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Signed Total k-independence in Digraphs

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Abstract. Let $k \geq 2$ be an integer. A function $f : V(D) \rightarrow \{-1, 1\}$ defined on the vertex set $V(D)$ of a digraph D is a signed total k-independence function if $\sum_{x \in N^-(v)} f(x) \leq k - 1$ for each $v \in V(D)$, where $N^-(v)$ consists of all vertices of D from which arcs go into v. The weight of a signed total k-independence function f is defined by $w(f) = \sum_{x \in V(D)} f(x)$. The maximum of weights $w(f)$, taken over all signed total k-independence functions f on D, is the signed total k-independence number $\alpha_{st}^k(D)$ of D.

In this work, we mainly present upper bounds on $\alpha_{st}^k(D)$, as for example $\alpha_{st}^k(D) \leq n - 2\lceil(\Delta^- + 1 - k)/2\rceil$ and

$$\alpha_{st}^k(D) \leq \frac{\Delta^+ + 2k - \delta^+ - 2}{\Delta^+ + \delta^+} \cdot n,$$

where n is the order, Δ^- the maximum indegree and Δ^+ and δ^+ are the maximum and minimum outdegree of the digraph D. Some of our results imply well-known properties on the signed total 2-independence number of graphs.

1. Terminology and Introduction

In this paper, all digraphs are finite without loops or multiple arcs. The vertex set and arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. The *order* $n = n(D)$ of a digraph D is the number of its vertices. If uv is an arc of D, then we write $u \rightarrow v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For a vertex v of a digraph D, we denote the set of in-neighbors and out-neighbors of v by $N^-(v) = N_D^-(v)$ and $N^+(v) = N_D^+(v)$, respectively. The numbers $d_D^-(v) = d^-(v) = |N^-(v)|$ and $d_D^+(v) = d^+(v) = |N^+(v)|$ are the *indegree* and *outdegree* of v, respectively. The *minimum indegree*, *maximum indegree*, *minimum outdegree* and *maximum outdegree* of D are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. A digraph D is called *inregular* or *r-inregular* if $\delta^-(D) = \Delta^-(D) = r$ and *outregular* or *r-outregular* if $\delta^+(D) = \Delta^+(D) = r$. We say that D is *regular* or *r-regular* if it is r-inregular and r-outregular. If $X \subseteq V(D)$ and $v \in V(D)$, then $E(X, v)$ is the set of arcs from X to v and $E(v, X)$ the set of arcs from v to X. If X and Y are two disjoint vertex sets of a digraph D, then $E(X, Y)$ is the set of arcs from X to Y. The number of vertices of odd indegree and even indegree are denoted by n_o and n_e, respectively. If $X \subseteq V(D)$ and f is a mapping from $V(D)$ into some set of numbers, then $f(X) = \sum_{x \in X} f(x)$. For a vertex v in $V(D)$, we denote $f(N^-(v))$ by $f[v]$ for notational convenience. The *associated digraph* $D(G)$ of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e.

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In this work, we initiate the concept of the signed total k-independence number of a digraph. For graphs G and $k = 2$, this parameter was introduced by Wang and Shan [5] as a certain dual to the signed total domination number. The signed total domination number was introduced by Zelinka [7]. A two-valued function $f : V(G) \to \{-1, 1\}$ is a *signed total 2-independence function* if $f(N(v)) \leq 1$ for each vertex $v \in V(G)$, where $N(v)$ is the neighborhood of the vertex v in the graph G. The sum $f(V(G))$ is called the weight $w(f)$ of f. The maximum of weights $w(f)$, taken over all signed total 2-independence functions f on G, is called the *signed total 2-independence number* of G, denoted by $\alpha_{st}^2(G)$. The signed total 2-independence number is called *negative decision number* by Wang [4], and its possible application in social networks was also presented. This parameter has been studied in [4, 5] and [6]. Detailed information on domination and independence can be found in the two books by Haynes, Hedetniemi and Slater [1, 2].

Let $k \geq 2$ be an integer. A two-valued function $f : V(D) \to \{-1, 1\}$ is a *signed total k-independence function* if $f[v] \leq k - 1$ for every $v \in V(D)$. The weight of a signed total k-independence function f is defined by $w(f) = f(V(D))$. The maximum of weights $w(f)$, taken over all signed total k-independence functions f on D, is called the *signed total k-independence number* of D, denoted by $\alpha_{st}^k(D)$. A signed total k-independence function of weight $\alpha_{st}^k(D)$ is called a *$\alpha_{st}^k(D)$-function*. If $k \geq n$, then obviously $\alpha_{st}^k(D) = n$. Therefore we assume throughout this paper that $k \leq n - 1$. The signed total k-independence number only exists for digraphs D with $\delta^-(D) \geq 1$. The signed total 2-independence number of a digraph is a dual to the signed total domination number in a certain sense. The signed total domination number for digraphs was introduced by Sheikholeslami in [3].

