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Topological Indices of the Bipartite Kneser Graph $H_{n,k}$

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Abstract. In this paper we use transitivity property of the automorphism group of the bipartite Kneser graph to calculate its Wiener, Szeged and PI indices.

1. Introduction

In this section we will use some of definitions and theorems in [1] and [2] to calculate the Wiener, the Szeged and PI-index of graphs.

Definition 1 Let G be a group which acts on a set X. Let us denote the action of $\sigma \in G$ on $x \in X$ by x^σ. Then G is said to act transitively on X if for every $x, y \in X$ there is $\sigma \in G$ such that $x^\sigma = y$.

Definition 2 Let $G = (V, E)$ be a graph. An automorphism σ of G is a one-to-one mapping from V to V which preserves adjacency, i. e. $e = uv$ is an edge of G if and only if $e^\sigma := u^\sigma v^\sigma$ is also an edge of G. The set of all the automorphisms of the graph G is a group under the usual composition of mappings. This group is denoted by $\mathbb{A}ut(G)$ and is a subgroup of the symmetric group on X.

From definition 2 it is clear that $\mathbb{A}ut(G)$ acts on the set V of vertices of G. This action induces an action on the set E of edges of G. In fact if $e = uv$ is an edge of G and $\sigma \in \mathbb{A}ut(G)$ then $e^\sigma = u^\sigma v^\sigma$ is an edge of G and this is a well-defined action $\mathbb{A}ut(G)$ on E.

Definition 3 Let $G = (V, E)$ be a graph. G is called vertex-transitive if $\mathbb{A}ut(G)$ acts transitively on the set X of vertices of G. If $\mathbb{A}ut(G)$ acts transitively on the set E of edges of G,then G called an edge-transitive graph.

The proofs of the following theorems can be found in [1] and here we state them without proofs.

Theorem 1 Let $G = (V, E)$ be a simple vertex-transitive graph and let $v \in V$ be a fixed vertex of G. Then

$$W(G) = (1/2)\, |V|\, d(v),$$

where

$$d(v) = \sum_{x \in V} d(v, x).$$

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Theorem 2 Let $G = (V, E)$ be a simple edge-transitive graph and let $e = uv$ be a fixed edge of G. Then the Szeged index of G is as follows:

$$Sz(G) = |E| n_u(e|G) n_v(e|G).$$

Theorem 3 Let $G = (V, E)$ be a simple edge-transitive graph and let $e = uv$ be a fixed edge of G. Then the PI-index of G is as follows:

$$PI(G) = |E|\Big(n_{eu}(e|G) + n_{ev}(e|G)\Big).$$

2. Computing the Wiener, the Szeged and PI-index of the Bipartite Kneser Graph

Definition 4 For a positive integer $k \geq 2$, let X be any set of cardinality n and V be the set of all k-subsets and $(n-k)$-subsets of X which are denoted by X_k and X_{n-k} respectively. The bipartite Kneser graph $H_{n,k}$ has V as its vertex set, and vertices A, B are connected if and only if $A \subset B$ or $B \subset A$. If $n = 2k$ it is obvious that we don't have any edges, and $H_{n,k}$ would be the null graph hence we assume $n \geq 2k + 1$.

From the above fact we can show vertex and edge transitivity of the bipartite Kneser graph.
The complete bipartite graph on n vertices is the bipartite Kneser graph $H_{n,1}$. The bipartite Kneser graph $H_{2n-1,n-1}$ is known as the double odd graph $2O_n$.

Therefore $H_{n,k}$ has $2\binom{n}{k}$ vertices, it is regular of degree $\binom{n-k}{k}$. The number of edges of $H_{n,k}$ is $\binom{n-k}{k}\binom{n}{k}$. If σ is a permutation of Ω and $A \subseteq \Omega$ then A^σ is defined by: $A^\sigma = \{a^\sigma | a \in A\}$ which is again a subset of Ω of cardinality $|A|$. Therefore each permutation of Ω induces a permutation on the set of vertices of $H_{n,k}$. If AB is an edge of $H_{n,k}$ then A and B are subset of Ω with cardinality k and $n-k$ respectively, where $A \subset B$ and for any permutation σ of Ω we have $A^\sigma \subset B^\sigma$ if and only if $A \subset B$, which proves that σ is an element of $\mathbb{A}ut(H_{n,k})$. Therefore we have proved the following theorem:

Theorem 4 The automorphism group of the bipartite Kneser graph $H_{n,k}$ contains a subgroup isomorphic to the symmetric group on n letters.

