Existence of Solutions or Nonlinear nth-order Differential Equations and Inclusions with Nonlocal and Integral Boundary Conditions via Fixed Point Theory

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Abstract. In this paper, a class of boundary value problems of nonlinear nth-order differential equations and inclusions with nonlocal and integral boundary conditions is studied. New existence results are obtained by means of some fixed point theorems. Examples are given for the illustration of the the results.

1. Introduction

We discuss the existence of solutions for the following boundary value problems of nonlinear nth-order differential equations and inclusions with nonlocal and integral boundary conditions:

$$
\begin{cases}
u^{(n)}(t) = f(t, u(t)), \quad t \in [0, 1], \\
u(0) = u_0 + g(u), \\
u'(0) = 0, \quad u''(0) = 0, \dots, u^{(n-2)}(0) = 0, \\
\delta_1 u(1) + \delta_2 u'(1) = \sum_{i=1}^{m} \kappa_i \int_0^{\zeta_i} u'(s)ds, \quad 0 < \zeta_i < 1,
\end{cases}
\tag{1}
$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $g : C([0, 1], \mathbb{R}) \to \mathbb{R}$ are given continuous functions, $\delta_1, \delta_2, \kappa_i, \zeta_i$, $(i = 1, 2, \ldots, m)$ are real constants and

$$
\begin{cases}
u^{(n)}(t) \in F(t, u(t)), & \text{a.e. } t \in [0, 1], \\
\\
u(0) = u_0 + g(u), \\
\\
u'(0) = 0, \quad u''(0) = 0, \ldots, u^{(n-2)}(0) = 0, \\
\delta_1 u(1) + \delta_2 u'(1) = \displaystyle\sum_{i=1}^{m} \kappa_i \int_0^{\zeta_i} u'(s) ds, \quad 0 < \zeta_i < 1,
\end{cases}
\tag{2}
$$

where $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, and $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$.

The existence results for the problem (1) are based on Banach's contraction principle and a fixed point theorem due to D. O'Regan [12], while the results for the problem (2) rely on the Nonlinear Alternative for contractive maps and a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values.

Nonlocal conditions were initiated by Bitsadze [1]. As remarked by Byszewski [3–5], a nonlocal condition can be more useful than a standard initial condition to describe some physical phenomena. For example, $g(x)$ may be given by $g(x) = \sum_{i=1}^{p} c_i x(t_i)$, where c_i are given constants and t_i are the interior points of the interval under consideration. For more details on initial and boundary value problems with nonlocal conditions, we refer to a survey paper by Ntouyas [11].

The paper is organized as follows. Section 2 is devoted to a lemma which plays a crucial role in the sequel. Section 3 contains the existence and uniqueness results for the problem (1), while the existence results for the problem (2) are presented in Section 4.

2. An Auxiliary Lemma

Lemma 2.1. *Let* $\delta_1 + (n-1)\delta_2 \neq \displaystyle\sum_{i=1}^{m} \kappa_i \zeta_i^n$. *For any* $y \in C([0, 1], \mathbb{R})$, *the unique solution of the boundary value problem*

$$
\begin{cases}
u^{(n)}(t) = y(t), & t \in [0, 1], \\
\\
u(0) = u_0 + g(u), \\
\\
u'(0) = 0, \quad u''(0) = 0, \ldots, u^{(n-2)}(0) = 0, \\
\delta_1 u(1) + \delta_2 u'(1) = \displaystyle\sum_{i=1}^{m} \kappa_i \int_0^{\zeta_i} u'(s) ds, \quad 0 < \zeta_i < 1,
\end{cases}
\tag{3}
$$

is given by

$$
\begin{aligned}
u(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s) ds + \Lambda t^{n-1} \left\{ \sum_{i=1}^{m} \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} y(s) ds \right. \\
&\quad \left. - \delta_1 \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s) ds - \delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} y(s) ds \right\} \\
&\quad + [1 - \delta_1 \Lambda t^{n-1}][u_0 + g(u)],
\end{aligned}
\tag{4}
$$

where

$$
\Lambda = \frac{1}{\delta_1 + (n-1)\delta_2 - \displaystyle\sum_{i=1}^{m} \kappa_i \zeta_i^{n-1}}.
\tag{5}
$$

Proof. It is well known that the solution of the differential equation in (3) can be written as

$$u(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)ds + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-2} t^{n-2} + c_{n-1} t^{n-1}, \tag{6}$$

where $c_i, i = 0, 1, \ldots n - 1$ are arbitrary real constants. Using the boundary condition $u(0) = u_0 + g(u)$, we get $c_0 = u_0 + g(u)$. Using the boundary conditions $u'(0) = u''(0) = \ldots = u^{(n-2)}(0) = 0$ in (6), we find that $c_1 = c_2 = \ldots = c_{n-2} = 0$ and applying the boundary condition: $\delta_1 u(1) + \delta_2 u'(1) = \sum_{i=1}^m \kappa_i \int_0^{\zeta_i} u'(s)ds$, we obtain

$$\begin{aligned} c_{n-1} &= \Lambda \left(\sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} y(s)ds - \delta_1 \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s)ds \right. \\ &\quad \left. -\delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} y(s)ds \right) - \delta_1 \Lambda [u_0 + g(u)], \end{aligned}$$

where Λ defined by (5). Substituting the values of $c_0, c_1, c_2, \ldots, c_{n-2}$ and c_{n-1} in (6), we get (4). $\qquad\square$

3. Existence Results - The single-valued Case

We denote by $C = C([0, 1], \mathbb{R})$ the Banach space of all continuous functions from $[0, 1] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0, 1]\}$.