Throughout this paper, if f is a $\alpha_{st}^k(D)$-function, then we let P and M denote the sets of those vertices in D which assigned under f the values 1 and -1, respectively, and we let $|P| = p$ and $|M| = m$. Thus $w(f) = |P| - |M| = n - 2m = 2p - n$.

We mainly present upper bounds on $\alpha_{st}^k(D)$. In addition, we prove some Nordhaus-Gaddum type inequalities. A lot of examples demonstrate the sharpness of the obtained bounds. Some of our results imply well-known properties on the signed total 2-independence number of graphs given by Wang [4], Wang, Shan [5] and Wang, Tong, Volkmann [6].

Since $N^-_{D(G)}(v) = N_G(v)$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Proposition 1.1. *Let $k \geq 2$ be an integer. If $D(G)$ is the associated digraph of a graph G with $\delta(G) \geq 1$, then we have $\alpha_{st}^k(D(G)) = \alpha_{st}^k(G)$.*

2. Upper Bounds

Theorem 2.1. *If $k \geq 2$ is an integer and D a digraph of order $n \geq k + 1$ with $\delta^-(D) \geq 1$, then*

$$2k - 2 - n \leq \alpha_{st}^k(D) \leq n - 2\left\lceil \frac{\Delta^-(D) + 1 - k}{2} \right\rceil.$$

Proof. Let $w \in V(D)$ be a vertex of maximum indegree $d^-(w) = \Delta^- = \Delta^-(D)$, and let f be a $\alpha_{st}^k(D)$-function.

The condition $f[w] \leq k-1$ leads to $|E(P, w)| - |E(M, w)| \leq k-1$, and since w is a vertex of maximum indegree, we have $|E(P, w)| + |E(M, w)| = \Delta^-$. Combining the last two inequalities, we deduce that $2|E(M, w)| \geq \Delta^- - k + 1$. It follows that

$$m \geq |E(M, w)| = \frac{2|E(M, w)|}{2} \geq \frac{\Delta^- + 1 - k}{2}$$

and so $m \geq \lceil(\Delta^- + 1 - k)/2\rceil$. This yields the upper bound

$$\alpha_{st}^k(D) = n - 2m \leq n - 2\left\lceil \frac{\Delta^- + 1 - k}{2} \right\rceil.$$

For the lower bound define the function $f : V(D) \rightarrow \{-1, 1\}$ by $f(a_1) = f(a_2) = \ldots = f(a_{k-1}) = 1$ for an arbitrary set of $k-1$ vertices $A = \{a_1, a_2, \ldots, a_{k-1}\}$ and $f(x) = -1$ for each vertex $x \in V(D) - A$. Obviously, f is a signed total k-independence function on D of weight $2k-2-n$ and thus $\alpha_{st}^k(D) \geq 2k-2-n$. □

Let K_n^* be the complete digraph of order n. If $n \geq 3$, then it is straightforward to verify that $\alpha_{st}^{n-1}(K_n^*) = n-4$. Thus the lower bound in Theorem 2.1 is sharp.

Example 2.2. *Let $k \geq 2$ be an integer, and let $K_{1,\Delta}$ be the star with the center w of degree $\Delta \geq k$ and the leaves $v_1, v_2, \ldots, v_\Delta$. Now let D be the associated digraph of $K_{1,\Delta}$. Then $\Delta^-(D) = \Delta$ and $\delta^-(D) = 1$.*

Assume first that $\Delta - k$ is even. Define the function $f : V(D) \rightarrow \{-1, 1\}$ by $f(w) = f(v_1) = f(v_2) = \ldots = f(v_{(\Delta+k-2)/2}) = 1$ and $f(x) = -1$ otherwise. Then

$$f[w] = \frac{\Delta + k - 2}{2} - \frac{\Delta + 2 - k}{2} = k - 2$$

and $f[x] = 1 \leq k-1$ for $x \neq w$. Therefore f is a signed total k-independence function on D with $w(f) = k-1$. Hence Theorem 2.1 implies that

$$k - 1 \leq \alpha_{st}^k(D) \leq n(D) - 2\left\lceil \frac{\Delta + 1 - k}{2} \right\rceil = k - 1$$

and thus $\alpha_{st}^k(D) = k - 1$.

