho m+\lambda i+\alpha)}\omega_1^m t^{\rho m}, \tag{56}$$

where (in view of (29))

$$\sigma_3(m) = \sum_{k_1+\cdots+k_i=m}\prod_{j=1}^{i}\sigma_1\left(k_j\right) \tag{57}$$

and

$$\Omega(m;\sigma_3,\sigma_2;\omega_1,\omega_2) = \sum_{n=0}^{m}\sigma_3(m-n)\,\sigma_2(n)\left(\frac{\omega_2}{\omega_1}\right)^n. \tag{58}$$

We can briefly write now

$$\mathcal{J}^{\sigma_3}_{\rho,\lambda i,0+;\omega_1}\left[x^{\alpha-1}\mathcal{F}^{\sigma_2}_{\rho,\alpha}(\omega_2 t^\rho)\right](t) = t^{\alpha-1}t^{\lambda i}\mathcal{F}^{\Omega(m;\sigma_3,\sigma_2;\omega_1,\omega_2)}_{\rho,\lambda i+\alpha}(\omega_1 t^\rho), \tag{59}$$

and substituting (59) into (24), we get (55). This completes the proof. □

The expression defining the coefficient $\Omega(m;\sigma_3,\sigma_2;\omega_1,\omega_2)$ above suggets that we can obtain closed form solutions by suitably choosing the coefficients σ_1 and σ_2 such that the finite sum (58) is summable.

If we set $\omega_1 = \omega_2 = \omega$ in (58), so that

$$\sigma_1(m) = \frac{(\rho_1)_m}{m!} \text{ and } \sigma_2(n) = \frac{(\rho_2)_n}{n!},$$

and $\sigma_3(m-n)$ is given by

$$\sigma_3(m-n) = \frac{(i\rho_1)_{m-n}}{(m-n)!}.$$

By applying the Chu-Vandermonde identity [15, p. 387, Eqn. (15.4.24)], we find that

$$\Omega(m;\sigma_3,\sigma_2;\omega_1,\omega_2) = \frac{(i\rho_1+\rho_2)_m}{m!}.$$

Hence, we have the following corollary.

Corollary 2.9. *([6]) If $c>0, \beta>0$, $\alpha, \rho_1, \rho_2, \rho, \omega, \lambda \in \mathbb{C}$, $\Re(\lambda)>0, \Re(\rho)>0$, then the solution of equation*

$$N(t) - N_0 t^{\alpha-1}E^{\rho_2}_{\rho,\alpha}(\omega t^\rho) = -c^\beta\left(\mathbf{E}^{\rho_1}_{\rho,\lambda,\omega;0+}N\right)(t) \tag{60}$$

is given by

$$N(t) = N_0 t^{\alpha-1}\left[E^{\rho_2}_{\rho,\alpha}(\omega t^\rho) + \sum_{i=1}^{\infty}(-1)^i c^{\beta i}t^{\lambda i}E^{i\rho_1+\rho_2}_{\rho,\lambda i+\alpha}(\omega t^\rho)\right]. \tag{61}$$

3. Extensions of Generalized Fractional Kinetic Equations

In this section, we will investigate further extensions of the generalized fractional kinetic equation (23).

Theorem 3.1. *If* $a_i > 0, \lambda_i, \rho, \omega \in \mathbb{C}\,(\Re(\lambda_i) > 0, \Re(\rho) > 0, i = 1, \cdots, n)$ *and* $f(t) \in L(0,\infty)$, *then the solution of the extended form of the generalized fractional kinetic equation*

$$N(t) - N_0 f(t) = -\sum_{i=1}^{n} a_i \mathcal{J}_{\rho,\lambda_i;0+,\omega} N(t) \tag{62}$$

is given by

$$N(t) = N_0 \left[f(t) + \sum_{k=1}^{\infty} (-1)^k \sum_{k_1+\cdots+k_n=k} \frac{k!}{k_1!\cdots k_n!} \prod_{i=1}^{n} a_i^{k_i} \left(\mathcal{J}_{\rho,\sum_{i=1}^{n}\lambda_i k_i,0+;\omega} f \right)(t) \right], \tag{63}$$

provided that the series on the right side of (63) *exists, where the integral operator* $\left(\mathcal{J}_{\rho,\sum_{i=1}^{n}\lambda_i k_i,0+;\omega} f\right)(t)$ *is defined by* (14).