In view of Lemma 2.1, we define an operator $F : C \to C$ by

$$\begin{aligned} (Fu)(t) &= \left(1 - \delta_1 \Lambda t^{n-1}\right)[u_0 + g(u)] + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, u(s))ds \\ &\quad + \Lambda t^{n-1} \left\{ \sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} f(s, u(s))ds \right. \\ &\quad \left. -\delta_1 \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s, u(s))ds - \delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f(s, u(s))ds \right\} \end{aligned} \tag{7}$$

where Λ given by (5). Observe that the problem (1) has solutions if and only if the operator equation $Fu = u$ has fixed points.

Define two operators from $C \to C$, respectively, by

$$\begin{aligned} (F_1 u)(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, u(s))ds \\ &\quad + \Lambda t^{n-1} \left\{ \sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} f(s, u(s))ds \right. \\ &\quad \left. -\delta_1 \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s, u(s))ds - \delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f(s, u(s))ds \right\}, \end{aligned} \tag{8}$$

and

$$(F_2 u)(t) = \left(1 - \delta_1 \Lambda t^{n-1}\right)[u_0 + g(u)]. \tag{9}$$

Clearly

$$(Fu)(t) = (F_1 u)(t) + (F_2 u)(t), \quad t \in [0, 1]. \tag{10}$$

For computational convenience, we set the notations:

$$p_0 := \frac{1}{n!} + |\Lambda|\left(\frac{\sum_{i=1}^{m}\kappa_i\zeta_i^n}{n!} + \frac{|\delta_1|}{n!} + \frac{|\delta_2|}{(n-1)!}\right) \tag{11}$$

and

$$k_0 := 1 + |\delta_1||\Lambda|. \tag{12}$$

Theorem 3.1. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(A_1) $|f(t, x) - f(t, y)| \le L|x - y|, \forall t \in [0, 1], L > 0, x, y \in \mathbb{R}$;

(A_2) *there exist a positive constant $\ell < k_0^{-1}$ and a continuous function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(z) \le \ell z$ and $|g(x) - g(y)| \le \phi(\|x - y\|)$ for all $x, y \in C([0, 1])$.*

(A_3) $\gamma = Lp_0 + \ell k_0 < 1.$

Then the boundary value problem (1) has a unique solution.

Proof. For $u, v \in C$ and for each $t \in [0, 1]$, from the definition of F and assumptions (A_1) and (A_2), we obtain

$$
\begin{aligned}
|(Fu)(t) - (Fv)(t)| \quad \le \quad & \left\{\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|f(s, u(s)) - f(s, v(s))|ds\right. \\
& + |\Lambda t^{n-1}|\left(\sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!}|f(s, u(s)) - f(s, v(s))|ds\right. \\
& + |\delta_1|\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}|f(s, u(s)) - f(s, v(s))|ds \\
& \left.\left. + |\delta_2|\int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}|f(s, u(s)) - f(s, v(s))|ds\right)\right\} \\
& + |1 - \delta_1\Lambda t^{n-1}||g(u) - g(v)| \\
\le \quad & L\left\{\frac{1}{n!} + |\Lambda|\left(\frac{\sum_{i=1}^m \kappa_i \zeta_i^n}{n!} + \frac{|\delta_1|}{n!} + \frac{|\delta_2|}{(n-1)!}\right)\right\}\|u - v\| \\
& + \ell[1 + |\delta_1||\Lambda|]\|x - y\|,
\end{aligned}
$$

which, in view of (11) and (12) together with (A_3), becomes

$$\|Fu - Fv\| \le \gamma\|u - v\|.$$

As $\gamma < 1$ by (A_3), F is a contraction map from the Banach space C into itself. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). $\square$

Next, we introduce a fixed point theorem due to O'Regan [12], which will be used to establish the next main result.

Lemma 3.2. *Denote by U an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$. Also assume that $F(\bar{U})$ is bounded and that $F : \bar{U} \to C$ is given by $F = F_1 + F_2$, in which $F_1 : \bar{U} \to E$ is continuous and completely continuous and $F_2 : \bar{U} \to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \le \phi(\|x - y\|)$ for all $x, y \in \bar{U}$). Then, either*

(C1) F has a fixed point $u \in \bar{U}$; or

(C2) there exist a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$, where $\bar{U}$ and ∂U, respectively, represent the closure and boundary of U.