Assume second that $\Delta - k \geq 1$ is odd. Define the function $f : V(D) \rightarrow \{-1, 1\}$ by $f(w) = f(v_1) = f(v_2) = \ldots = f(v_{(\Delta+k-1)/2}) = 1$ and $f(x) = -1$ otherwise. Then

$$f[w] = \frac{\Delta + k - 1}{2} - \frac{\Delta + 1 - k}{2} = k - 1$$

and $f[x] = 1 \leq k-1$ for $x \neq w$. Therefore f is a signed total k-independence function on D with $w(f) = k$. Hence Theorem 2.1 implies that

$$k \leq \alpha_{st}^k(D) \leq n(D) - 2\left\lceil \frac{\Delta + 1 - k}{2} \right\rceil = k$$

and thus $\alpha_{st}^k(D) = k$.

Example 2.2 demonstrates that the upper bound in Theorem 2.1 is sharp.

Corollary 2.3. *([6]) If G is a graph of order n without isolated vertices and maximum degree Δ, then $\alpha_{st}^2(G) \leq n - 2\lfloor \Delta/2 \rfloor$.*

Proof. Since $\Delta = \Delta^-(D(G))$, it follows from Proposition 1.1 and Theorem 2.1 that

$$\alpha_{st}^2(G) = \alpha_{st}^2(D(G)) \leq n - 2\left\lceil \frac{\Delta^-(D(G)) - 1}{2} \right\rceil = n - 2\left\lfloor \frac{\Delta}{2} \right\rfloor. \ \square$$

Corollary 2.4. *Let $k \geq 2$ be an integer. If D is a digraph of order $n \geq k+1$ with $\delta^-(D) \geq 1$, then $\alpha_{st}^k(D) = n$ if and only if $\Delta^-(D) \leq k-1$.*

Proof. If $\Delta^-(D) \leq k-1$, then $f : V(D) \rightarrow \{-1, 1\}$ with $f(v) = 1$ for each vertex $v \in V(D)$ is a signed total k-independence function on D of weight n and thus $\alpha_{st}^k(D) = n$.

Conversely, assume that $\alpha_{st}^k(D) = n$. If we suppose that $\Delta^-(D) \geq k$, then Theorem 2.1 leads to the contradiction $n = \alpha_{st}^k(D) \leq n - 2$. Therefore $\Delta^-(D) \leq k-1$, and the proof is complete. □

Theorem 2.5. *Let $k \geq 2$ be an even integer. If D is a digraph of order $n \geq k+1$ with $\delta^+, \delta^- \geq 1$, then*

$$\alpha_{st}^k(D) \leq \min\left\{ \frac{n(\Delta^+ + k - 2) + n_o - |A(D)|}{\Delta^+}, \frac{n(k - 2 - \delta^+) + n_o + |A(D)|}{\delta^+} \right\}.$$

Proof. Let V_o and V_e be the vertex sets of odd and even indegree, respectively. Now let f be a $\alpha_{st}^k(D)$-function. The conditions $f[v] \leq k-1$ and k even imply that $f[v] \leq k-2$ for $v \in V_e$. It follows that

$$\sum_{v\in V(D)} f[v] = \sum_{v\in V_o} f[v] + \sum_{v\in V_e} f[v] \leq n_o(k-1) + (n-n_o)(k-2) = n(k-2) + n_o$$

and thus

$$\begin{aligned} n(k-2)+n_o &\geq \sum_{v\in V(D)} f[v] = \sum_{v\in V(D)} d^+(v)f(v) = \sum_{v\in P} d^+(v) - \sum_{v\in M} d^+(v) \\ &= \sum_{v\in V(D)} d^+(v) - 2\sum_{v\in M} d^+(v) = 2\sum_{v\in P} d^+(v) - \sum_{v\in V(D)} d^+(v) \\ &= |A(D)| - 2\sum_{v\in M} d^+(v) = 2\sum_{v\in P} d^+(v) - |A(D)|. \end{aligned}$$

It follows that

$$n(k-2) + n_o \geq |A(D)| - 2(n-p)\Delta^+ \quad (1)$$

as well as

$$n(k-2) + n_o \geq 2p\delta^+ - |A(D)| \quad (2)$$

and so

$$2p \leq \frac{kn + 2n\Delta^+ - |A(D)| + n_o - 2n}{\Delta^+} \quad (3)$$

and

$$2p \leq \frac{kn + |A(D)| + n_o - 2n}{\delta^+}. \quad (4)$$

Using (3) and (4), we obtain

$$\alpha_{st}^k(D) = 2p - n \leq \frac{n(\Delta^+ + k - 2) - |A(D)| + n_o}{\Delta^+}$$

and

$$\alpha_{st}^k(D) = 2p - n \leq \frac{n(k - 2 - \delta^+) + |A(D)| + n_o}{\delta^+},$$

and the last two inequalities lead to the desired result. □

Corollary 2.6. *Let $k \geq 2$ be an even integer. If D is a digraph of order $n \geq k+1$ with $\delta^+, \delta^- \geq 1$, then*

$$\alpha_{st}^k(D) \leq \frac{n(\Delta^+ + 2k - \delta^+ - 4) + 2n_o}{\Delta^+ + \delta^+}.$$