Proof. Applying the Laplace transform to (62) and using (21), we get

$$\mathcal{L}[N(t)](s) - N_0 \mathcal{L}[f(t)](s) = -\sum_{i=1}^{n} a_i s^{-\lambda_i} \mathcal{F}(\omega s^{-\rho}) \mathcal{L}[N(t)](s). \tag{64}$$

Solving for $\mathcal{L}[N(t)](s)$, it gives

$$\begin{aligned} \mathcal{L}[N(t)](s) &= N_0 \left(1 + \mathcal{F}(\omega s^{-\rho}) \sum_{i=1}^{n} a_i s^{-\lambda_i} \right)^{-1} \mathcal{L}[f(t)](s) \\ &= N_0 \sum_{k=0}^{\infty} (-1)^k \mathcal{F}^k(\omega s^{-\rho}) \left(\sum_{i=1}^{n} a_i s^{-\lambda_i} \right)^k \mathcal{L}[f(t)](s). \\ &\left(\left| \mathcal{F}(\omega s^{-\rho}) \left(a_1 s^{-\lambda_1} + \cdots + a_n s^{-\lambda_n} \right) \right| < 1; \Re(s) > 0 \right) \end{aligned} \tag{65}$$

We note (in view of the particular case (12)) that

$$\mathcal{F}^k(\omega s^{-\rho}) = \sum_{l=0}^{\infty} \Omega(l;k)\, \omega^l s^{-\rho l} \quad (k = 1, 2, 3, \cdots), \tag{66}$$

where the coefficients $\Omega(l;k)$ are given by

$$\Omega(l;k) = \sum_{l_1+l_2+\cdots+l_k=l} \sigma(l_1)\cdots\sigma(l_k). \tag{67}$$

On the other hand, we also note that

$$\left(\sum_{i=1}^{n} a_i s^{-\lambda_i} \right)^k = \sum_{k_1+\cdots+k_n=k} \frac{k!}{k_1!\cdots k_n!} \prod_{i=1}^{n} a_i^{k_i} s^{-\sum_{i=1}^{n}\lambda_i k_i}, \tag{68}$$

$(k = 1, 2, 3, \cdots)$

where the summation is taken over all non-negative integers $k_1, \cdots, k_n$, such that $\sum_{i=1}^{n} k_i = k$.

Substituting (66) and (68) into (65), we obtain

$$\begin{aligned}\mathcal{L}[N(t)](s) &= N_0\mathcal{L}[f(t)](s) + N_0\sum_{k=1}^{\infty}(-1)^k\sum_{k_1+\cdots+k_n=k}\frac{k!}{k_1!\cdots k_n!}\prod_{i=1}^{n}a_i^{k_i}\\ &\quad\times\sum_{l=0}^{\infty}\Omega(l;k)\,\omega^l s^{-\rho l-\sum_{i=1}^{n}\lambda_i k_i}\mathcal{L}[f(t)](s)\\ &= N_0\mathcal{L}[f(t)](s) + N_0\sum_{k=1}^{\infty}(-1)^k\sum_{k_1+\cdots+k_n=k}\frac{k!}{k_1!\cdots k_n!}\prod_{i=1}^{n}a_i^{k_i}\\ &\quad\times\sum_{l=0}^{\infty}\Omega(l;k)\,\omega^l\frac{\mathcal{L}\left[t^{\rho l+\sum_{i=1}^{n}\lambda_i k_i-1}\right]\mathcal{L}[f(t)](s)}{\Gamma\left(\rho l+\sum_{i=1}^{n}\lambda_i k_i\right)}.\end{aligned} \tag{69}$$