Let

$$\Omega_r = \{u \in C([0, 1], \mathbb{R}) : \|u\| < r\},$$

and denote the maximum number by

$$M_r = \max\{|f(t, u)| : (t, u) \in [0, 1] \times [-r, r]\}.$$

Theorem 3.3. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that (A_1) and (A_2) hold. In addition we assume that*

(A_4) $g(0) = 0$;

(A_5) there exists a nonnegative function $p \in C([0, 1], \mathbb{R})$ and a nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ such that

$$|f(t, u)| \leq p(t)\psi(|u|) \text{ for any } (t, u) \in [0, 1] \times \mathbb{R};$$

(A_6) $\displaystyle\sup_{r \in (0, \infty)} \frac{r}{k_0|u_0| + p_0\psi(r)\|p\|} > \frac{1}{1 - k_0\ell}$, where p_0, k_0 are defined by (11) and (12) respectively.

Then the boundary value problem (1) has at least one solution on $[0, 1]$.

Proof. Consider the operator $F : C \to C$ defined by (10), that is,

$$(Fu)(t) = (F_1u)(t) + (F_2u)(t), \quad t \in [0, 1],$$

where the operators F_1 and F_2 are defined respectively in (8) and (9).
From (A_6) there exists a number $r_0 > 0$ such that

$$\frac{r_0}{k_0|u_0| + p_0\psi(r_0)\|p\|} > \frac{1}{1 - k_0\ell}. \tag{13}$$

We shall prove that the operators F_1 and F_2 satisfy all the conditions of Lemma 3.2.
Step 1. *The operator F_1 is continuous and completely continuous.* We first show that $F_1(\bar{\Omega}_{r_0})$ is bounded. For any $u \in \bar{\Omega}_{r_0}$ we have

$$
\begin{aligned}
\|F_1u\| &\leq \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|f(s, u(s))|ds \\
&\quad + |\Lambda t^{n-1}|\Bigg\{\sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!}|f(s, u(s))|ds \\
&\quad + |\delta_1| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}|f(s, u(s))|ds + |\delta_2| \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}|f(s, u(s))|ds\Bigg\} \\
&\leq M_{r_0}\Bigg\{\frac{1}{n!} + |\Lambda|\Bigg(\frac{\sum_{i=1}^m |\kappa_i|\zeta_i^n}{(n+1)!} + \frac{|\delta_1|}{n!} + \frac{|\delta_2|}{(n-1)!}\Bigg)\Bigg\}\|p\|.
\end{aligned}
$$

This proves that $F_1(\bar{\Omega}_{r_0})$ is uniformly bounded.

Next, for any $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, we have

$$
\begin{aligned}
&|(F_1 u)(t_2) - (F_1 u)(t_1)| \\
\leq\ & \left| \frac{1}{(n-1)!} \int_0^{t_1} [(t_2 - s)^{n-1} - (t_1 - s)^{n-1}] f(s, u(s)) ds + \int_{t_1}^{t_2} (t_2 - s)^{n-1} f(s, u(s)) ds \right| \\
& + |\Lambda||t_2^{n-1} - t_1^{n-1}| \Bigg(\sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} |f(s, u(s))| ds \\
& + |\delta_1| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} |f(s, u(s))| ds + |\delta_2| \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} |f(s, u(s))| ds \Bigg) \\
\leq\ & \frac{M_{r_0}}{n!} |2(t_2 - t_1)^n + t_1^n - t_2^n| \\
& + M_{r_0} |\Lambda||t_2^{n-1} - t_1^{n-1}| \left(\frac{\sum_{i=1}^m |\kappa_i|\zeta_i^n}{n!} + \frac{|\delta_1|}{n!} + \frac{|\delta_2|}{(n-1)!} \right) \|p\|,
\end{aligned}
$$

which is independent of u, and tends to zero as $t_2 - t_1 \to 0$. Thus, F_1 is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $F_1(\bar{\Omega}_{r_0})$ is a relatively compact set. Now, let $u_n \subset \bar{\Omega}_{r_0}$ with $\|u_n - u\| \to 0$. Then the limit $\|u_n(t) - u(t)\| \to 0$ uniformly on $[0, 1]$. From the uniform continuity of $f(t, u)$ on the compact set $[0, 1] \times [-r_0, r_0]$, it follows that $\|f(t, u_n(t)) - f(t, u(t))\| \to 0$ is uniformly on $[0, 1]$. Hence $\|F_1 u_n - F_1 u\| \to 0$ as $n \to \infty$ which proves the continuity of F_1. Hence Step 1 is completely established.

Step 2. *The operator $F_2 : \bar{\Omega}_{r_0} \to C([0, 1], \mathbb{R})$ is contractive.* This is a consequence of (A_2).

Step 3. *The set $F(\bar{\Omega}_{r_0})$ is bounded.* For any $u \in \bar{\Omega}_{r_0}$, we find by (A_2) and (A_4) that

$$\|F_2(u)\| \leq (1 + |\delta_1| \|\Lambda\|)(|u_0| + \ell r_0),$$

which, together with the boundedness of the set $F_1(\bar{\Omega}_{r_0})$ implies that the set $F(\bar{\Omega}_{r_0})$ is bounded.