Lemma 1 The bipartite Kneser graph is both vertex and edge transitive.

Proof. Let Ω be a set of size n. Without loss of generality we may assume $\Omega = \{1, 2, ..., n\}$. Let the bipartite Kneser graph be defined on Ω. Consider two distinct vertices A and B of $H_{n,k}$. We may assume $A = \{1, 2, ..., k\}(or\, \{1, 2, ..., n-k\}), B = \{1', 2', ..., k'\}(or\, \{1', 2', ..., (n-k)'\})$. Then we set $\Omega - A = \{k+1, ..., n\}(or\, \{n-k+1, ..., n\}$ and $\Omega - B = \{(k+1)', ..., n'\}(or\, \{(n-k+1)', ..., n'\}$ and both are subsets of Ω. Then $\pi : \Omega \to \Omega$ defined by $i \to i'$ is an element of the symmetric group S_n which induces an element of $\mathbb{A}ut(H_{n,k})$ and $A^\pi = B$. This proves that $H_{n,k}$ is vertex-transitive. Now assume AB and CD are distinct edges of $H_{n,k}$. To prove edge-transitivity of $H_{n,k}$ it is enough to show that there is a permutation π on Ω such that $A^\pi = C$ and $B^\pi = D$. Without loss of generality we may assume that $A = \{1, 2, ..., k-1, k\}$, $B = \{1, 2, ..., n-k\}$, $C = \{1', 2', ..., k'\}$, $D = \{1', 2', ..., , (n-k)'\}$. Then we set $\Omega - (A \cup B) = \{n-k+1, ..., n\}$ and $\Omega - (C \cup D) = \{(n-k+1)', ..., n'\}$ and both are subsets Ω. Now the permutation $\pi : \Omega \to \Omega$ defined by $i \to i'$ has the required property and the lemma is proved. □

Since in the case of $n = 2k+1$, $H_{n,k}$ is the double odd graph and in [13] we calculated the Wiener, Szeged and PI indices of this graph, therefore here we will assume $n \geq 2k + 2$.

Lemma 2 For a positive integer $k \geq 2$, let $n \geq 3k$, then for any two vertices like u and v in $H_{n,k}$ we have: $d(u, v) \leq 3$

Proof. Let u, v be two distinct vertices in $H_{n,k}$. We consider two cases:

(1) $u \subset v$ or $v \subset u$

In this case we have $d(u, v) = 1$.

(2) $u \not\subseteq v$ and $v \not\subseteq u$.

Therefore $|u \cap v| = i$ where $0 \leq i \leq k-1$

Let $\Omega = \{1, 2, ..., n\}$ and u, v be two distinct subset of Ω. Without loss of generality we can assume $u \in X_k$. Now we consider two cases for v.

(a) $v \in X_k$. Without loss of generality we can assume $u = \{1, 2, ..., i, i{+}1, ..., k\}$ and $v = \{1, 2, ..., i, k{+}1, ..., 2k{-}i\}$ such that $0 \le i \le k-1$. We consider $c = \{1, 2, ..., i, i+1, ..., k, k+1, ..., 2k-i, 2k-i+1, ..., n-k\}$ which is possible because $n \ge 3k$. Therefore ucv is a shortest path of length 2 from u to v.

(b) $v \in X_{n-k}$, without of generality we can assume $u = \{1, 2, ..., i, i+1, ..., k\}$ and $v = \{1, 2, ..., i, k+1, ..., n-i\}$ such that $0 \le i \le k-1$. We consider $c = \{1, 2, ..., k, k+1, ..., n-k\}$ and $d = \{k+1, ..., 2k\}$ which is possible because $n \ge 3k$ therefore $ucdv$ is a shortest path of length 3 from u to v. □

Remark 1 If $A \in V$ and $2k + 2 \le n \le 3k - 1$, then it is obvious that A have equal distance with vertices like B such that $k - (i + 1)(n - 2k) \le |A \cap B| \le k - i(n - 2k) - 1$, where $0 \le i \le m$ and $m = [k/(n - 2k)]$.

Lemma 3 Let $A \in X_k$, $B \in X_{n-k}$ and $m = [k/(n-2k)]$ such that $k-(i+1)(n-2k) \le |A \cap B| \le k-i(n-2k)-1$ where $0 \le i \le m$ then $d(A, B) = 2i + 3$.