Proof. According to (1) and (2), we have

$$2p\Delta^+ \leq n(2\Delta^+ + k - 2) - |A(D)| + n_o$$

and

$$2p\delta^+ \leq n(k-2) + |A(D)| + n_o.$$

Adding these two inequalities, we arrive at

$$2p \leq \frac{2n(\Delta^+ + k - 2) + 2n_o}{\Delta^+ + \delta^+}.$$

and this yields to the desired bound immediately. □

Corollary 2.7. *If $k \geq 2$ is an even integer and D an r-outregular digraph of order $n \geq k+1$ with $r \geq 1$ and $\delta^- \geq 1$, then*

$$\alpha_{st}^k(D) \leq \frac{n(k-2)+n_o}{r}.$$

Corollary 2.8. *Let $k \geq 2$ be an even integer and D an r-regular digraph of order $n \geq k+1$. If $r \geq 2$ is even, then*

$$\alpha_{st}^k(D) \leq \frac{n(k-2)}{r}.$$

In the case that k is odd, we obtain the next results analogously to the proofs of Theorem 2.5 and Corollary 2.6.

Theorem 2.9. *Let $k \geq 3$ be an odd integer. If D is a digraph of order $n \geq k+1$ with $\delta^+, \delta^- \geq 1$, then*

$$\alpha_{st}^k(D) \leq \min\left\{\frac{n(\Delta^+ + k - 2) - |A(D)| + n_e}{\Delta^+}, \frac{n(k-2-\delta^+) + |A(D)| + n_e}{\delta^+}\right\}.$$

Corollary 2.10. *Let $k \geq 3$ be an odd integer. If D is a digraph of order $n \geq k+1$ with $\delta^+, \delta^- \geq 1$, then*

$$\alpha_{st}^k(D) \leq \frac{n(\Delta^+ + 2k - \delta^+ - 4) + 2n_e}{\Delta^+ + \delta^+}.$$

Corollary 2.11. *Let $k \geq 3$ be an odd integer. If D is an r-outregular digraph of order $n \geq k+1$ with $r \geq 1$ and $\delta^- \geq 1$, then*

$$\alpha_{st}^k(D) \leq \frac{n(k-2)+n_e}{r}.$$

Corollary 2.12. *Let $k \geq 3$ be an odd integer and D an r-regular digraph of order $n \geq k+1$. If $r \geq 1$ is odd, then*

$$\alpha_{st}^k(D) \leq \frac{n(k-2)}{r}.$$

Example 2.13. *Let $u_1, u_2, \ldots, u_p$ and $v_1, v_2, \ldots, v_p$ be the partite sets of the complete bipartite digraph $K_{p,p}^*$, and let k be an integer such that $2 \leq k \leq p$.*

Assume that $k = 2t$ is even and $p = 2s+1$ is odd. Define the function $f : V(K_{p,p}^*) \to \{-1, 1\}$ by $f(u_1) = f(u_2) = \ldots = f(u_{t+s}) = f(v_1) = f(v_2) = \ldots = f(v_{t+s}) = 1$ and $f(x) = -1$ otherwise. Then $f[x] = t + s - (s+1-t) = 2t-1 = k-1$ for each vertex $x \in V(K_{p,p}^*)$. Therefore f is a signed total k-independence function on $K_{p,p}^*$ with $w(f) = 2(k-1)$. Hence Corollary 2.7 implies that

$$2(k-1) \leq \alpha_{st}^k(K_{p,p}^*) \leq \frac{2p(k-2)+2p}{p} = 2(k-1)$$

and thus $\alpha_{st}^k(K_{p,p}^*) = 2(k-1)$ when k is even and p is odd.

Assume that $k = 2t$ and $p = 2s$ are even. Define $f : V(K_{p,p}^*) \to \{-1, 1\}$ by $f(u_1) = f(u_2) = \ldots = f(u_{t+s-1}) = f(v_1) = f(v_2) = \ldots = f(v_{t+s-1}) = 1$ and $f(x) = -1$ otherwise. Then $f[x] = t+s-1-(s+1-t) = 2t-2 = k-2$ for each vertex $x \in V(K_{p,p}^*)$. Therefore f is a signed total k-independence function on $K_{p,p}^*$ with $w(f) = 2(k-2)$. Hence Corollary 2.8 implies that

$$2(k-2) \leq \alpha_{st}^k(K_{p,p}^*) \leq \frac{2p(k-2)}{p} = 2(k-2)$$

and thus $\alpha_{st}^k(K_{p,p}^*) = 2(k-2)$ when k and p are even.

