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Degree Distance of Tensor Product and Strong Product of Graphs

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Abstract. In this paper, we determine the degree distance of $G \times K_{r_0,r_1,\ldots,r_{n-1}}$ and $G \boxtimes K_{r_0,r_1,\ldots,r_{n-1}}$, where $\times$ and $\boxtimes$ denote the tensor product and strong product of graphs, respectively, and $K_{r_0,r_1,\ldots,r_{n-1}}$ denotes the complete multipartite graph with partite sets $V_0, V_1, \ldots, V_{n-1}$ where $|V_j| = r_j$, $0 \leq j \leq n-1$ and $n \geq 3$. Using the formulae obtained here, we have obtained the exact value of the degree distance of some classes of graphs.

1. Introduction

In this paper, all graphs considered are simple, connected and finite. Let $G = (V(G), E(G))$ be a connected graph of order n. For any $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u,v)$, is the length of the shortest (u,v)-path in G. The degree of a vertex $w \in V(G)$ is denoted by $d_G(w)$. For $S \subseteq V(G)$, $\langle S \rangle$ denotes the *subgraph of G induced by S*. For two subsets $S, T \subset V(G)$, by $d_G(S,T)$, we mean the sum of the distances, in G, from each vertex of S to every vertex of T, that is, $d_G(S,T) = \sum_{u \in S,\, v \in T} d_G(u,v)$. For $S, T \subseteq V(G)$, let $D(S,T)$, denote the sum $\sum_{x \in S,\, y \in T} d_G(x,y)[d_G(x) + d_G(y)]$. Let P_n and C_n denote the path and the cycle on n vertices, respectively. We call K_3 a triangle. Notation and definitions which are not given here can be found in [1] or [2].

A *topological index* is a numerical quantity related to a graph that is invariant under graph automorphisms. A topological index related to distance is called a "distance-based topological index". The *Wiener index* $W(G)$ is the first distance-based topological index defined as $W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v)$ with the summation runs over all pairs of vertices of G. The topological indices and graph invariants based on distances between vertices of a graph are widely used in mathematical chemistry[19]. The Wiener index is one of the most used topological indices with high correlation with many physical and chemical indices of molecular compounds. It is used in the study of paraffin boiling points [20].

There are some topological indices based on degrees such as the first and second Zagreb indices of molecular graphs. The first and second kinds of Zagreb indices have been introduced more than 30 years ago by Gutman and Trinajstić in [10] (see also [9]). The developement and uses of these indices can be found in [11] and [14]. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of a graph G are defined as $M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)] = \sum_{v \in V(G)} d_G^2(v)$ and $M_2(G) = \sum_{uv \in E(G)} [d_G(u) d_G(v)]$.

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The degree distance was introduced by Dobrynin and Kochetova [6] and Gutman [8] as a weighted version of the Wiener index. The *degree distance* of G, denoted by $DD(G)$,is defined as $DD(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)[d_G(u) + d_G(v)] = \frac{1}{2} \sum_{u,v\in V(G)} d_G(u,v)[d_G(u) + d_G(v)]$ with the summation runs over all pairs of vertices of G. The degree distance is a structure descriptor based on molecular topology, of quantitative relations between structure and activity. Its physico chemical applications range from the prediction of boiling points to the calculation of velocity of ultrasound in organic materials. In [3], it has been demonstrated that the Wiener index and the degree distance are closely mutually related for certain classes of molecular graphs. In [18], Ioan Tomescu proved one of the conjectures and disproved the other, made by Dobrynin and Kochetova [6] on the minimum and maximum values of the degree distance of a graph.

The *tensor product* of G_1 and G_2, denoted by $G_1 \times G_2$, has the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and the edge set $E(G_1 \times G_2) = \{(u,x)(v,y) : uv \in E(G_1) \text{ and } xy \in E(G_2)\}$. The *cartesian product* of the graphs G_1 and G_2, denoted by $G_1 \square G_2$, has the vertex set $V(G_1 \square G_2) = V(G_1) \times V(G_2)$ and $(u,x)(v,y)$ is an edge of $G_1 \square G_2$ if $u = v$ and $xy \in E(G_2)$ or, $uv \in E(G_1)$ and $x = y$. The *strong product* of the graphs G_1 and G_2, denoted by $G_1 \boxtimes G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ and $(u,x)(v,y)$ is an edge whenever (i) $u = v$ and $xy \in E(G_2)$ or, (ii) $uv \in E(G_1)$ and $x = y$ or, (iii) $uv \in E(G_1)$ and $xy \in E(G_2)$. In fact, $G_1 \boxtimes G_2 = G_1 \times G_2 \oplus G_1 \square G_2$, where $\oplus$ denotes the edge disjoint union of two graphs. The *wreath product* of the graphs G_1 and G_2, denoted by $G_1 \circ G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ and $(u,x)(v,y)$ is an edge whenever (i) $uv \in E(G_1)$ or (ii) $u = v$ and $xy \in E(G_2)$.

In [16], the degree distance of the graphs $G_1 \square G_2$ and $G_1 \circ G_2$, have been obtained.

In this paper, we obtain the degree distance of the tensor product $G \times K_{r_0,r_1,\ldots,r_{n-1}}$ and the strong product $G \boxtimes K_{r_0,r_1,\ldots,r_{n-1}}$, where $K_{r_0,r_1,\ldots,r_{n-1}}$ is the complete n-partite graph with $r = \sum_{j=0}^{n-1} r_j$ and $n \geq 3$. In $K_{r_0,r_1,\ldots,r_{n-1}}$, if $r_0 = r_1 = \ldots = r_{n-1} = s$, then we denote $K_{r_0,r_1,\ldots,r_{n-1}}$ by $K_{n(s)}$. Using the formulae obtained here, we have obtained the exact degree distance of some classes of graphs.

2. Degree Distance of the Tensor Product of Graphs

In this section, we compute the degree distance of $G \times K_{r_0,r_1,\ldots,r_{n-1}}$.

Let G be a simple nontrivial connected graph with $V(G) = \{u_0, u_1, \ldots, u_{m-1}\}$, $m \geq 2$. In $K_{r_0,r_1,\ldots,r_{n-1}}$, let $r = \sum_{j=0}^{n-1} r_j$ and $n \geq 3$. Let $V_0, V_1, \ldots, V_{n-1}$ be the partite sets of $K_{r_0,r_1,\ldots,r_{n-1}}$ and let $|V_j| = r_j$, $0 \leq j \leq n-1$. In the graph $G \times K_{r_0,r_1,\ldots,r_{n-1}}$, let $Z_{ij} = u_i \times V_j$, $u_i \in V(G)$ and $0 \leq j \leq n-1$. We call Z_{ij}, the $(i,j)^{th}$ *block* of $G \times K_{r_0,r_1,\ldots,r_{n-1}}$ (do not be confused with the block of a graph.) Clearly $S_i = \bigcup_{j=0}^{n-1} Z_{ij}$ is an independent set of $G \times K_{r_0,r_1,\ldots,r_{n-1}}$ and $V(G \times K_{r_0,r_1,\ldots,r_{n-1}}) = \bigcup_{i=0}^{m-1} S_i$. We call S_i a *layer* of $G \times K_{r_0,r_1,\ldots,r_{n-1}}$. Throughout the paper, we denote $K_{r_0,r_1,\ldots,r_{n-1}}$ by K and $\epsilon(K(\widehat{r_j}))$ denote the number of edges in $K \setminus V_j$.