Proof. We use induction on i. If $i = 0$, then $k - (n - 2k) \le |A \cap B| \le k - 1$ by Remark 1 it is enough we assume $|A \cap B| = k - 1$. Without loss of generality we can assume $A = \{1, 2, ..., k-1, k\}$ and $B = \{1, 2, ..., k-1, k+1, ..., n-k+1\}$. We consider $c = \{1, 2, ..., k, ..., n-k\}$ and $d = \{1, 2, ..., k-1, k+1\}$ hence $AcdB$ is a shortest path of length 3 from A to B. Therefore by induction we assume the lemma is true for $i - 1$ and prove it for $i - 1$. Hence we assume $A \in X_k$, $B \in X_{n-k}$ and $k - (i + 1)(n - 2k) \le |A \cap B| \le k - i(n - 2k) - 1$. By Remark 1 it is enough we assume $|A \cap B| = k - i(n - 2k) - 1$. Without loss of generality we can assume $A = \{1, 2, ..., k - i(n - 2k) - 1, ..., k\}$ and $B = \{1, 2, ..., k - i(n - 2k) - 1, k + 1, ..., n - k + i(n - 2k) + 1\}$ where $0 \le i \le m$. We consider $d = \{1, 2, ..., k - i(n - 2k) - 1, k - i(n - 2k), k + 1, ..., n - k + i(n - 2k)\}$ then we observe that $|A \cap d| = k - i(n - 2k)$ and $|B - d| = 1$ therefore by induction hypothesis we have $d(A, d) = 2i + 1$ and by the properties of bipartite graphs we have $d(B, d) = 2$ which is possible because $2 \le n - 2k \le k - 1$, $|B - d| = 1$ and $|B| = |d| = n - k$ hence $d(A, B) = 2i + 3$, therefore the lemma is proved. □

The following tables are used in further results. Let $2k+2 \le n \le 3k-1$ and $m = [k/(n-2k)]$. By definition of the bipartite Kneser graph, Remark 1 and Lemma 3 we obtain results in Tables 1-3:

Table 1: Distance $d(A, B)$ and corresponding$|A \cap B|$ for $A \in X_k$ and $B \in X_{n-k}$.

$d(A, B)$	1	3	...	3	5	...	5	...	2m+3	...	2m+3
$\lvert A \cap B\rvert$	k	k-1	...	k-(n-2k)	k-(n-2k)-1	...	k-2(n-2k)	...	k-m(n-2k)-1	...	0

Table 2: Distance $d(A, B)$ and corresponding$|A \cap B|$ for $A \in X_k$ and $B \in X_k$.

$d(A, B)$	0	2	...	2	4	...	4	...	2m+2	...	2m+2
$\lvert A \cap B\rvert$	k	k-1	...	k-(n-2k)	k-(n-2k)-1	...	k-2(n-2k)	...	k-m(n-2k)-1	...	0

Table 3: Distance $d(A, B)$ and corresponding$|A \cap B|$ for $A, B \in X_{n-k}$.

$d(A, B)$	0	2	...	2	4	...	4	...	2m+2	...	2m+2
$\lvert A \cap B\rvert$	n-k	n-k-1	...	n-k-(n-2k)	k-1	...	n-k-2(n-2k)	...	n-k-m(n-2k)-1	...	n-2k

Theorem 5 For a positive integer $k \ge 2$, let $n \ge 2k + 2$ and $m = [k/(n - 2k)]$:

(1) If $n \geq 3k$ then we have

$$W(H_{n,k}) = \binom{n}{k}\left(\binom{n-k}{k} + 2\left(\binom{n}{k} - 1\right) + 3\left(\binom{n}{k} - \binom{n-k}{k}\right)\right),$$

(2) If $2k + 2 \leq n \leq 3k - 1$, then we have

$$W(H_{n,k}) = \binom{n}{k}\Bigg(\Big(\sum_{i=1}^{m+1} 2i \sum_{j=1}^{n-2k} \binom{k}{k-(i-1)(n-2k)-j}\binom{n-k}{(i-1)(n-2k)+j}\Big)+$$

$$\Big(\binom{n-k}{n-2k} + \sum_{i=1}^{m+1}(2i+1)\sum_{j=1}^{n-2k}\binom{k}{k-j-(i-1)(n-2k)}\binom{n-k}{i(n-2k)+j}\Big)\Bigg).$$