Assume that $k = 2t+1$ and $p = 2s+1$ are odd. Define $f : V(K_{p,p}^*) \to \{-1, 1\}$ by $f(u_1) = f(u_2) = \ldots = f(u_{t+s}) = f(v_1) = f(v_2) = \ldots = f(v_{t+s}) = 1$ and $f(x) = -1$ otherwise. Then $f[x] = t + s - (s+1-t) = 2t-1 = k-2$ for each vertex $x \in V(K_{p,p}^*)$. Therefore f is a signed total k-independence function on $K_{p,p}^*$ with $w(f) = 2(k-2)$. Hence Corollary 2.12 implies that

$$2(k-2) \leq \alpha_{st}^k(K_{p,p}^*) \leq \frac{2p(k-2)}{p} = 2(k-2)$$

and thus $\alpha_{st}^k(K_{p,p}^*) = 2(k-2)$ *when* k *and* p *are odd.*

Assume that $k = 2t+1$ *is odd and* $p = 2s$ *is even. Define* $f : V(K_{p,p}^*) \to \{-1, 1\}$ *by* $f(u_1) = f(u_2) = \ldots = f(u_{t+s}) = f(v_1) = f(v_2) = \ldots = f(v_{t+s}) = 1$ *and* $f(x) = -1$ *otherwise. Then* $f[x] = t + s - (s-t) = 2t = k-1$ *for each vertex* $x \in V(K_{p,p}^*)$*. Therefore* f *is a signed total k-independence function on* $K_{p,p}^*$ *with* $w(f) = 2(k-1)$*. Hence Corollary 2.11 implies that*

$$2(k-1) \le \alpha_{st}^k(K_{p,p}^*) \le \frac{2p(k-2)+2p}{p} = 2(k-1)$$

and thus $\alpha_{st}^k(K_{p,p}^*) = 2(k-1)$ *when* k *is odd and* p *is even.*

Example 2.13 shows that Corollaries 2.7, 2.8, 2.11 and 2.12 and therefore Theorems 2.5 and 2.9 as well as Corollaries 2.6 and 2.10 are sharp.

Corollary 2.14. *([5]) Let* G *be a graph of order* n *without isolated vertices, maximum degree* Δ *and minimum degree* δ*. If* $n_0(G)$ *is the number of vertices of odd degree, then*

$$\alpha_{st}^2(G) \le \frac{n(\Delta-\delta)+2n_0(G)}{\Delta+\delta}.$$

Proof. Since $\delta = \delta^+(D(G))$, $\Delta = \Delta^+(D(G))$, $n = n(D(G))$ and $n_0 = n_0(G)$, it follows from Corollary 2.6 and Proposition 1.1 that

$$\alpha_{st}^2(G) = \alpha_{st}^2(D(G)) \le \frac{n(\Delta^+(D(G)) - \delta^+(D(G))) + 2n_0}{\Delta^+(D(G)) + \delta^+(D(G))} = \frac{n(\Delta-\delta)+2n_0(G)}{\Delta+\delta}. \quad \square$$

Corollary 2.15. *([4, 5]) If* G *is an* r*-regular graph of order* n *with* $r \ge 1$*, then* $\alpha_{st}^2(G) \le n/r$ *when* r *is odd and* $\alpha_{st}^2(G) \le 0$ *when* r *is even.*

Theorem 2.16. $k \ge 2$ *be an integer. If* D *is a digraph of order* $n \ge k+1$ *and minimum indegree* $\delta^- \ge k-1$*, then*

$$\alpha_{st}^k(D) \le \frac{n}{\Delta^+}\left(\Delta^+ - 2\left\lceil\frac{\delta^-+1-k}{2}\right\rceil\right).$$

Proof. Let f be a $\alpha_{st}^k(D)$-fuction. As $f[x] \le k-1$, we deduce that $|E(P,x)| - |E(M,x)| \le k-1$ for each vertex $x \in V(D)$. It follows that

$$\delta^- \le d^-(x) = |E(P,x)| + |E(M,x)| \le 2|E(M,x)| + k - 1$$

and so $|E(M,x)| \ge \lceil(\delta^- + 1 - k)/2\rceil$ for each vertex $x \in V(D)$. This leads to

$$\begin{aligned} n\left\lceil\frac{\delta^-+1-k}{2}\right\rceil &\le \sum_{x\in V(D)} |E(M,x)| = \sum_{x\in M} |E(M,x)| + \sum_{x\in P} |E(M,x)| \\ &= \sum_{x\in M} |E(x,M)| + \sum_{x\in M} |E(x,P)| = \sum_{x\in M} d^+(x) \le m\Delta^+ \end{aligned}$$

and thus

$$m \ge \frac{n}{\Delta^+}\left\lceil\frac{\delta^-+1-k}{2}\right\rceil.$$

It follows that

$$\alpha_{st}^k(D) = n - 2m \le \frac{n}{\Delta^+}\left(\Delta^+ - 2\left\lceil\frac{\delta^-+1-k}{2}\right\rceil\right). \quad \square$$

Counting the arcs from P to M, we obtain the next theorem analogously to the proof of Theorem 2.16.