It is clear that the degree of the vertex $(x,y) \in G \times K$ is $d_{G\times K}(x,y) = d_G(x)d_K(y)$; very often we shall use this fact for our computation.

Lemma 2.1. *Let G be a nontrivial connected graph. Let Z_{ij} and Z_{pq} be two blocks in $H = G \times K$. Then*

(a) $d_H(Z_{ij}, Z_{iq}) = \begin{cases} 2r_j(r_j - 1), & \text{if } j = q, \\ 2r_j r_q, & \text{if } j \neq q, \end{cases}$

(b) if $u_i u_p \in E(G)$,

$$d_H(Z_{ij}, Z_{pq}) = \begin{cases} r_j r_q, & \text{if } j \neq q, \\ 2r_j^2, & \text{if } j = q \text{ and } u_i u_p \text{ lies on a triangle of } G, \\ 3r_j^2, & \text{if } j = q \text{ and } u_i u_p \text{ does not lie on a triangle of } G, \end{cases}$$

(*c*) *if* $u_iu_p \notin E(G)$,

$$d_H(Z_{ij}, Z_{pq}) = \begin{cases} r_jr_qd_G(u_i, u_p), & \text{if } j \neq q, \\ r_j^2 d_G(u_i, u_p), & \text{if } j = q. \end{cases}$$

Proof. Let Z_{ij} and Z_{pq} be two blocks in $H = G \times K$.

Proof of (a).

Suppose $i = p,\ j = q$. By the nature of the graph H and $G \neq K_1$, any two vertices of Z_{ij} are at distance 2. There are $r_j(r_j - 1)$ pairs of distinct vertices in Z_{ij} and hence $d_H(Z_{ij}, Z_{ij}) = 2r_j(r_j - 1)$.

Suppose $i = p,\ j \neq q$. In H, distance between a vertex of Z_{ij} and a vertex of Z_{iq} is 2. There are r_jr_q such pairs of vertices and hence $d_H(Z_{ij}, Z_{iq}) = 2r_jr_q$.

Proof of (b). $u_iu_p \in E(G)$.

Suppose $j \neq q$. If $u_iu_p \in E(G)$, distance between a vertex of Z_{ij} and a vertex of Z_{pq} in H is 1. There are r_jr_q such pairs of vertices and hence $d_H(Z_{ij}, Z_{pq}) = r_jr_q$.

Suppose $j = q$ and u_iu_p lies on a triangle of G.

If $u_iu_p \in E(G)$ and u_iu_p lies on a triangle of G, distance between a vertex of Z_{ij} and a vertex of Z_{pj} in H is 2. There are r_j^2 such pairs of vertices and hence $d_H(Z_{ij}, Z_{pj}) = 2r_j^2$.

Suppose $j = q$ and u_iu_p does not lie on a triangle of G.

If $u_iu_p \in E(G)$ and u_iu_p does not lie on a triangle of G, distance between a vertex of Z_{ij} and a vertex of Z_{pj} in H is 3. There are r_j^2 such pairs of vertices and hence $d_H(Z_{ij}, Z_{pj}) = 3r_j^2$.

Proof of (c). $u_iu_p \notin E(G)$.

Suppose $j \neq q$.

If $u_iu_p \notin E(G)$, distance between u_i and u_p in G is $d_G(u_i, u_p)$ and hence the distance between a vertex of Z_{ij} and a vertex of Z_{pq} in H is $d_G(u_i, u_p)$. There are r_jr_q such pairs of vertice and hence $d_H(Z_{ij}, Z_{pq}) = r_jr_qd_G(u_i, u_p)$.

Suppose $j = q$.

As above, the distance between a vertex of Z_{ij} and a vertex of Z_{pj} in H is $d_G(u_i, u_p)$. There are r_j^2 such pairs of vertices and hence $d_H(Z_{ij}, Z_{pj}) = r_j^2 d_G(u_i, u_p)$.

From the definition of the tensor product, the following lemma follows.

Lemma 2.2. *Let G be a nontrivial connected graph. Let $(u_i, v_j) \in V(H)$ and let $v_j \in V_j$. Then the degree of (u_i, v_j) is $d_H((u_i, v_j)) = d_G(u_i)d_K(v_j) = d_G(u_i)(r - r_j)$.*

Theorem 2.3. *Let G be a nontrivial connected graph and let $K = K_{r_0, r_1, \ldots, r_{n-1}}$, $n \geq 3$, denote the complete n-partite graph. Let E_1(resp.E_2) denote the set of edges which lie (resp. do not lie) on a triangle of G. Then $DD(G \times K) = \Big\{8(r-1)\epsilon(G) + 2rDD(G) + rM_1(G) + rD_0(G)\Big\}\epsilon(K) - \Big\{M_1(G) + D_0(G)\Big\}\sum\limits_{j=0}^{n-1} r_j\epsilon(K(\widehat{r_j}))$, where $r = \sum\limits_{j=0}^{n-1} r_j$, $D_0(G) = \sum\limits_{u_iu_p \in E_2} \Big[d_G(u_i) + d_G(u_p)\Big]$ and $DD(G)$ and $M_1(G)$ denote the degree distance and the first Zagreb index of G, and $\epsilon(K(\widehat{r_j}))$ is the number of edges in $K - V_j$.*

Proof. Let $H = G \times K_{r_0, r_1, \ldots, r_{n-1}} = G \times K$. Then

$$\begin{aligned} DD(H) &= \frac{1}{2}\Big\{\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} D(Z_{ij}, Z_{ij}) + \sum_{i=0}^{m-1}\sum_{\substack{j\,q=0 \\ j\neq q}}^{n-1} D(Z_{ij}, Z_{iq}) + \sum_{j=0}^{n-1}\sum_{\substack{i\,p=0 \\ i\neq p}}^{m-1} D(Z_{ij}, Z_{pj}) + \sum_{\substack{i\,p=0 \\ i\neq p}}^{m-1}\sum_{\substack{j\,q=0 \\ j\neq q}}^{n-1} D(Z_{ij}, Z_{pq})\Big\} \\ &= \frac{1}{2}[A_1 + A_2 + A_3 + A_4], \qquad (1) \end{aligned}$$

where A_1-A_4 are the sums of the above terms, in order.

We shall calculate A_1 to A_4 of (1) separately.