Taking the inverse Laplace transform of (69) and applying the convolution theorem of the Laplace transforms, we have

$$\begin{aligned}N(t) &= N_0 f(t) + N_0\sum_{k=1}^{\infty}(-1)^k\sum_{k_1+\cdots+k_n=k}\frac{k!}{k_1!\cdots k_n!}\prod_{i=1}^{n}a_i^{k_i}\\ &\quad\times\sum_{l=0}^{\infty}\frac{\Omega(l;k)\,\omega^l}{\Gamma\left(\rho l+\sum_{i=1}^{n}\lambda_i k_i\right)}\int_0^t (t-x)^{\rho l+\sum_{i=1}^{n}\lambda_i k_i-1}f(x)\,\mathrm{d}x\\ &= N_0 f(t) + N_0\sum_{k=1}^{\infty}(-1)^k\sum_{k_1+\cdots+k_n=k}\frac{k!}{k_1!\cdots k_n!}\prod_{i=1}^{n}a_i^{k_i}\\ &\quad\times\int_0^t (t-x)^{\sum_{i=1}^{n}\lambda_i k_i-1}\mathcal{F}_{\rho,\sum_{i=1}^{n}\lambda_i k_i}\left[\omega(t-x)^{\rho}\right]f(x)\,\mathrm{d}x\\ &= N_0\left[f(t)+\sum_{k=1}^{\infty}(-1)^k\sum_{k_1+\cdots+k_n=k}\frac{k!}{k_1!\cdots k_n!}\prod_{i=1}^{n}a_i^{k_i}\left(\mathcal{J}_{\rho,\sum_{i=1}^{n}\lambda_i k_i,0+;\omega}f\right)(t)\right],\end{aligned} \tag{70}$$

which proves Theorem 3.1. □

Remark 3.2. *Some special cases of* (62) *have been studied in [3, p. 28, Theorem 1] and [4, p. 89, Theorem 1].*

Corollary 3.3. *If* $a_i > 0, \lambda_i, \rho, \omega \in \mathbb{C}\ (\Re(\lambda_i) > 0, \Re(\rho) > 0, i = 1, \cdots, n)$ *and* $f(t) \in L(0,\infty)$, *then the solution of the fractional kinetic equation*

$$N(t) - N_0 f(t) = -\sum_{i=1}^{n} a_i \mathbf{E}^{\gamma}_{\rho,\lambda_i,\omega;0+}N(t) \tag{71}$$

is given by

$$N(t) = N_0\left[f(t)+\sum_{k=1}^{\infty}(-1)^k\sum_{k_1+\cdots+k_n=k}\frac{k!}{k_1!\cdots k_n!}\prod_{i=1}^{n}a_i^{k_i}\left(\mathbf{E}^{\gamma k}_{\rho,\sum_{i=1}^{n}\lambda_i k_i,0+;\omega}f\right)(t)\right], \tag{72}$$

where the fractional integral operator $\left(\mathbf{E}^{\gamma k}_{\rho,\sum_{i=1}^{n}\lambda_i k_i,0+;\omega}f\right)(t)$ *is defined by* (9).

If we set $\gamma = 0$ in Corollary 3.3, then the fractional integral operator $\mathbf{E}^{\gamma}_{\rho,\lambda_i,\omega;0+}$ reduces to the Riemann-Liouville fractional integral operator ${}_0D_t^{-\lambda_i}$. We thus have the following result.

Corollary 3.4. *If $a_i > 0, \Re(\lambda_i) > 0$ $(i = 1, \cdots, n)$ and $f(t) \in L(0, \infty)$, then the solution of the fractional kinetic equation*

$$N(t) - N_0 f(t) = -\sum_{i=1}^{n} a_i \left({}_0D_t^{-\lambda_i} N\right)(t) \tag{73}$$

is given by

$$N(t) = N_0 \left[f(t) + \sum_{k=1}^{\infty} (-1)^k \sum_{k_1+\cdots+k_n=k} \frac{k!}{k_1! \cdots k_n!} \prod_{i=1}^{n} a_i^{k_i} \left({}_0D_t^{-\sum_{i=1}^{n} \lambda_i k_i} f\right)(t) \right]. \tag{74}$$

Remark 3.5. *It may be observed that Chaurasia and Singh [4, p. 89, Theorem 1.] have given the solution of* (73) *in a markedly different form by using the Sumudu transform.*

By using the same notations and methods employed in the proof of Theorem 2.8, we can easily derive the following theorem.