Proof. By Lemma 1, $H_{n,k}$ is vertex-transitive and by Theorem 1 :

$$W(H_{n,k}) = \binom{n}{k} d(A)$$

where A is a fixed vertex of $H_{n,k}$ and $d(A) = \sum_B d(A,B)$, where B is a subset of Ω with cardinality k or $n-k$.

proof (1) Let $u \in X_k$. By Lemma 2 the number of vertices like $v \in V$ such that $d(u,v) = i$, $0 \leq i \leq 3$ is calculated as follows:
if $d(u,v) = 0$, then the number of choices for v is 1, if $d(u,v) = 1$ then by properties of bipartite graphs we must have $v \in X_{n-k}$, hence the number of choices for v is $\binom{n-k}{k}$. If $d(u,v) = 2$ then we must have $v \in X_k$ hence the number of choices for v is $\binom{n}{k} - 1$, because $n \geq 3k$ it is obvious that if $d(u,v) = 3$, then we have $v \in X_{n-k}$ hence the number of choices for v is $\binom{n}{k} - \binom{n-k}{k}$ where $\binom{n-k}{k}$ is the number of vertices like $w \in V$ such that $d(u,w) = 1$ and $\binom{n}{k}$ is the number of vertices in X_{n-k}. Therefore we have

$$W(H_{n,k}) = \binom{n}{k}\left(\binom{n-k}{k} + 2\left(\binom{n}{k} - 1\right) + 3\left(\binom{n}{k} - \binom{n-k}{k}\right)\right),$$

proof (2) Let $u \in X_k$. By Tables 1, 2, 3 the number of vertices like $v \in V$ such that $d(u,v) = i$, $0 \leq i \leq 2m+3$ is calculated as follows:
if $d(u,v) = 0$, then the number of choices for v is 1, if $d(u,v) = 1$ then by properties of bipartite graphs we must have $v \in X_{n-k}$, hence the number of choices for v is $\binom{n-k}{k}$. Now if $d(u,v)$ is even then by Tables 2 the number of choices for v is $\left(\sum_{i=1}^{m+1}\sum_{j=1}^{n-2k} \binom{k}{k-(i-1)(n-2k)-j}\binom{n-k}{(i-1)(n-2k)+j}\right)$ and if $d(u,v)$ is odd then by Table 1 the number of choices for v is $\Big(\sum_{i=1}^{m+1}\sum_{j=1}^{n-2k}$ $\binom{k}{k-j-(i-1)(n-2k)}\binom{n-k}{i(n-2k)+j}\Big)$. Therefore we have

$$W(H_{n,k}) = \binom{n}{k}\Bigg(\Big(\sum_{i=1}^{m+1} 2i \sum_{j=1}^{n-2k} \binom{k}{k-(i-1)(n-2k)-j}\binom{n-k}{(i-1)(n-2k)+j}\Big)+$$

$$\Big(\binom{n-k}{n-2k} + \sum_{i=1}^{m+1}(2i+1)\sum_{j=1}^{n-2k}\binom{k}{k-j-(i-1)(n-2k)}\binom{n-k}{i(n-2k)+j}\Big)\Bigg).$$

□

Lemma 4 Let $e = uv \in E(H_{n,k})$.

(a) If $n \geq 3k$ to calculate $n_u(e|H_{n,k})$ it is enough to calculate vertices like z in V such that $d(u,z) \leq 2$ and $d(u,z) < d(v,z)$,

(b) If $2k+2 \leq n \leq 3k-1$, to calculate $n_u(e|H_{n,k})$ it is enough to calculate vertices like z in V such that $d(u,z) \leq 2m+2$ and $d(u,z) < d(v,z)$ where $m = [n/(n-2k)]$.

Proof. For vertices like u, v, z such that $uv \in E(H_{n,k})$ we have 4 possibilities:

(1) If $d(u,z) = 0$, then $z = u$, therefore $z \in N_u(e|O_k)$,

(2) If $d(u,z) = 1$, then by Lemma 2 and by properties of bipartite graphs we have $d(v,z) = 0$ *or* 2. Now if $d(v,z) = 0$ then $z \notin N_u(e|O_k)$ otherwise $z \in N_u(e|O_k)$,

(3) If $d(u,z) = 2$, then by Lemma 2 and by properties of bipartite graphs we have $d(v,z) = 1$ *or* 3. Now if $d(v,z) = 1$ then $z \notin N_u(e|O_k)$ otherwise $z \in N_u(e|O_k)$,

(4) If $d(u,z) = 3$, then by Lemma 2 and by properties of bipartite graphs we have $d(v,z) = 0$ *or* 2 then $z \notin N_u(e|O_k)$.