First we compute $A_1 = \sum\limits_{i=0}^{m-1}\sum\limits_{j=0}^{n-1} D(Z_{ij}, Z_{ij})$. For this, first we calculate $\sum\limits_{j=0}^{n-1} D(Z_{ij}, Z_{ij})$. As G is connected and nontrivial, any pair of distinct vertices in Z_{ij} are at distance 2; also there are $r_j(r_j-1)$ such pairs of vertices and we have

$$\sum_{j=0}^{n-1} D(Z_{ij},Z_{ij}) = \sum_{j=0}^{n-1} 2r_j(r_j-1)\Big\{d_G(u_i)(r-r_j)+d_G(u_i)(r-r_j)\Big\},\quad \text{by Lemmas 2.1 and 2.2,}$$

$$= 4d_G(u_i)\sum_{j=0}^{n-1}\Big\{r_j^2(r-r_j)-r_j(r-r_j)\Big\} = 4d_G(u_i)\Big\{\sum_{j=0}^{n-1} r_j^2(r-r_j)-\Big(r^2-\sum_{j=0}^{n-1}r_j^2\Big)\Big\},\ \text{ as } r=\sum_{j=0}^{n-1}r_j,$$

$$= 4d_G(u_i)\Big\{\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2 r_q-\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q\Big\},\ \text{ as } r-r_j=\sum_{\substack{q=0\\ q\neq j}}^{n-1} r_q \text{ and } r^2-\sum_{j=0}^{n-1}r_j^2=\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q. \tag{2}$$

Now, by using (2), we have

$$A_1 = 8\epsilon(G)\Big\{\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2 r_q-\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q\Big\}. \tag{3}$$

Next we compute $A_2=\sum_{i=0}^{m-1}\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} D(Z_{ij},Z_{iq})$. For this, first we calculate $\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} D(Z_{ij},Z_{iq})$. As there are r_jr_q pairs of vertices with the first vertex in Z_{ij} and the second vertex in Z_{iq}, $j\neq q$ and they are at distance 2 in H, we have

$$\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} D(Z_{ij},Z_{iq}) = \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} 2r_jr_q\Big\{d_G(u_i)(r-r_j)+d_G(u_i)(r-r_q)\Big\},\ \text{ by Lemmas 2.1 and 2.2,}$$

$$= 2d_G(u_i)\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_jr_q\Big\{2r-r_j-r_q\Big\} = 4d_G(u_i)\Big\{r\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_jr_q-\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2r_q\Big\}. \tag{4}$$

Now, using (4), we get

$$A_2 = 8\epsilon(G)\Big\{r\sum_{\substack{j q=0\\ j\neq q}}^{n-1} r_jr_q-\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2r_q\Big\}. \tag{5}$$

Next we calculate $A_3=\sum_{j=0}^{n-1}\sum_{\substack{i,p=0\\ i\neq p}}^{m-1} D(Z_{ij},Z_{pj})$. For this, first we obtain $\sum_{\substack{i,p=0\\ i\neq p}}^{m-1} D(Z_{ij},Z_{pj})$.

Let $E_1=\{uv\in E(G)|\, uv \text{ is on a } K_3 \text{ of } G\}$ and $E_2=E(G)-E_1$ and hence $|E_1\cup E_2|=\epsilon(G)$.

$$\sum_{\substack{i,p=0\\ i\neq p}}^{m-1} D(Z_{ij},Z_{pj}) = \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\notin E(G)}}^{m-1} D(Z_{ij},Z_{pj}) + \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\in E(G)}}^{m-1} D(Z_{ij},Z_{pj})$$

$$= \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\notin E(G)}}^{m-1} D(Z_{ij},Z_{pj}) + \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\in E_1}}^{m-1} D(Z_{ij},Z_{pj}) + \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\in E_2}}^{m-1} D(Z_{ij},Z_{pj})$$

$$= \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\notin E(G)}}^{m-1} d_G(u_i,u_p)r_j^2\Big[d_G(u_i)(r-r_j)+d_G(u_p)(r-r_j)\Big] + \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\in E_1}}^{m-1} 2r_j^2\Big[d_G(u_i)(r-r_j)$$

$$+ d_G(u_p)(r - r_j)\Big] + \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\in E_2}}^{m-1} 3r_j^2\Big[d_G(u_i)(r - r_j) + d_G(u_p)(r - r_j)\Big], \text{ by Lemmas 2.1 and 2.2,}$$

$$= r_j^2(r - r_j)\Big\{ \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\notin E(G)}}^{m-1} d_G(u_i, u_p)\Big[d_G(u_i) + d_G(u_p)\Big] + \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\in E_1}}^{m-1} \Big(d_G(u_i, u_p) + 1\Big)\Big[d_G(u_i) + d_G(u_p)\Big]$$

$$+ \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\in E_2}}^{m-1} \Big(d_G(u_i, u_p) + 2\Big)\Big[d_G(u_i) + d_G(u_p)\Big]\Big\}, \text{ since } d_G(u_i, u_p) = 1,$$

$$= r_j^2(r - r_j)\Big\{ \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\notin E(G)}}^{m-1} d_G(u_i, u_p)[d_G(u_i) + d_G(u_p)] + \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\in E_1}}^{m-1} d_G(u_i, u_p)\Big[d_G(u_i) + d_G(u_p)]$$

$$+ \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\in E_2}}^{m-1} d_G(u_i, u_p)[d_G(u_i) + d_G(u_p)]\Big\} + r_j^2(r - r_j)\Big\{ \sum_{\substack{i\,p=0\\ i\neq p\\ u_iu_p\in E_1}}^{m-1} \Big[d_G(u_i) + d_G(u_p)\Big]$$

$$+ \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\in E_2}}^{m-1} \Big[d_G(u_i) + d_G(u_p)\Big]\Big\} + r_j^2(r - r_j) \sum_{\substack{i,p=0\\ i\neq p\\ u_iu_p\in E_2}}^{m-1} \Big[d_G(u_i) + d_G(u_p)\Big]$$

$$= \Big(2DD(G) + 2M_1(G) + 2 \sum_{u_iu_p\in E_2} \Big[d_G(u_i) + d_G(u_p)\Big]\Big)r_j^2(r - r_j). \tag{6}$$

Using (6), we get
$$A_3 = \Big\{2DD(G) + 2M_1(G) + \sum_{u_iu_p\in E_2} \Big[d_G(u_i) + d_G(u_p)\Big]\Big\} \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2 r_q. \tag{7}$$

Finally, we calculate $A_4 = \sum\limits_{\substack{i,p=0\\ i\neq p}}^{m-1} \sum\limits_{\substack{j,q=0\\ j\neq q}}^{n-1} D(Z_{ij}, Z_{pq})$. For this, first we compute $\sum\limits_{\substack{j,q=0\\ j\neq q}}^{n-1} D(Z_{ij}, Z_{pq})$. As there are r_jr_q pairs of vertices of Z_{ij} and Z_{pq} with its first vertex in Z_{ij} and the second vertex in Z_{pq} at distance $d_G(u_i, u_p)$, we have,

$$\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} D(Z_{ij}, Z_{pq}) = \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} d_G(u_i, u_p) r_j r_q\Big[d_G(u_i)(r - r_j) + d_G(u_p)(r - r_q)\Big], \text{ by Lemmas 2.1 and 2.2,}$$

$$= r d_G(u_i, u_p)\Big[d_G(u_i) + d_G(u_p)\Big] \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q - d_G(u_i, u_p)\Big[d_G(u_i) + d_G(u_p)\Big] \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2 r_q. \tag{8}$$

Using (8), we get
$$A_4 = 2DD(G)\Big\{r \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q - \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2 r_q\Big\}. \tag{9}$$