Step 4. *Here, it is shown that the case (C2) in Lemma 3.2 does not occur.* On the contrary, we suppose that (C2) holds. Then, there exist $\lambda \in (0, 1)$ and $u \in \partial \Omega_{r_0}$ such that $u = \lambda F u$. So, we have $\|u\| = r_0$ and

$$
\begin{aligned}
(F u)(t) =\ & \lambda \left(1 - \delta_1 \Lambda t^{n-1} \right)[u_0 + g(u)] + \lambda \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, u(s)) ds \\
& + \lambda \Lambda t^{n-1} \Bigg\{ \sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} f(s, u(s)) ds \\
& - \delta_1 \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s, u(s)) ds - \delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f(s, u(s)) ds \Bigg\}.
\end{aligned}
$$

In view of the hypotheses $(A_4) - (A_6)$, we have

$$
r_0 \leq \psi(r_0) \left\{ \frac{1}{n!} + |\Lambda| \left(\frac{\sum_{i=1}^m \kappa_i \zeta_i^n}{(n+1)!} + \frac{|\delta_1|}{n!} + \frac{|\delta_2|}{(n-1)!} \right) \right\} \|p\| + (1 + |\delta_1| \|\Lambda\|)(|u_0| + \ell r_0),
$$

which implies that

$$r_0 \leq k_0 \ell r_0 + k_0 |u_0| + p_0 \psi(r_0) \|p\|.$$

Thus,

$$\frac{r_0}{k_0 |u_0| + p_0 \psi(r_0) \|p\|} \leq \frac{1}{1 - k_0 \ell},$$

which contradicts (13).

Thus the operators F_1 and F_2 satisfy all the conditions of Lemma 3.2. Hence, the operator F has at least one fixed point $u \in \bar{\Omega}_{r_0}$, which is the solution of the problem (1). □

Now we present some illustrative examples for our results obtained for the single-valued case.

Example 3.4. *Consider the following boundary value problem*

$$
\begin{cases}
u'''(t) = \dfrac{1}{(t+2)^2} \cdot \dfrac{|u|}{1+|u|} + 1 + \sin^2 t, \quad 0 < t < 1, \\[2mm]
u(0) = 1 + \dfrac{1}{16} u(\xi), \quad u'(0) = 0, \\[2mm]
u(1) + u'(1) = \displaystyle\sum_{i=1}^{3} \kappa_i \int_0^{\zeta_i} u'(s)ds, \quad 0 < \zeta_i < 1.
\end{cases}
\tag{14}
$$

Here $\ell = 1/16$, $\delta_1 = 1, \delta_2 = 1, \zeta_1 = 1/4, \zeta_2 = 1/2, \zeta_3 = 3/4, \kappa_1 = 1, \kappa_2 = 1/3, \kappa_3 = 2/3$ and $f(t,u) = \dfrac{1}{(t+2)^2} \cdot \dfrac{|u|}{1+|u|} + 1 + \sin^2 t$. As $|f(t,u) - f(t,v)| \le \dfrac{1}{4}|u-v|$, therefore (A_1) is satisfied with $L = \dfrac{1}{4}$. Further

$$
\gamma = L\left\{ \frac{1}{n!} + |\Lambda|\left(\frac{\sum_{i=1}^{m} \kappa_i \zeta_i^n}{n!} + \frac{|\delta_1|}{n!} + \frac{|\delta_2|}{(n-1)!} \right) \right\} + \ell[1 + |\delta_1||\Lambda|] = \frac{2311}{11424} < 1.
$$

Thus, by Theorem 3.1, problem (14) has a unique solution on $[0,1]$.

Example 3.5. *Let $\theta > 0$ and consider the following boundary value problem*

$$
\begin{cases}
u'''(t) = \theta t^2 \sin^2 x, \quad 0 < t < 1, \\[2mm]
u(0) = \dfrac{6}{7} + \ell u(\xi), \quad u'(0) = 0, \\[2mm]
u(1) + u'(1) = 64 \displaystyle\int_0^{1/4} u'(s)ds + 27 \int_0^{1/3} u'(s)ds + 8 \int_0^{1/2} u'(s)ds.
\end{cases}
\tag{15}
$$

We shall prove that the problem (15) admits at least one solution provided that $|\ell| < 1$ and $0 < \theta < \dfrac{18}{13}\left(1 - \dfrac{7}{6}|\ell|\right)^2$.

In order to show the validity of this claim, we need to verify that all conditions in Theorem 3.3 are satisfied. Note that here $f(t,u) = \theta t^2 \sin^2 u, u_0 = 6/7, g(u) = \ell u(\xi), \delta_1 = 1, \delta_2 = 1, \kappa_1 = 64, \kappa_2 = 27, \kappa_3 = 8, \zeta_1 = 1/4, \zeta_2 = 1/3, \zeta_3 = 1/2$.