(b) For vertices like u, v, z such that $uv \in E(H_{n,k})$ we have:

(1) If $d(u,z) = 0$, then $z = u$, therefore $z \in N_u(e|O_k)$,

(2) If $d(u,z) = 1$, then by Tables 1, 2, 3 we have $d(v,z) = 0, 2, ..., 2m+2$ now if $d(v,z) = 0$ then $z \notin N_u(e|O_k)$ otherwise $z \in N_u(e|O_k)$,

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(2m+3) If $d(u,z) = 2m+2$, then by Tables 1, 2, 3 we have $d(v,z) = 1, 3, ..., 2m+3$ now if $d(v,z) = 1, 3, ..., 2m+1$ then $z \notin N_u(e|O_k)$ otherwise $z \in N_u(e|O_k)$,

(2m+4) If $d(u,z) = 2m+3$, then by Tables 1, 2, 3 we have $d(v,z) = 0, 2, ..., 2m+2$ then $z \notin N_u(e|O_k)$. □

Theorem 6 For a positive integer $k \geq 2$ let $n \geq 2k+2$. The Szeged index of $H_{n,k}$ is:

(1) If $n \geq 3k$ then we have

$$Sz(H_{n,k}) = \binom{n}{k}\binom{n-k}{k}\Big(E_0 + E_1 + E_2\Big)^2,$$

where $E_0 = 1$, $E_1 = \binom{n-k}{k} - 1$ and $E_2 = \binom{n}{k} - 1 - E_1$.

(2) If $2k+2 \leq n \leq 3k-1$, then we have

$$Sz(H_{n,k}) = \binom{n}{k}\binom{n-k}{k}\Big(\sum_{i=0}^{2m+2} F_i\Big)^2.$$

where $F_0 = 1$, $F_1 = \binom{n-k}{k} - 1$, and

$$F_i = \begin{cases} \sum_{j=1}^{n-2k} \binom{k}{k-j-(i-1)(n-2k)}\binom{n-2k}{n-2k+(i-1)(n-2k)+j} - F_{i-1} & if\ i \geq 3, i\ is\ odd, \\ \sum_{j=1}^{n-2k} \binom{k}{k-j-(i-1)(n-2k)}\binom{n-k}{(i-1)(n-2k)+j} - F_{i-1} & if\ i \geq 2, i\ is\ even. \end{cases}$$

Proof. Since by Lemma 1, $H_{n,k}$ is edge-transitive, we can use Theorem 2 to write

$$Sz(H_{n,k}) = \binom{n}{k}\binom{n-k}{k} n_u(e|H_{n,k}) n_v(e|H_{n,k}),$$

where $e = uv$ is a fixed edge of $H_{n,k}$ and $u \in X_k$, $v \in X_{n-k}$ or conversely. Since $H_{n,k}$ is a symmetric graph therefore $n_u(e|H_{n,k}) = n_v(e|H_{n,k})$, hence

$$Sz(H_{n,k}) = \binom{n}{k}\binom{n-k}{k}\Big(n_u(e|H_{n,k})\Big)^2.$$

We proceed to calculate $n_u(e|H_{n,k})$. We define E_i, $0 \leq i \leq 2$ and F_i, $0 \leq i \leq 2m+2$ where $m = [n/(n-2k)]$, as the number of vertices like $x \in V$ such that $d(u,x) = i$ and $d(u,x) < d(v,x)$

proof (1) By Lemma 4 and properties of bipartite graphs it is enough to calculate E_0, E_1 and E_2. It is obvious that $E_0 = 1$ and $E_1 = \binom{n-k}{k} - 1$. $E_2 = \binom{n}{k} - 1 - E_1$ because by assumption if we assume $u = \{1, ..., k\}$ then for other vertices like $w \in X_k$ we have $d(u,w) = 2$, but for the number of these vertices like $z \in V$ we have $d(v,z) = 1$, therefore this number must be ommited. Then we have

$$Sz(H_{n,k}) = \binom{n}{k}\binom{n-k}{k}\Big(E_0 + E_1 + E_2\Big)^2,$$

where $E_0 = 1$, $E_1 = \binom{n-k}{k} - 1$ and $E_2 = \binom{n}{k} - 1 - E_1$.