Using (3), (5), (7) and (9) in (1), we have

$$DD(H) = 4\epsilon(G)\Big((r-1)\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q\Big) + \Big(M_1(G) + \sum_{u_iu_p\in E_2}[d_G(u_i)+d_G(u_p)]\Big)\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2 r_q + rDD(G)\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q$$

$$= \Big(4(r-1)\epsilon(G) + rDD(G)\Big)\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q + \Big(M_1(G) + \sum_{u_iu_p\in E_2}[d_G(u_i)+d_G(u_p)]\Big)\Big(\frac{r}{2}\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q - \frac{1}{2}\sum_{\substack{j,q,k=0\\ j\neq q\neq k}}^{n-1} r_j r_q r_k\Big),$$

using the identity $2\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2 r_q = \Big(\sum_{j=0}^{n-1} r_j\Big)\Big(\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q\Big) - \sum_{\substack{j,q,k=0\\ j\neq q\neq k}}^{n-1} r_j r_q r_k$ and $r = \sum_{j=0}^{n-1} r_j$,

$$= \Big\{8(r-1)\epsilon(G) + 2rDD(G) + rM_1(G) + rD_0(G)\Big\}\epsilon(K) - \Big\{M_1(G) + D_0(G)\Big\}\sum_{j=0}^{n-1} r_j\epsilon(K(\widehat{r_j})),$$

where $D_0(G) = \sum_{u_iu_p\in E_2}[d_G(u_i)+d_G(u_p)]$.

We use the following Remark in the next corollary.

Remark 2.4. *The sum* $\sum_{\substack{j,q,k=0\\ j\neq q\neq k}}^{n-1} r_j r_q r_k$*, when* $r_0 = r_1 = \ldots = r_{n-1} = s$*, can be given as*

$$\begin{aligned}\sum_{\substack{j,q,k=0\\ j\neq q\neq k}}^{n-1} r_j r_q r_k &= 2\sum_{j=0}^{n-1} r_j\epsilon(K(\widehat{r_j}))\\ &= 2r_0\epsilon(K-V_0) + 2r_1\epsilon(K-V_1) + \cdots + 2r_{n-1}\epsilon(K-V_{n-1})\\ &= 2s\Big[n\epsilon(K-V_0)\Big], \quad \text{since } K-V_0 \simeq K-V_i,\ i=1,2,\ldots,n-1\\ &= n(n-1)(n-2)s^3.\end{aligned}$$

If $r_j = s$, $0 \le j \le n-1$, in Theorem 2.3, we have the following corollary, using the Remark 2.4.

Corollary 2.5. *Let* G *be a nontrivial connected graph with* $|V(G)| = m$*. Let* E_1 *denote the set of edges of* G *which lie on a triangle and* $E_2 = E(G) - E_1$*. Then* $DD(G\times K_{n(s)}) = n(n-1)s^2\Big[4\epsilon(G)(ns-1) + nsDD(G) + sM_1(G) + sD_0(G)\Big]$*, where* $n \ge 3$ *and* $D_0(G) = \sum_{u_iu_p\in E_2}[d_G(u_i)+d_G(u_p)]$*,* $DD(G)$ *and* $M_1(G)$ *denote the degree distance and the first Zagreb index of* G*, respectively.*

As $K_n \simeq K_{n(1)}$, we have the following corollary.

Corollary 2.6. *Let* G *be a nontrivial connected graph with* $|V(G)| = m$*. Let* E_1 *denote the set of edges of* G *which lie on a triangle and* $E_2 = E(G) - E_1$*. Then* $DD(G\times K_n) = n(n-1)\Big[4\epsilon(G)(n-1) + nDD(G) + M_1(G) + D_0(G)\Big]$*, where* $n \ge 3$ *and* $D_0(G) = \sum_{u_iu_p\in E_2}[d_G(u_i)+d_G(u_p)]$ *and* $DD(G)$ *and* $M_1(G)$ *denote the degree distance and the first Zagreb index of* G*, respectively.*

A graph is *chordal* if every cycle of length at least 4 has a chord, that is, an edge joining a pair of nonconsecutive vertices of a cycle of length at least 4. If G is a 2-edge connected chordal graph, then in the above notation, $E_2 = \emptyset$ and hence we have $D_0(G) = 0$; consequently we have the following corollary.

Corollary 2.7. *Let G be a 2-edge connected chordal graph. Then* $DD(G \times K) = \Big\{8(r-1)\epsilon(G) + 2rDD(G) + rM_1(G)\Big\}\epsilon(K) - M_1(G)\sum_{j=0}^{n-1} r_j\epsilon(K(\widehat{r_j}))$, *where* $DD(G)$ *and* $M_1(G)$ *denote the degree distance and the first Zagreb index of* G*, respectively.*

In particular, if $G = K_m$, $m \geq 3$, then the exact value of $DD(G \times K)$ can be given as $DD(K_m \times K) = m(m-1)\Big\{\Big[3rm + r - 4\Big]\epsilon(K) - (m-1)\sum_{j=0}^{n-1} r_j\epsilon(K(\widehat{r_j}))\Big\}$. Further, if $r_0 = r_1 = \ldots = r_{n-1} = s$, then $DD(K_m \times K_{n(s)}) = mn(m-1)(n-1)s^2\Big\{s(mn + m + n - 1) - 2\Big\}$ and if $s = 1$, then $DD(K_m \times K_n) = mn(m-1)(n-1)\Big\{mn + m + n - 3\Big\}$, where $n \geq 3$.

For a triangle free graph, in the above notation, $E_1 = \emptyset$ and hence $E_2 = E(G)$; consequently, $D_0(G) = \sum_{u_iu_p \in E(G)} [d_G(u_i) + d_G(u_p)] = M_1(G)$. Using this in Theorem 2.3 we get the following corollary.

Corollary 2.8. *Let G be a nontrivial connected triangle free graph. Then* $DD(G \times K) = \Big\{8(r-1)\epsilon(G) + 2rDD(G) + 2rM_1(G)\Big\}\epsilon(K) - 2M_1(G)\sum_{j=0}^{n-1} r_j\epsilon(K(\widehat{r_j}))$, *where* $DD(G)$ *and* $M_1(G)$ *denote the degree distance and the first Zagreb index of* G*, respectively.*

The following lemma is proved in [8].

Lemma 2.9. *Let G be a tree on m vertices. Then* $DD(G) = 4W(G) - m(m-1)$, *where* $W(G)$ *is the Wiener index of* G.

If G is a tree on m vertices, by Lemma 2.9, $DD(G \times K) = \Big\{2(m-1)[4(r-1) - rm] + 2r[4W(G) + M_1(G)]\Big\}\epsilon(K) - 2M_1(G)\sum_{j=0}^{n-1} r_j\epsilon(K(\widehat{r_j}))$, where $W(G)$ and $M_1(G)$ denote the Wiener index and the first Zagreb index of G, respectively.

If $r_j = s$, $0 \leq j \leq n-1$, in Corollary 2.8, we have

Corollary 2.10. *Let G be a nontrivial connected triangle free graph. Then* $DD(G \times K_{n(s)}) = n(n-1)s^2\Big(4\epsilon(G)(ns - 1) + nsDD(G) + 2sM_1(G)\Big)$, *where* $n \geq 3$ *and* $DD(G)$ *and* $M_1(G)$ *denote the degree distance and the first Zagreb index of* G*, respectively.*

In particular, if G is a tree on m vertices, by Lemma 2.9, $DD(G \times K_{n(s)}) = n(n-1)s^2\Big\{(m-1)\Big[4ns - nsm - 4\Big] + 2s\Big[2nW(G) + M_1(G)\Big]\Big\}$, where $W(G)$ and $M_1(G)$ denote the Wiener index and the first Zagreb index of G, respectively.