Theorem 2.17. *Let $k \geq 2$ be an integer. If D is a digraph of order $n \geq k+1$ with $\delta^+, \delta^- \geq 1$, then*

$$\alpha_{st}^k(D) \leq \frac{n}{\delta^+}\left(2\left\lfloor \frac{\Delta^- + k - 1}{2} \right\rfloor - \delta^+\right).$$

Note that Theorems 2.16 and 2.17 also imply Corollaries 2.8 and 2.12 immediately.

Theorem 2.18. *Let $k \geq 2$ be an integer and D a digraph of order $n \geq k+1$ with $\delta^- \geq 1$. If $\delta^+ - \lfloor(\Delta^- + k - 1)/2\rfloor \geq 0$, then*

$$\alpha_{st}^k(D) \leq n + k - 2 + \delta^+ - \left\lfloor \frac{\Delta^- + k - 1}{2} \right\rfloor - \sqrt{\left(k - 2 + \delta^+ - \left\lfloor \frac{\Delta^- + k - 1}{2} \right\rfloor\right)^2 + 4n\left(\delta^+ - \left\lfloor \frac{\Delta^- + k - 1}{2} \right\rfloor\right)}.$$

Proof. *Let f be a $\alpha_{st}^k(D)$-function. The condition $f[x] \leq k-1$ implies that $|E(P,x)| + 1 - k \leq |E(M,x)|$ for each vertex $x \in P$. It follows that*

$$\Delta^- \geq d^-(x) = |E(P,x)| + |E(M,x)| \geq 2|E(P,x)| + 1 - k$$

and so $|E(P,x)| \leq \lfloor(\Delta^- + k - 1)/2\rfloor$ for each $x \in P$. Hence we deduce that

$$|E(D[P])| = \sum_{x \in P} |E(P,x)| \leq p \left\lfloor \frac{\Delta^- + k - 1}{2} \right\rfloor$$

and thus

$$|E(P,M)| = \sum_{x \in P} d^+(x) - |E(D[P])| \geq p\delta^+ - p\left\lfloor \frac{\Delta^- + k - 1}{2} \right\rfloor = (n-m)\left(\delta^+ - \left\lfloor \frac{\Delta^- + k - 1}{2} \right\rfloor\right). \quad (5)$$

Because of $f[x] \leq k-1$, each vertex of M has most $m + k - 2$ in-neighbors in P. and so $|E(P,M)| \leq m(m+k-2)$. Using (5), we conclude that

$$(n-m)\left(\delta^+ - \left\lfloor \frac{\Delta^- + k - 1}{2} \right\rfloor\right) \leq |E(P,M)| \leq m(m+k-2)$$

and therefore

$$m^2 + m\left(k - 2 + \delta^+ - \left\lfloor \frac{\Delta^- + k - 1}{2} \right\rfloor\right) - n\left(\delta^+ - \left\lfloor \frac{\Delta + k - 1}{2} \right\rfloor\right) \geq 0.$$

This leads to

$$m \geq -\frac{1}{2}\left(k - 2 + \delta^+ - \left\lfloor \frac{\Delta^+ + k - 1}{2} \right\rfloor\right) + \sqrt{\frac{1}{4}\left(k - 2 + \delta^+ - \left\lfloor \frac{\Delta^+ + k - 1}{2} \right\rfloor\right)^2 + n\left(\delta^+ - \left\lfloor \frac{\Delta^+ + k - 1}{2} \right\rfloor\right)},$$

and we obtain the desired bound as follows

$$\alpha_s^k(D) = n - 2m \leq n + k - 2 + \delta^+ - \left\lfloor \frac{\Delta^+ + k - 1}{2} \right\rfloor - \sqrt{\left(k - 2 + \delta^+ - \left\lfloor \frac{\Delta^+ + k - 1}{2} \right\rfloor\right)^2 + 4n\left(\delta^+ - \left\lfloor \frac{\Delta^+ + k - 1}{2} \right\rfloor\right)}. \quad \square$$

3. Nordhaus-Gaddum Type Results

The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u, v the arc uv belongs to $\overline{D}$ if and only if uv does not belong to D. As an application of Theorem 2.1 and Corollaries 2.4, 2.7, 2.8, 2.11 and 2.12, we shall prove some Nordhaus-Gaddum type results.

Theorem 3.1. *Let $k \geq 2$ be an integer. If D is a digraph of order $n \geq k+1$ such that $\delta^-(D), \delta^-(\overline{D}) \geq 1$, then*

$$\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq n + 2k - 1$$

with equality only if D is inregular.