The function $g(u) = \ell u(\xi)$ is contractive because $|g(x) - g(y)| < |\ell| \cdot \|x - y\|$ for any $x, y \in C([0,1])$. Moreover $g(0) = 0$. Hence the condition (A_4) is satisfied. With $p(t) = \theta t$ and $\psi(u) = u^2$, the condition (A_5) is satisfied, that is,

$$
|f(t,u)| \le |\theta t^2 \sin^2 u| \le \theta t u^2, \quad \text{for any } (t,u) \in [0,1] \times \mathbb{R}.
$$

With the given values, it is found that $p_0 = \dfrac{13}{36}$, $k_0 = \dfrac{7}{6}$, $\|p\| = \theta$ and hence we obtain the estimation:

$$
\sup_{r \in (0,\infty)} \frac{r}{k_0|u_0| + p_0\psi(r)\|p\|} = \sup_{r \in (0,\infty)} \frac{r}{1 + \frac{13\theta}{36}r^2} = \frac{1}{2}\sqrt{\frac{36}{13\theta}} > \frac{1}{1 - \frac{7}{6}|\ell|},
$$

provided $|\ell| < 1$ and $0 < \theta < \dfrac{9}{13}\left(1 - \dfrac{7}{6}|\ell|\right)^2$. This means that (A_6) is satisfied as long as both $|\ell| < 1$ and $0 < \theta < \dfrac{9}{13}\left(1 - \dfrac{7}{6}|\ell|\right)^2$ hold. Therefore, according to Theorem 3.3, we can conclude that problem (15) has at least one solution on $[0,1]$.

4. Existence Results - The Multi-valued Case

We begin this section with some preliminary concepts of multi-valued maps [6, 8].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x \in \mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x \in \mathbb{B}} \{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by *FixG*. A multivalued map $G : [0;1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Let $L^1([0,1], \mathbb{R})$ be the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

Definition 4.1. *A function $u \in C^{n-1}([0,1], \mathbb{R})$ is a solution of the problem (2) if $u(0) = u_0 + g(u)$, $u'(0) = 0$, $u''(0) = 0, \ldots, u^{(n-2)}(0) = 0$, $\delta_1 u(1) + \delta_2 u'(1) = \sum_{i=1}^{m} \kappa_i \int_0^{\zeta_i} u'(s)ds$, and there exists a function $f \in L^1([0,1], \mathbb{R})$ such that $f(t) \in F(t, u(t))$ a.e. on $[0,1]$ andi*

$$
\begin{aligned}
u(t) \quad = \quad & \left(1 - \delta_1 \Lambda t^{n-1}\right)[u_0 + g(u)] + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds \\
& + \Lambda t^{n-1} \bigg\{ \sum_{i=1}^{m} \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} f(s)ds \\
& - \delta_1 \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s)ds - \delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f(s)ds \bigg\}.
\end{aligned}
$$

4.1. The Carathéodory Case

Definition 4.2. *A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) *$t \longmapsto F(t, u)$ is measurable for each $x \in \mathbb{R}$;*

(ii) *$x \longmapsto F(t, u)$ is upper semicontinuous for almost all $t \in [0,1]$;*

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) *for each $\delta_1 > 0$, there exists $\varphi_{\delta_1} \in L^1([0,1], \mathbb{R}^+)$ such that*

$$\|F(t, u)\| = \sup\{|v| : v \in F(t, u)\} \le \varphi_{\delta_1}(t)$$

for all $\|u\| \le \delta_1$ and for a. e. $t \in [0,1]$.

For each $u \in C([0,1], \mathbb{R})$, define the set of selections of F by

$$S_{F,u} := \{v \in L^1([0,1], \mathbb{R}) : v(t) \in F(t, u(t)) \text{ for a.e. } t \in [0,1]\}.$$

We define the graph of G to be the set $Gr(G) = \{(x, y) \in X \times Y, y \in G(x)\}$ and recall two results for closed graphs and upper-semicontinuity.

Lemma 4.3. *([6, Proposition 1.2]) If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ and $\{y_n\}_{n \in \mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 4.4. *([10]) Let X be a Banach space. Let $F : [0,1] \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1], X)$ to $C([0,1], X)$. Then the operator*

$$\Theta \circ S_F : C([0,1], X) \to \mathcal{P}_{cp,c}(C([0,1], X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.

To prove our main result in this section we will use the following form of the Nonlinear Alternative for contractive maps [13, Corollary 3.8].