If $s = 1$ in the Corollary 2.10, we have

Corollary 2.11. *Let G be a nontrivial connected triangle free graph. Then* $DD(G \times K_n) = n(n-1)\Big[4\epsilon(G)(n-1) + nDD(G) + 2M_1(G)\Big]$, *where* $n \geq 3$ *and* $M_1(G)$ *is the first Zagreb index of* G.

In particular, if G is a tree on m vertices, by Lemma 2.9, $DD(G \times K_n) = n(n-1)\Big\{(m-1)\Big[4n - nm - 4\Big] + 2\Big[2nW(G) + M_1(G)\Big]\Big\}$, where $W(G)$ and $M_1(G)$ denote the Wiener index and the first Zagreb index of G, respectively. The following lemma is proved in [7].

Lemma 2.12. *Let G be a connected graph with m vertices and diameter two. Then* $DD(G) = 4(m-1)\epsilon(G) - M_1(G)$ *where* $M_1(G)$ *is the first Zagreb index of* G.

Using Lemma 2.12 in Theorem 2.3, we have the following corollary.

Corollary 2.13. *Let G be a connected graph with $m \geq 2$ vertices and diameter two. Let E_1 denote the set of edges which lie on a triangle and $E_2 = E(G) - E_1$. Then $DD(G \times K) = \Big\{8(rm-1)\epsilon(G) - rM_1(G) + rD_0(G)\Big\}\epsilon(K) - \Big\{M_1(G) + D_0(G)\Big\}\sum_{j=0}^{n-1} r_j\epsilon(K(\widehat{r_j}))$, where $D_0(G) = \sum_{u_iu_p \in E_2}\Big[d_G(u_i) + d_G(u_p)\Big]$ and $M_1(G)$ is the first Zagreb index of G.*

For our future reference we quote the following Lemmas.

Lemma 2.14. *([17]). Let P_n and C_n denote the path and the cycle on n vertices, respectively.*

1. *For $n \geq 2$, $W(P_n) = \frac{1}{6}n(n^2-1)$.*
2. *For $n \geq 3$, $W(C_n) = \begin{cases} \frac{n^3}{8}, & \text{if } n \text{ is even} \\ \frac{n(n^2-1)}{8}, & \text{if } n \text{ is odd.} \end{cases}$*

Lemma 2.15. *([7, 18]). Let P_n and C_n denote the path and the cycle on n vertices, respectively.*

1. *For $n \geq 2$, $DD(P_n) = \frac{1}{3}n(n-1)(2n-1)$.*
2. *For $n \geq 3$, $DD(C_n) = \begin{cases} \frac{n^3}{2}, & \text{if } n \text{ is even} \\ \frac{n(n^2-1)}{2}, & \text{if } n \text{ is odd.} \end{cases}$*

From [16], we have $DD(K_{n(s)}) = n(n-1)(ns+s-2)s^2$ and $DD(Q_k) = 2^{2k-1}k^2$, where Q_k, $k \geq 1$ is the hypercube of dimension k. It can be easily verified that $DD(K_n) = n(n-1)^2$, and $W(K_n) = \frac{1}{2}n(n-1)$ and $W(K_{n,n}) = n(3n-2)$. From [13], we have $M_1(C_n) = 4n$, $n \geq 3$, $M_1(P_n) = 4n-6$, $n > 1$, $M_1(P_1) = 0$ and $M_1(K_n) = n(n-1)^2$. Also from [13], we have $M_1(Q_k) = 2^k k^2$, $k \geq 1$ and $M_1(K_{r_0, r_1, \ldots, r_{n-1}}) = \Big(\sum_{j=0}^{n-1} r_j\Big)^3 + \Big(\sum_{j=0}^{n-1} r_j^3\Big) - 2\Big(\sum_{j=0}^{n-1} r_j\Big)\Big(\sum_{j=0}^{n-1} r_j^2\Big)$. Using Theorem 2.3, Lemmas 2.14 and 2.15, we obtain the exact degree distance of the following graphs.

1. For $m \geq 2$, $n \geq 3$, $DD(P_m \times K_{n(s)}) = \dfrac{n(n-1)s^2}{3}\Big\{(m-1)\Big[12(ns-1) + nsm(2m-1)\Big] + 12s(2m-3)\Big\}$.
2. For $m \geq 2$, $n \geq 3$, $DD(P_m \times K_n) = \dfrac{n(n-1)}{3}\Big\{2nm^3 - 3nm^2 + 13mn + 12m - 12n - 24\Big\}$.
3. For $m \geq 3$, $n \geq 3$,
$$DD(C_m \times K_{n(s)}) = \begin{cases} \dfrac{mn(n-1)s^2}{2}\Big[nsm^2 + 8ns + 16s - 8\Big], & \text{if } m \text{ is even,} \\ \dfrac{mn(n-1)s^2}{2}\Big[nsm^2 + 7ns + 16s - 8\Big], & \text{if } m \text{ is odd.} \end{cases}$$
4. For $m \geq 3$, $n \geq 3$,
$$DD(C_m \times K_n) = \begin{cases} \dfrac{mn(n-1)}{2}\Big[nm^2 + 8n + 8\Big], & \text{if } m \text{ is even,} \\ \dfrac{mn(n-1)}{2}\Big[nm^2 + 7n + 8\Big], & \text{if } m \text{ is odd.} \end{cases}$$
5. For $m \geq 2$, $n \geq 3$, $DD(K_m \times K_{n(s)}) = mn(m-1)(n-1)s^2\Big[s(mn+m+n-1) - 2\Big]$.
6. For $m \geq 2$, $n \geq 3$, $DD(K_m \times K_n) = mn(m-1)(n-1)\Big[mn + m + n - 3\Big]$.
7. For $m \geq 1$, $n \geq 3$, $DD(K_{m,m} \times K_{n(s)}) = 2n(n-1)m^2s^2\Big[3mns + ms - 2\Big]$.
8. For $m \geq 1$, $n \geq 3$, $DD(K_{m,m} \times K_n) = 2n(n-1)m^2\Big[3mn + m - 2\Big]$.
9. For $m \geq 1$, $n \geq 3$, $DD(Q_m \times K_{n(s)}) = 2^m mn(n-1)s^2\Big[2(ns-1) + 2^{m-1}smn + 2ms\Big]$.
10. For $m \geq 1$, $n \geq 3$, $DD(Q_m \times K_n) = 2^{m+1}mn(n-1)\Big[m + n - 1 + 2^{m-2}mn\Big]$.

3. Degree Distance of the Strong Product of Graphs

In this section, we compute the degree distance of $G \boxtimes K_{r_0,r_1,\ldots,r_{n-1}}$.

Let $V(G) = \{u_0, u_1, \ldots, u_{m-1}\}$, $m \geq 2$. $K_{r_0,r_1,\ldots,r_{n-1}}$, V_j, Z_{ij}, $\epsilon(K(\widehat{r_j}))$ are as defined in the Section 2.