Proof. As $\Delta^-(D) + \Delta^-(\overline{D}) \geq n - 1$, Theorem 2.1 implies that

$$\begin{aligned} \alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) &\leq n - 2\left\lceil \frac{\Delta^-(D) + 1 - k}{2} \right\rceil + n - 2\left\lceil \frac{\Delta^-(\overline{D}) + 1 - k}{2} \right\rceil & (6) \\ &\leq n - \Delta^-(D) + k - 1 + n - \Delta^-(\overline{D}) + k - 1 \\ &= 2n + 2k - 2 - \Delta^-(D) - \Delta^-(\overline{D}) & (7) \\ &\leq n + 2k - 1 \end{aligned}$$

and this is the desired Nordhaus-Gaddum bound. Let $d_D^-(u) = \delta^-(D)$. If D is not inregular, then $\delta^-(D) < \Delta^-(D)$ and therefore

$$\begin{aligned} \Delta^-(D) + \Delta^-(\overline{D}) &\geq \Delta^-(D) + d_{\overline{D}}^-(u) = \Delta^-(D) + d_{\overline{D}}^-(u) + d_D^-(u) - d_D^-(u) \\ &= \Delta^-(D) + n - 1 - d_D^-(u) = \Delta^-(D) + n - 1 - \delta^-(D) \geq n. \end{aligned}$$

Using this inequality chain and (7), we obtain in the case that D is not inregular the better bound $\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq n + 2k - 2$. This completes the proof. □

For regular digraphs we shall improve the Nordhaus-Gaddum bound given in Theorem 3.1.

Theorem 3.2. *Let $k \geq 2$ be an integer, and let D be an r-regular digraph of order $n \geq k+1$ such that $r \geq 1$ and $n - r - 1 \geq 1$. If $r \geq k$ or $n - r - 1 \geq k$, then*

$$\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq n + 2k - 3.$$

Proof. Note that $\overline{D}$ is $(n - r - 1)$-regular.

Case 1. Assume that $k \geq 2$ is even.

Subcase 1.1. Assume that r and $n - r - 1$ are even. Then (6) implies that

$$\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq n - (r + 2 - k) + n - (n - r - 1 + 2 - k) = n + 2k - 3.$$

Subcase 1.2. Assume that $r \geq k$ and $n - r - 1 \geq k$. Furthermore, assume that r or $n - r - 1$ is odd, say r is odd. Since k is even and $r \geq k$, we observe that $k + 1 \leq r \leq n - k - 1$ and thus $n \geq 2k + 2$. Corollary 2.7 implies that

$$\begin{aligned} \alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) &\leq n(k-1)\left(\frac{1}{r} + \frac{1}{n-r-1}\right) \\ &\leq n(k-1)\max\left\{\frac{1}{k+1} + \frac{1}{n-k-2}, \frac{1}{n-k-1} + \frac{1}{k}\right\} \\ &\leq n(k-1)\left(\frac{1}{n-k-1} + \frac{1}{k}\right). & (8) \end{aligned}$$

Now we show that

$$n(k-1)\left(\frac{1}{n-k-1} + \frac{1}{k}\right) < n + 2k - 2. \tag{9}$$

Inequality (9) is equivalent to

$$nk^2 + n^2 + 2k > n + 2k^3 + 2kn. \tag{10}$$

Since $n \geq 2k+2$, we deduce that

$$nk^2 + n^2 + 2k \geq (2k+2)k^2 + n(2k+2) + 2k = 2k^3 + 2k^2 + 2kn + 2n + 2k > 2k^3 + 2kn + n.$$

Therefore (10) and so (9) are proved. The inequalities (8) and (9) show that $\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq n + 2k - 3$ in that case.

Subcase 1.3. Assume that $r \geq k$ and $n - r - 1 \leq k - 1$ or $r \leq k - 1$ and $n - r - 1 \geq k$, say $r \geq k$ and $n - r - 1 \leq k - 1$. Note that $n = (n - r - 1) + r + 1 \leq k - 1 + r + 1 = r + k$.

Subcase 1.3.1. Assume that r is even. It follows from Corollaries 2.4 and 2.8 that

$$\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq \frac{n(k-2)}{r} + n. \tag{11}$$

Since $n \leq r + k$ and $r \geq k$, we observe that

$$n(k-2) \leq (r+k)(k-2) = r(k-2) + k(k-2) \leq r(k-2) + r(k-1) = r(2k-3).$$

Using this inequality chain and (11), we obtain

$$\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq \frac{n(k-2)}{r} + n \leq n + 2k - 3.$$

Subcase 1.3.2. Assume that r is odd. Since k is even, we see that $r \geq k+1$. It follows from Corollaries 2.4 and 2.7 that

$$\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq \frac{n(k-1)}{r} + n. \tag{12}$$

Since $n \leq r + k$ and $r \geq k + 1$, we observe that

$$n(k-1) \leq (r+k)(k-1) = r(k-1) + k(k-1) < r(k-1) + r(k-1) = r(2k-2).$$

Using this inequality chain and (12), we obtain

$$\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq \frac{n(k-1)}{r} + n < \frac{r(2k-2)}{r} + n = n + 2k - 2$$

and thus $\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq n + 2k - 3$.