Theorem 4.5. *Let X be a Banach space, and D a bounded neighborhood of $0 \in X$. Let $Z_1 : X \to \mathcal{P}_{cp,c}(X)$ and $Z_2 : \bar{D} \to \mathcal{P}_{cp,c}(X)$ two multi-valued operators satisfying*

  (a) *Z_1 is contraction, and*

  (b) *Z_2 is u.s.c and compact.*

Then, if $G = Z_1 + Z_2$, either

  (i) *G has a fixed point in $\bar{D}$ or*

  (ii) *there is a point $u \in \partial D$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.*

Theorem 4.6. *Assume that*

(H_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory multivalued map;*

(H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,1], \mathbb{R}^+)$ such that*
$$\|F(t,u)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,u)\} \le p(t)\psi(\|u\|) \text{ for each } (t,u) \in [0,1] \times \mathbb{R};$$

(H_3) *there exists a constant $L_g < k_0^{-1}$ such that*
$$|g(u) - g(v)| \le L_g|u - v|, \quad \forall u, v \in \mathbb{R};$$

(H_4) *there exists a number $M > 0$ such that*

$$\frac{(1 - L_g k_0)M}{\psi(M|)p_0\|p\| + (1 + |\delta_1\|\Lambda|)|u_0|} > 1, \tag{16}$$

  where p_0 and k_0 are given by (11) and (12) respectively.

Then the boundary value problem (2) has at least one solution on $[0,1]$.

Proof. Transform the problem (2) into a fixed point problem. Consider the operator $\mathcal{N} : C([0,1], \mathbb{R}) \longrightarrow \mathcal{P}(C([0,1], \mathbb{R}))$ defined by

$$\mathcal{N}(u) = \left\{ \begin{array}{l} h \in C([0,1], \mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \left(1 - \delta_1 \Lambda t^{n-1}\right)[u_0 + g(u)] + \displaystyle\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds \\ + \Lambda t^{n-1}\left\{ \displaystyle\sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} f(s)ds \right. \\ \left. - \delta_1 \displaystyle\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s)ds - \delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f(s)ds \right\}, \end{array} \right. \end{array} \right.$$

for $f \in S_{F,u}$.

Now, we define an operator $\mathcal{A} : C([0,1], \mathbb{R}) \longrightarrow C([0,1], \mathbb{R})$ as

$$\mathcal{A}u(t) = \left(1 - \delta_1 \Lambda t^{n-1}\right)[u_0 + g(u)], \tag{17}$$

and a multi-valued operator $\mathcal{B} : C([0,1], \mathbb{R}) \longrightarrow \mathcal{P}(C([0,1], \mathbb{R}))$ by

$$\mathcal{B}(u) = \left\{ \begin{array}{l} h \in C([0,1], \mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds \\ + \Lambda t^{n-1} \left\{ \displaystyle\sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} f(s) ds \right. \\ \left. -\delta_1 \displaystyle\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s) ds \right. \\ \left. -\delta_2 \displaystyle\int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f(s) ds \right\}. \end{array} \right. \end{array} \right\} \tag{18}$$

Clearly $\mathcal{N} = \mathcal{A} + \mathcal{B}$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.5. The proof consists of a sequence of steps and claims.

Step 1: We show that $\mathcal{A}$ *is a contraction on* $C([0,1], \mathbb{R})$. Let $u, v \in C([0,1], \mathbb{R})$. Then

$$\begin{aligned} |\mathcal{A}u(t) - \mathcal{A}v(t)| &= \left| \left(1 - \delta_1 \Lambda t^{n-1}\right)[g(u) - g(v)] \right| \\ &\leq L_g k_0 |u - v|. \end{aligned}$$

Taking supremum for $t \in [0,1]$,

$$\|\mathcal{A}u - \mathcal{A}v\| \leq L_g k_0 \|u - v\|.$$

This shows that $\mathcal{A}$ is a contraction as $L_g k_0 < 1$.

Step 2: We shall show that *the operator $\mathcal{B}$ is compact and convex valued and it is completely continuous*. This will be given in several claims.

CLAIM I: $\mathcal{B}$ *maps bounded sets into bounded sets in* $C([0,1], \mathbb{R})$. To see this, let $B_r = \{u \in C([0,1], \mathbb{R}) : \|u\| \leq r\}$ be a bounded set in $C([0,1], \mathbb{R})$. Then, for each $h \in \mathcal{B}(u), u \in B_r$, there exists $f \in S_{F,u}$ such that

$$\begin{aligned} |h(t)| &\leq \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |f(s)| ds + |\Lambda t^{n-1}| \left\{ \sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^n}{(n-1)!} |f(s)| ds \right. \\ &\quad \left. + |\delta_1| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} |f(s)| ds + |\delta_2| \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} |f(s)| ds \right\} \\ &\leq \psi(\|u\|) \left\{ \frac{t^n}{n!} + |\Lambda t^{n-1}| \left(\frac{\sum_{i=1}^m |\kappa_i| \zeta_i^n}{n!} + \frac{|\delta_1|}{n!} + \frac{|\delta_2|}{(n-1)!} \right) \right\} \|p\| \end{aligned}$$

which, by taking maximum on the interval $[0,1]$ together with (11), yields

$$\|h\| \leq \psi(r) p_0 \|p\|.$$

CLAIM II: Next we show that $\mathcal{B}$ *maps bounded sets into equi-continuous sets*. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and