The proof of the following lemmas follow easily from the properties and structure of $G \boxtimes K_{r_0,r_1,\ldots,r_{n-1}}$ and hence we give them without proof.

Lemma 3.1. *Let G be a nontrivial connected graph. Let Z_{ij} be the $(i, j)^{th}$ block in $H = G \boxtimes K$. Then the degree of a vertex (u_i, v_j) in Z_{ij} in H is*

$$d_H((u_i, v_j)) = d_G(u_i) + (r - r_j) + (r - r_j)d_G(u_i).$$

Lemma 3.2. *Let G be a nontrivial connected graph. Let $H = G \boxtimes K$. Let Z_{ij} and Z_{pq} be as defined above. Then*

(a) $d_H(Z_{ij}, Z_{iq}) = \begin{cases} 2r_j(r_j - 1), & \text{if } j = q, \\ r_j r_q, & \text{if } j \neq q, \end{cases}$

(b) if $u_i u_p \in E(G)$,

$d_H(Z_{ij}, Z_{pq}) = \begin{cases} (2r_j - 1)r_j, & \text{if } j = q, \\ r_j r_q, & \text{if } j \neq q, \end{cases}$

(c) if $u_i u_p \notin E(G)$,

$d_H(Z_{ij}, Z_{pq}) = \begin{cases} r_j^2 d_G(u_i, u_p), & \text{if } j = q, \\ r_j r_q d_G(u_i, u_p), & \text{if } j \neq q. \end{cases}$

Proof. Let Z_{ij} and Z_{pq} be two blocks in $H = G \boxtimes K$.

Proof of (a).

Suppose $i = p$, $j = q$. By the nature of the graph H, any two vertices of Z_{ij} are at distance 2. There are $r_j(r_j - 1)$ pairs of distinct vertices in Z_{ij}. Hence $d_H(Z_{ij}, Z_{ij}) = 2r_j(r_j - 1)$.

Suppose $i = p$, $j \neq q$. In H, distance between a vertex of Z_{ij} and a vertex of Z_{iq} is 1. There are $r_j r_q$ such pairs of vertices. Hence $d_H(Z_{ij}, Z_{iq}) = r_j r_q$.

Proof of (b). $u_i u_p \in E(G)$.

Suppose $j = q$. If $u_i u_p \in E(G)$, distance in H, between a vertex of Z_{ij} and its corresponding vertex in Z_{pj} in H is 1 and for the rest of the $(r_j - 1)$ vertices of Z_{pj} in H is 2. Therefore the sum of the distances from a vertex of Z_{ij} to every vertex of Z_{pj} in H is $2(r_j - 1) + 1 = 2r_j - 1$. There are r_j vertices in Z_{ij}. Hence $d_H(Z_{ij}, Z_{pj}) = (2r_j - 1)r_j$.

Suppose $j \neq q$. If $u_i u_p \in E(G)$, distance between a vertex of Z_{ij} and a vertex of Z_{pq} in H is 1. There are $r_j r_q$ such pairs of vertices and hence $d_H(Z_{ij}, Z_{pq}) = r_j r_q$.

Proof of (c). $u_i u_p \notin E(G)$.

Suppose $j = q$. As $u_i u_p \notin E(G)$, the distance between a vertex of Z_{ij} and a vertex of Z_{pj} in H is $d_G(u_i, u_p) \geq 2$. There are r_j^2 such pairs of vertices and hence $d_H(Z_{ij}, Z_{pj}) = r_j^2 d_G(u_i, u_p)$.

Suppose $j \neq q$. If $u_i u_p \notin E(G)$, distance between u_i and u_p in G is $d_G(u_i, u_p)$ and hence the distance between a vertex of Z_{ij} and a vertex of Z_{pq} in H is $d_G(u_i, u_p)$. There are $r_j r_q$ such pairs of vertices and hence $d_H(Z_{ij}, Z_{pq}) = r_j r_q d_G(u_i, u_p)$.

Theorem 3.3. *Let G be a nontrivial connected graph with $|V(G)| = m$ and let $K_{r_0,r_1,\ldots,r_{n-1}}$, $n \geq 3$, denote the complete n-partite graph. Then $DD(G \boxtimes K_{r_0,r_1,\ldots,r_{n-1}}) = \Big\{4\epsilon(G) + DD(G) + M_1(G)\Big\}r^2 - \Big\{4\epsilon(G) + M_1(G)\Big\}r + \Big\{8(r-2)\epsilon(G) + m(3r - 4) + (r - 4)M_1(G) + 2rDD(G) + 4rW(G)\Big\}\epsilon(K) - \Big\{m + 4\epsilon(G) + M_1(G)\Big\} \sum_{j=0}^{n-1} r_j \epsilon(K(\widehat{r_j}))$, where $r = \sum_{j=0}^{n-1} r_j$ and $DD(G)$, $W(G)$ and $M_1(G)$ are the degree distance, the Wiener index and the first Zagreb index of G, respectively.*

Proof. Let $K = K_{r_0,r_1,\ldots,r_{n-1}}$ and let $H = G \boxtimes K$.

$$DD(H) = \frac{1}{2}\Big\{\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} D(Z_{ij}, Z_{ij}) + \sum_{i=0}^{m-1}\sum_{\substack{j,q=0 \\ j\neq q}}^{n-1} D(Z_{ij}, Z_{iq}) + \sum_{j=0}^{n-1}\sum_{\substack{i,p=0 \\ i\neq p}}^{m-1} D(Z_{ij}, Z_{pj}) + \sum_{\substack{i,p=0 \\ i\neq p}}^{m-1}\sum_{\substack{j,q=0 \\ j\neq q}}^{n-1} D(Z_{ij}, Z_{pq})\Big\}$$

$$= \frac{1}{2}[A_1 + A_2 + A_3 + A_4], \tag{10}$$

where A_1-A_4 are the sums of the above terms, in order.

We shall calculate A_1 to A_4 of (10) separately.

First we calculate $A_1 = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} D(Z_{ij}, Z_{ij})$. For this, first we compute $\sum_{j=0}^{n-1} D(Z_{ij}, Z_{ij})$. Any pair of distinct vertices in Z_{ij} are at distance 2 and we can find $r_j(r_j - 1)$ such pairs of vertices. Consequently, we have

$$\sum_{j=0}^{n-1} D(Z_{ij}, Z_{ij}) = \sum_{j=0}^{n-1} 2r_j(r_j - 1)\Big\{2\Big[d_G(u_i) + (r - r_j) + d_G(u_i)(r - r_j)\Big]\Big\}, \text{ by Lemmas 3.1 and 3.2,}$$

$$= 4\Big\{d_G(u_i)\sum_{j=0}^{n-1} r_j(r_j - 1) + \Big(1 + d_G(u_i)\Big)\sum_{j=0}^{n-1}\Big[r_j^2(r - r_j) - r_j(r - r_j)\Big]\Big\}$$

$$= 4d_G(u_i)\sum_{j=0}^{n-1} r_j(r_j - 1) + 4\Big(1 + d_G(u_i)\Big)\Big\{\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2 r_q - \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q\Big\}, \text{ as } r - r_j = \sum_{\substack{q=0\\ q\neq j}}^{n-1} r_q. \tag{11}$$