Case 2. Assume that $k \geq 3$ is odd.

Subcase 2.1. Assume that r and $n - r - 1$ are odd. Then (6) implies as in Subcase 1.1 that

$$\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq n + 2k - 3.$$

Subcase 2.2. Assume that $r \geq k$ and $n - r - 1 \geq k$. Furthermore, assume that r or $n - r - 1$ is even, say r is even. Since k is odd and $r \geq k$, we observe that $k + 1 \leq r \leq n - k - 1$ and thus $n \geq 2k + 2$. Corollary 2.11 implies that

$$\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq n(k-1)\left(\frac{1}{r} + \frac{1}{n-r-1}\right)$$

Now we obtain $\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq n + 2k - 3$ as in Subcase 1.2.

Subcase 2.3. Assume that $r \geq k$ and $n - r - 1 \leq k - 1$ or $r \leq k - 1$ and $n - r - 1 \geq k$, say $r \geq k$ and $n - r - 1 \leq k - 1$. Note that $n \leq r + k$.

Subcase 2.3.1. Assume that r is odd. Then $n(k-2) \leq r(2k-3)$, and it follows from Corollaries 2.4 and 2.12 that

$$\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq \frac{n(k-2)}{r} + n \leq n + 2k - 3.$$

Subcase 2.3.2. Assume that r is even. Since k is odd, we see that $r \geq k+1$. Then $n(k-1) < r(2k-2)$, and we deduce from from Corollaries 2.4 and 2.11 that

$$\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq \frac{n(k-1)}{r} + n < \frac{r(2k-2)}{r} + n = n + 2k - 2$$

and thus $\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq n + 2k - 3$. □

Example 3.3. *Let $k \geq 3$ be an odd integer, and let H be the graph of order $n = 2k+1$ with vertex set*

$$\{w, z, u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_{k-1}\}$$

such that w is adjacent to $z, u_1, u_2, \dots, u_k$, z is adjacent to $v_1, v_2, \dots, v_{k-1}$, each vertex u_i is adjacent to each vertex v_j for $1 \leq i \leq k$ and $1 \leq j \leq k-1$, u_i is adjacent to u_{i+1} for each $i \in \{2, 4, \dots, k-1\}$ and u_1 is adjacent to z. Now let $D(H)$ be the associated digraph of H. It is evident that $D(H)$ is $(k+1)$-regular and so $\overline{D(H)}$ is $(k-1)$-regular. Define $f : V(D(H)) \to \{-1, 1\}$ by $f(w) = f(z) = -1$ and $f(x) = 1$ for $x \in V(D(H)) - \{w, z\}$. Since every vertex x of $D(H)$ has at least one in-neighbor in $\{w, z\}$, we observe that $f[x] \leq k-1$ for each vertex x. Therefore f is a signed total k-independence function on $D(H)$ with $w(f) = 2k-3$. Hence Corollary 2.11 leads to

$$2k - 3 \leq \alpha_{st}^k(D(H)) \leq \left\lfloor \frac{n(k-1)}{k+1} \right\rfloor = \left\lfloor \frac{(2k+1)(k-1)}{k+1} \right\rfloor = \left\lfloor \frac{(2k-3)(k+1)+2}{k+1} \right\rfloor = 2k - 3$$

and thus $\alpha_{st}^k(D(H)) = 2k-3$. Applying Corollary 2.4, we obtain

$$\alpha_{st}^k(D(H)) + \alpha_{st}^k(\overline{D(H)}) = n + 2k - 3.$$

Example 3.3 demonstrates that Theorem 3.2 is sharp, at least for k odd. If $\Delta(D) \leq k-1$ and $\Delta(\overline{D}) \leq k-1$, then Corollary 2.4 implies that $\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) = 2n$. The next example will show that in this case the Nordhaus-Gaddum bound $\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) \leq n + 2k - 3$ in Theorem 3.2 is not valid in general.

Example 3.4. *Let $k \geq 3$ be an integer. If D is a $(k-1)$-regular digraph of order $n = 2(k-1)$, then $\overline{D}$ is $(k-2)$-regular. It follows from Corollary 2.4 that*

$$\alpha_{st}^k(D) + \alpha_{st}^k(\overline{D}) = 2n = n + 2k - 2.$$

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