$u \in B_r$, where B_r is a bounded set of $C([0, 1], \mathbb{R})$. For each $h \in B(u)$, we obtain

$$
\begin{aligned}
|h(t_2) - h(t_1)| &\leq \left| \frac{1}{(n-1)!} \int_0^{t_1} [(t_2 - s)^{n-1} - (t_1 - s)^{n-1}] f(s) ds \right. \\
&\quad + \left. \int_{t_1}^{t_2} (t_2 - s)^{n-1} f(s) ds \right| \\
&\quad + |\Lambda||t_2^{n-1} - t_1^{n-1}| \left(\sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} |f(s)| ds \right. \\
&\quad + |\delta_1| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} |f(s)| ds + |\delta_2| \left. \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} |f(s)| ds \right) \\
&\leq \frac{\psi(r)}{n!} |2(t_2 - t_1)^n + t_1^n - t_2^n| \|p\| \\
&\quad + \psi(r)|\Lambda||t_2^{n-1} - t_1^{n-1}| \left(\frac{\sum_{i=1}^m |\kappa_i| \zeta_i^n}{n!} + \frac{|\delta_1|}{n!} + \frac{|\delta_2|}{(n-1)!} \right) \|p\|.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $u \in B$ as $t_2 - t_1 \to 0$. As $\mathcal{B}$ satisfies the above two assumptions, therefore it follows by the Arzelá-Ascoli theorem that the operator $\mathcal{B} : C([0, 1], \mathbb{R}) \to \mathcal{P}(C([0, 1], \mathbb{R}))$ is completely continuous.

By Lemma 4.3, $\mathcal{B}$ will be upper semi-continuous (u.s.c.) if we prove that it has a closed graph since $\mathcal{B}$ is already shown to be completely continuous.

CLAIM III ($\mathcal{B}$ *has a closed graph.*) Let $u_n \to u_*$, $h_n \in \mathcal{B}(u_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{B}(u_*)$. Associated with $h_n \in \mathcal{B}(u_n)$, there exists $f_n \in S_{F,u_n}$ such that for each $t \in [0, 1]$,

$$
\begin{aligned}
h_n(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f_n(s) ds + \Lambda t^{n-1} \left\{ \sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} f_n(s) ds \right. \\
&\quad \left. - \delta_1 \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f_n(s) ds - \delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f_n(s) ds \right\}.
\end{aligned}
$$

Thus we have to show that there exists $f_* \in S_{F,u_*}$ such that for each $t \in [0, 1]$,

$$
\begin{aligned}
h_*(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f_*(s) ds + \Lambda t^{n-1} \left\{ \sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} f_*(s) ds \right. \\
&\quad \left. - \delta_1 \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f_*(s) ds - \delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f_*(s) ds \right\}.
\end{aligned}
$$

Let us consider the continuous linear operator $\Theta : L^1([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ given by

$$
\begin{aligned}
f \mapsto \Theta(f)(t) &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds + \Lambda t^{n-1} \left\{ \sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} f(s) ds \right. \\
&\quad \left. - \delta_1 \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s) ds - \delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f(s) ds \right\}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\|h_n(t) - h_*(t)\| \;=\; & \left| \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}(f_n(s) - f_*(s))ds \right. \\
& + \Lambda t^{n-1}\left\{ \sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!}(f_n(s) - f_*(s))ds \right. \\
& - \delta_1 \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}(f_n(s) - f_*(s))ds \\
& \left. \left. - \delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f(f_n(s) - f_*(s))ds \right\} \right| \;\to\; 0,
\end{aligned}
$$

as $n \to \infty$. Thus, it follows by Lemma 4.4 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $u_n \to u_*$, therefore, we have

$$
\begin{aligned}
h_*(t) \;=\; & \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f_*(s)ds + \Lambda t^{n-1}\left\{ \sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} f_*(s)ds \right. \\
& \left. - \delta_1 \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f_*(s)ds - \delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f_*(s)ds \right\},
\end{aligned}
$$

for some $f_* \in S_{F,u_*}$. Hence $\mathcal{B}$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathcal{B}$ is compact valued.

Therefore the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.5. Thus, the conclusion of Theorem 4.5 yields either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $u \in \lambda\mathcal{A}(u) + \lambda\mathcal{B}(u)$ for $\lambda \in (0,1)$, then there exists $f \in S_{F,u}$ such that

$$
\begin{aligned}
u(t) \;=\; & \lambda\left(1 - \delta_1 \Lambda t^{n-1}\right)[u_0 + g(u)] + \lambda \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds \\
& + \lambda \Lambda t^{n-1}\left\{ \sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} f(s)ds \right. \\
& \left. - \delta_1 \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(s)ds - \delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f(s)ds \right\}.
\end{aligned} \tag{19}
$$