Using (11), we get

$$A_1 = 8\epsilon(G)\Big(r(r-1) - \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q\Big) + 4\Big(m + 2\epsilon(G)\Big)\Big[\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2 r_q - \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q\Big],$$

$$\text{since } \sum_{j=0}^{n-1} r_j = r \text{ and } \sum_{j=0}^{n-1} r_j^2 = r^2 - \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q$$

$$= 8\epsilon(G)r(r-1) - \Big(16\epsilon(G) + 4m\Big)\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q + 4\Big(m + 2\epsilon(G)\Big)\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2 r_q. \tag{12}$$

Next we calculate $A_2 = \sum_{i=0}^{m-1}\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} D(Z_{ij}, Z_{iq})$. For this, first we compute $\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} D(Z_{ij}, Z_{iq})$. As there are $r_j r_q$ pairs of vertices with the first vertex in Z_{ij} and the second vertex in Z_{iq} and they are at distance 1 in H, we have

$$\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} D(Z_{ij}, Z_{iq}) = \sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j r_q\Big\{\Big[d(u_i) + (r - r_j) + d(u_i)(r - r_j)\Big] + \Big[d(u_i) + (r - r_q) + d(u_i)(r - r_q)\Big]\Big\},$$

$$\text{since } \Big\langle Z_{ij} \cup Z_{iq}\Big\rangle \text{ is a complete bipartite graph}$$

$$= 2d(u_i)\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j r_q + 2r\Big(d(u_i) + 1\Big)\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j r_q - 2\Big(d(u_i) + 1\Big)\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j^2 r_q$$

$$= 2\Big((r+1)d(u_i) + r\Big)\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j r_q - 2\Big(d(u_i) + 1\Big)\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j^2 r_q. \tag{13}$$

Using (13), we get

$$A_2 = \Big(4(r+1)\epsilon(G) + 2rm\Big)\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j r_q - 2\Big(2\epsilon(G) + m\Big)\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j^2 r_q. \tag{14}$$

Next we calculate $A_3 = \sum_{j=0}^{n-1}\sum_{\substack{i\,p=0\\ i\neq p}}^{m-1} D(Z_{ij}, Z_{pj})$. For this, initially we compute $\sum_{\substack{i\,p=0\\ i\neq p}}^{m-1} D(Z_{ij}, Z_{pj})$. Since the sum of the distances in H from each vertex of Z_{ij} to every vertex of Z_{pj} is $(2r_j - 1)r_j$, if $u_iu_p \in E(G)$ and the sum of the distances in H from each vertex of Z_{ij} to every vertex of Z_{pj} is $r_j^2 d_G(u_i, u_p)$, if $u_iu_p \notin E(G)$, we have

$$\begin{aligned}
\sum_{\substack{i\,p=0\\ i\neq p}}^{m-1} D(Z_{ij}, Z_{pj}) &= \sum_{\substack{i\,p=0\\ i\neq p\\ u_iu_p\in E(G)}}^{m-1} D(Z_{ij}, Z_{pj}) + \sum_{\substack{i\,p=0\\ i\neq p\\ u_iu_p\notin E(G)}}^{m-1} D(Z_{ij}, Z_{pj})\\
&= \sum_{\substack{i\,p=0\\ i\neq p\\ u_iu_p\in E(G)}}^{m-1} (2r_j - 1)r_j\Big\{\Big[d_G(u_i) + (r - r_j) + d_G(u_i)(r - r_j)\Big] + \Big[d_G(u_p) + (r - r_j) + d_G(u_p)(r - r_j)\Big]\Big\}\\
&+ \sum_{\substack{i\,p=0\\ i\neq p\\ u_iu_p\notin E(G)}}^{m-1} r_j^2 d_G(u_i, u_p)\Big\{\Big[d_G(u_i) + (r - r_j) + d_G(u_i)(r - r_j)\Big] + \Big[d_G(u_p) + (r - r_j) + d_G(u_p)(r - r_j)\Big]\Big\},\\
&\quad \text{by Lemmas 3.1 and 3.2,}\\
&= \sum_{\substack{i\,p=0\\ i\neq p\\ u_iu_p\in E(G)}}^{m-1} \Big(\Big[1 + d_G(u_i, u_p)\Big]r_j - 1\Big)r_j\Big\{\Big[d_G(u_i) + d_G(u_p)\Big] + 2(r - r_j) + \Big[d_G(u_i) + d_G(u_p)\Big](r - r_j)\Big\}\\
&\quad + \sum_{\substack{i\,p=0\\ i\neq p\\ u_iu_p\notin E(G)}}^{m-1} r_j^2 d_G(u_i, u_p)\Big\{\Big[d_G(u_i) + d_G(u_p)\Big] + 2(r - r_j) + \Big[d_G(u_i) + d_G(u_p)\Big](r - r_j)\Big\},\\
&\quad \text{since } 2 = d_G(u_i, u_p) + 1, \text{ when } u_iu_p \in E(G)\\
&= 2M_1(G)\Big(r_j^2 - r_j\Big)\Big[1 + (r - r_j)\Big] + \Big(r_j^2 + r_j^2(r - r_j)\Big)\Big(\sum_{\substack{i\,p=0\\ i\neq p\\ u_iu_p\in E(G)}}^{m-1} d_G(u_i, u_p)\Big[d_G(u_i) + d_G(u_p)\Big]\\
&\quad + \sum_{\substack{i\,p=0\\ i\neq p\\ u_iu_p\notin E(G)}}^{m-1} d_G(u_i, u_p)\Big[d_G(u_i) + d_G(u_p)\Big]\Big) + 2r_j^2(r - r_j)\Big(\sum_{\substack{i\,p=0\\ i\neq p\\ u_iu_p\in E(G)}}^{m-1} d_G(u_i, u_p) + \sum_{\substack{i\,p=0\\ i\neq p\\ u_iu_p\notin E(G)}}^{m-1} d_G(u_i, u_p)\Big)\\
&\quad + 4\Big[r_j^2(r - r_j) - r_j(r - r_j)\Big]\epsilon(G), \ \text{ since } M_1(G) = \sum_{u_iu_p\in E(G)}\Big[d_G(u_i) + d_G(u_p)\Big],\\
&= 2\Big[r_j^2 - r_j + r_j^2(r - r_j) - r_j(r - r_j)\Big]M_1(G) + 2\Big[r_j^2 + r_j^2(r - r_j)\Big]DD(G)\\
&\quad + 4r_j^2(r - r_j)W(G) + 4\Big[r_j^2(r - r_j) - r_j(r - r_j)\Big]\epsilon(G). \qquad (15)
\end{aligned}$$

Using (15), we get

$$A_3 = 2\Big(\sum_{j=0}^{n-1} r_j^2 - r + \sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j^2 r_q - \sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j r_q\Big)M_1(G) + 2\Big(\sum_{j=0}^{n-1} r_j^2 + \sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j^2 r_q\Big)DD(G) + 4\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j^2 r_q W(G)$$

$$+4\Big(\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j^2 r_q - \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q\Big)\epsilon(G), \quad \text{as } r=\sum_{j=0}^{n-1} r_j \text{ and } r-r_j=\sum_{\substack{q=0\\ j\neq q}}^{n-1} r_q$$

$$=2r^2\Big(DD(G)+M_1(G)\Big)-2rM_1(G)+2\Big(M_1(G)+DD(G)+2W(G)+2\epsilon(G)\Big)\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j^2 r_q$$

$$-2\Big(DD(G)+2M_1(G)+2\epsilon(G)\Big)\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j r_q \quad \text{as} \sum_{j=0}^{n-1} r_j^2 = r^2 - \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q. \tag{16}$$