Theorem 3.6. *If $a_i > 0, \lambda_i, \alpha, \rho, \omega_1, \omega_2 \in \mathbb{C}$ ($\Re(\alpha) > 0$, $\Re(\lambda_i) > 0$, $\Re(\rho) > 0$, $i = 1, \cdots, n$) and $f(t) \in L(0, \infty)$, then the solution of the generalized fractional kinetic equation*

$$N(t) - N_0 t^{\alpha-1} \mathcal{F}_{\rho,\alpha}^{\sigma_2}(\omega_2 t^{\rho}) = -\sum_{i=1}^{n} a_i \left(\mathcal{J}_{\rho,\lambda_i;0+;\omega_1}^{\sigma_1} N\right)(t) \tag{75}$$

is expressed by

$$N(t) = N_0 t^{\alpha-1} \mathcal{F}_{\rho,\alpha}^{\sigma_2}(\omega_2 t^{\rho}) + N_0 t^{\alpha-1} \sum_{k=1}^{\infty} (-1)^k \sum_{k_1+\cdots+k_n=k} \frac{k!}{k_1! \cdots k_n!} \prod_{i=1}^{n} \left(a_i t^{\lambda_i}\right)^{k_i} \mathcal{F}_{\rho,\alpha+\sum_{i=1}^{n} \lambda_i k_i}^{\Omega(m;\sigma_3,\sigma_2;\omega_1,\omega_2)}(\omega_1 t^{\rho}), \tag{76}$$

where the function $\mathcal{F}_{\rho,\lambda i+\alpha}^{\Omega(m;\sigma_3,\sigma_2;\omega_1,\omega_2)}(\omega_1 t^{\rho})$ is given by (56).

Following the substitutions $\sigma_1(m) = (\rho_1)_m / m!$ and $\sigma_2(n) = (\rho_2)_n / n!$ in Theorem 3.6 and using (58), we obtain the following corollary:

Corollary 3.7. *If $a_i > 0, \lambda_i, \alpha, \rho_1, \rho_2, \rho, \omega \in \mathbb{C}$ ($\Re(\lambda_i) > 0$, $\Re(\rho) > 0$, $\Re(\alpha) > 0$, $i = 1, \cdots, n$) and $f(t) \in L(0, \infty)$, then the solution of the generalized fractional kinetic equation*

$$N(t) - N_0 t^{\alpha-1} E_{\rho,\alpha}^{\rho_2}(\omega t^{\rho}) = -\sum_{i=1}^{n} a_i \left(\mathbf{E}_{\rho,\lambda_i;0+,\omega}^{\rho_1} N\right)(t) \tag{77}$$

is given by

$$N(t) = N_0 t^{\alpha-1} E_{\rho,\alpha}^{\rho_2}(\omega t^{\rho}) + N_0 t^{\alpha-1} \sum_{k=1}^{\infty} (-1)^k \sum_{k_1+\cdots+k_n=k} \frac{k!}{k_1! \cdots k_n!} \prod_{i=1}^{n} \left(a_i t^{\lambda_i}\right)^{k_i} E_{\rho,\alpha+\sum_{i=1}^{n} \lambda_i k_i}^{k\rho_1+\rho_2}(\omega t^{\rho}), \tag{78}$$

where $E_{\rho,\alpha+\sum_{i=1}^{n} \lambda_i k_i}^{k\rho_1+\rho_2}(\omega t^{\rho})$ is the generalized Mittag-Leffler function defined by (10).

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