Consequently, we have

$$
\begin{aligned}
|u(t)| \;\leq\; & \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |f(s)|ds + |\Lambda t^{n-1}|\left\{ \sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} |f(s)|ds \right. \\
& \left. + |\delta_1| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} |f(s)|ds + |\delta_2| \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} |f(s)|ds \right\} \\
& + (1 + |\delta_1||\Lambda|)(|u_0| + L_g\|u\|) \\
\leq\; & \psi(\|u\|)\left\{ \frac{t^n}{n!} + |\Lambda t^{n-1}|\left(\frac{\sum_{i=1}^m |\kappa_i|\zeta_i^n}{n!} + \frac{|\delta_1|}{n!} + \frac{|\delta_2|}{(n-1)!} \right) \right\}\|p\| \\
& + (1 + |\delta_1||\Lambda|)(|u_0| + L_g\|u\|) \\
\leq\; & \psi(\|u\|)\left\{ \frac{1}{n!} + |\Lambda|\left(\frac{\sum_{i=1}^m |\kappa_i|\zeta_i^n}{n!} + \frac{|\delta_1|}{n!} + \frac{|\delta_2|}{(n-1)!} \right) \right\}\|p\| \\
& + (1 + |\delta_1||\Lambda|)(|u_0| + L_g\|u\|).
\end{aligned} \tag{20}
$$

If condition (ii) of Theorem 4.5 holds, then there exists $\lambda \in (0, 1)$ and $u \in \partial B_M$ such that $u = \lambda \mathcal{N}(u)$. Then, u is a solution of (19) with $\|u\| = M$ and hence the inequality (20) implies that

$$\frac{(1 - L_g k_0)M}{\psi(M)p_0\|p\| + (1 + |\delta_1\|\Lambda|)|u_0|} \leq 1$$

which contradicts (16). Hence, $\mathcal{N}$ has a fixed point in $[0, 1]$ by Theorem 4.5, and consequently the problem (2) has a solution. This completes the proof. $\quad\square$

4.2. The Lower Semi-continuous Case

This section deals with the case when F is not necessarily convex valued. The combination of the nonlinear alternative of Leray-Schauder type and the selection theorem due to Bressan and Colombo [2] for lower semi-continuous maps with decomposable values is applied to establish the existence result for this case.

Let us mention some auxiliary facts. Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0, 1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ−algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0, 1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0, 1], \mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0, 1] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 4.7. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0, 1], \mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.*

Let $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0, 1] \times \mathbb{R}) \to \mathcal{P}(L^1([0, 1], \mathbb{R}))$ associated with F as

$$\mathcal{F}(u) = \{w \in L^1([0, 1], \mathbb{R}) : w(t) \in F(t, u(t)) \text{ for a.e. } t \in [0, 1]\},$$

which is called the Nemytskii operator associated with F.

Definition 4.8. *Let $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Lemma 4.9. *([7]) Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0, 1], \mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0, 1], \mathbb{R})$ such that $g(u) \in N(u)$ for every $u \in Y$.*

Theorem 4.10. *Assume that $(H_2), (H_3), (H_4)$ and the following condition holds:*

(H_5) *$F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*

    (a) $(t, u) \longmapsto F(t, u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,

    (b) $u \longmapsto F(t, u)$ is lower semicontinuous for each $t \in [0, 1]$.

Then the boundary value problem (2) has at least one solution on $[0, 1]$.

Proof. It follows from (H_2) and (H_5) that F is of l.s.c. type. Then, by Lemma 4.9, there exists a continuous function $f : C([0, 1], \mathbb{R}) \to L^1([0, 1], \mathbb{R})$ such that $f(u) \in \mathcal{F}(u)$ for all $u \in C([0, 1], \mathbb{R})$.

Consider the problem

$$
\begin{cases}
u^{(n)}(t) = f(u(t)), \ \ 0 < t < 1, \\[2mm]
u(0) = u_0 + g(u), \\[2mm]
u'(0) = 0, \ \ u''(0) = 0, \dots, u^{(n-2)}(0) = 0, \\[2mm]
\delta_1 u(1) + \delta_2 u'(1) = \displaystyle\sum_{i=1}^{m} \kappa_i \int_0^{\zeta_i} u'(s)ds, \ \ 0 < \zeta_i < 1.
\end{cases}
\tag{21}
$$

Observe that if $u \in C^{n-1}([0,1],\mathbb{R})$ is a solution of (21), then x is a solution to the problem (2). Now, we define operators $\mathcal{A}' : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ and $\mathcal{B}' : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ by

$$
\mathcal{A}'u(t) = \left(1 - \delta_1 \Lambda t^{n-1}\right)[u_0 + g(u)],
\tag{22}
$$

$$
\mathcal{B}'u(t) =
\begin{cases}
\displaystyle\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(u(s))ds \\[4mm]
+\Lambda t^{n-1}\Bigg\{ \displaystyle\sum_{i=1}^m \kappa_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{n-1}}{(n-1)!} f(u(s))ds \\[4mm]
-\delta_1 \displaystyle\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} f(u(s))ds - \delta_2 \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} f(u(s))ds \Bigg\}.
\end{cases}
\tag{23}
$$

It is obvious that the operators $\mathcal{A}'$ and $\mathcal{B}'$ are continuous. Also the argument used in the proof of Theorem 3.1 guarantees that $\mathcal{A}'$ and $\mathcal{B}'$ satisfy all the conditions of the Nonlinear Alternative for contractive maps in the single valued setting [9] and hence the problem (21) has a solution.
$\square$

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