Next we calculate $A_4=\sum_{\substack{i\,p=0\\ i\neq p}}^{m-1}\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} D(Z_{ij},Z_{pq})$. For this, initially we compute $\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} D(Z_{ij},Z_{pq})$. Since the sum of the distances in H from each vertex of Z_{ij} to every vertex of Z_{pq} is $r_j r_q d_G(u_i,u_p)$, we have

$$\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} D(Z_{ij},Z_{pq}) = \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q d_G(u_i,u_p)\Big\{\Big[d_G(u_i)+(r-r_j)+(r-r_j)d_G(u_i)\Big]+\Big[d_G(u_p)+(r-r_q)$$

$$+(r-r_q)d_G(u_p)\Big]\Big\}, \qquad \text{by Lemmas 3.1 and 3.2,}$$

$$=(1+r)d_G(u_i,u_p)\Big[d_G(u_i)+d_G(u_p)\Big]\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j r_q + 2r d_G(u_i,u_p)\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j r_q$$

$$-2d_G(u_i,u_p)\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j^2 r_q - d_G(u_i,u_p)\Big[d_G(u_i)+d_G(u_p)\Big]\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j^2 r_q. \tag{17}$$

Using (17), we get

$$A_4=\Big\{(1+r)\sum_{\substack{i,p=0\\ i\neq p}}^{m-1} d_G(u_i,u_p)\Big[d_G(u_i)+d_G(u_p)\Big]+2r\sum_{\substack{i\,p=0\\ i\neq p}}^{m-1} d_G(u_i,u_p)\Big\}\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j r_q - \Big\{2\sum_{\substack{i\,p=0\\ i\neq p}}^{m-1} d_G(u_i,u_p)$$

$$+\sum_{\substack{i,p=0\\ i\neq p}}^{m-1} d_G(u_i,u_p)\Big[d_G(u_i)+d_G(u_p)\Big]\Big\}\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j^2 r_q$$

$$=\Big(2(1+r)DD(G)+4rW(G)\Big)\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j r_q - \Big(4W(G)+2DD(G)\Big)\sum_{\substack{j\,q=0\\ j\neq q}}^{n-1} r_j^2 r_q. \tag{18}$$

Using (12), (14), (16) and (18) in (10), we have

$$DD(H)=\Big\{4\epsilon(G)+DD(G)+M_1(G)\Big\}r^2-\Big\{4\epsilon(G)+M_1(G)\Big\}r+\Big\{2(r-4)\epsilon(G)+m(r-2)-2M_1(G)+rDD(G)$$

$$+2rW(G)\Big\}\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q + \Big\{m+4\epsilon(G)+M_1(G)\Big\}\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2 r_q$$

$$=\Big\{4\epsilon(G)+DD(G)+M_1(G)\Big\}r^2-\Big\{4\epsilon(G)+M_1(G)\Big\}r+\Big\{2(r-4)\epsilon(G)+m(r-2)-2M_1(G)+rDD(G)$$

$$+ 2rW(G)\Big\} \sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q + \Big\{m + 4\epsilon(G) + M_1(G)\Big\}\Big(\frac{r}{2}\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q - \frac{1}{2}\sum_{\substack{j,q,k=0\\ j\neq q\neq k}}^{n-1} r_j r_q r_k\Big),$$

using the identity $2\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j^2 r_q = \Big(\sum_{j=0}^{n-1} r_j\Big)\Big(\sum_{\substack{j,q=0\\ j\neq q}}^{n-1} r_j r_q\Big) - \sum_{\substack{j,q,k=0\\ j\neq q\neq k}}^{n-1} r_j r_q r_k$ and $r = \sum_{j=0}^{n-1} r_j$,

$$= \Big\{4\epsilon(G) + DD(G) + M_1(G)\Big\}r^2 - \Big\{4\epsilon(G) + M_1(G)\Big\}r + \Big\{8(r-2)\epsilon(G) + m(3r-4)$$

$$+ (r-4)M_1(G) + 2rDD(G) + 4rW(G)\Big\}\epsilon(K) - \Big\{m + 4\epsilon(G) + M_1(G)\Big\}\sum_{j=0}^{n-1} r_j \epsilon(K(\widehat{r_j})),$$

where m denote the number of vertices of G.

If $r_j = s$, $0 \le j \le n-1$, in Theorem 3.3, we have the following corollary.

Corollary 3.4. *Let G be a nontrivial connected graph with $|V(G)| = m$. Then $DD(G \boxtimes K_{n(s)}) = \Big(4\epsilon(G) + DD(G) + M_1(G)\Big)n^2s^2 - \Big(4\epsilon(G) + M_1(G)\Big)ns + \Big[2\Big(ns+2s-4\Big)\epsilon(G) + nsDD(G) + 2nsW(G) + (s-2)M_1(G) + m\Big(ns+s-2\Big)\Big]n(n-1)s^2$, where $n \ge 3$ and $DD(G)$, $W(G)$ and $M_1(G)$ are the degree distance, the Wiener index and the first Zagreb index of G, respectively.*

In the above corollary, if $s = 1$, then we have the following corollary.

Corollary 3.5. *Let G be a be a nontrivial connected graph with $|V(G)| = m$. Then $DD(G \boxtimes K_n) = n^3DD(G) + 2n^2(n-1)\epsilon(G) + 2n^2(n-1)W(G) + mn(n-1)^2$, where $n \ge 3$ and $DD(G)$ and $W(G)$ are the degree distance and the Wiener index of G, respectively.*

From [4], we have $W(Q_k) = k4^{k-1}$, $k \ge 1$. Using Theorem 3.3, Lemmas 2.14 and 2.15, we obtain the exact degree distance of the following graphs.

1. For $m \ge 2$, $n \ge 3$, $DD(P_m \boxtimes K_n) = \frac{m(m-1)n^2}{3}\Big\{3mn - m - 1\Big\} + n(n-1)\Big\{3mn - 2n - m\Big\}$.
2. For $m \ge 3$, $n \ge 3$,
$$DD(C_m \boxtimes K_n) = \begin{cases} \frac{mn(3n-1)}{4}\Big[n(m^2+4) - 4\Big], & \text{if } m \text{ is even,}\\ \frac{mn(3n-1)}{4}\Big[n(m^2+3) - 4\Big], & \text{if } m \text{ is odd.}\end{cases}$$
3. For $m \ge 1$, $n \ge 3$, $DD(K_{m,m} \boxtimes K_n) = 2(3m-2)m^2n^3 + 2mn(n-1)\Big[4mn - n - 1\Big]$.
4. For $m \ge 1$, $n \ge 3$, $DD(Q_m \boxtimes K_n) = 2^{2m-1}n^2\Big[nm^2 + m(n-1)\Big] + 2^m n(n-1)\Big[mn + n - 1\Big]$.

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