proof (2) By Lemma 4 and properties of bipartite graphs it is enough to calculate F_i where $0 \leq i \leq 2m+2$ where $m = [n/(n-2k)]$. Without loss of generality we can assume $u \in X_k$ and $v \in X_{n-k}$. By Table 1 we have $F_0 = 1$ because $d(u,x) = 0$ if and only if $|u \cap x| = k$, $F_1 = \binom{n-k}{n-2k} - F_0$ where by Table 1, $\binom{n-k}{n-2k}$ is the number of choices for vertices like $y \in V$ such that $d(u,y) = 1$ and F_0 is the number of choices for vertices in V like w such that $d(w,v) = 0$ so this number must be ommited, $F_2 = \sum_{j=1}^{n-2k} \binom{k}{k-j-(n-2k)}\binom{n-k}{(n-2k)+j} - F_1$ where by Table 2, $\sum_{j=1}^{n-2k} \binom{k}{k-j-(n-2k)}\binom{n-k}{(n-2k)+j}$ is the number of choices for vertices like $a \in V$ such that $d(u,a) = 2$ and F_1 is the number of vertices like r in V such that $d(v,r) = 1$ so this number must be ommited, hence by Lemma 4 we must continue this method until F_{2m+2}. Then we have

$$F_i = \begin{cases} \sum_{j=1}^{n-2k} \binom{k}{k-j-(i-1)(n-2k)}\binom{n-2k}{n-2k+(i-1)(n-2k)+j} - F_{i-1} & if\ i \geq 3, i\ is\ odd, \\ \sum_{j=1}^{n-2k} \binom{k}{k-j-(i-1)(n-2k)}\binom{n-k}{(i-1)(n-2k)+j} - F_{i-1} & if\ i \geq 2, i\ is\ even. \end{cases}$$

Therefore we have

$$Sz(H_{n,k}) = \binom{n}{k}\binom{n-k}{k}\Big(\sum_{i=0}^{2m+2} F_i\Big)^2.$$

where $F_0 = 1$, $F_1 = \binom{n-k}{k} - 1$, and

$$F_i = \begin{cases} \sum_{j=1}^{n-2k} \binom{k}{k-j-(i-1)(n-2k)}\binom{n-2k}{n-2k+(i-1)(n-2k)+j} - F_{i-1} & if\ i \geq 3, i\ is\ odd, \\ \sum_{j=1}^{n-2k} \binom{k}{k-j-(i-1)(n-2k)}\binom{n-k}{(i-1)(n-2k)+j} - F_{i-1} & if\ i \geq 2, i\ is\ even. \end{cases}$$

□

Lemma 5 Let G be a connected graph, then we have

$$PI(G) = |E(G)|^2 - \sum_{e \in E(G)} N(e)$$

where $e = uv$ is a fixed edge of G and $N(e)$ is the number of edges equidistant from u and v.

Proof. By definition of $PI(G)$ we have

$$PI(G) = \sum_{e \in E(G)} \Big(n_{eu}(e|G) + n_{ev}(e|G)\Big)$$

Since $E(G) = n_{eu}(e|G) + n_{ev}(e|G) + N(e)$, hence $E(G) - N(e) = n_{eu}(e|G) + n_{ev}(e|G)$, and we have

$$PI(G) = \sum_{e \in E(G)} \Big(|E(G) - N(e)\Big) = |E(G)|^2 - \sum_{e \in E(G)} N(e).$$

□

Theorem 7 For a positive integer $k \geq 2$ let $n \geq 2k+2$. The PI-index of $H_{n,k}$ is:
(1) If $n \geq 3k$, then we have

$$PI(H_{n,k}) = 2\binom{n}{k}\binom{n-k}{k}\left(\binom{n}{k}-1\right),$$

(2) If $2k+2 \leq n \leq 3k-1$ and $m = [n/(n-2k)]$, then we have

$$PI(H_{n,k}) = \left(\binom{n}{k}\binom{n-k}{k}\right)^2 - \binom{n}{k}\binom{n-k}{k}(F_0 + F_2 + ... + F_{2m+2}),$$

where $F_0 = 1$, $F_1 = \binom{n-k}{n-2k} - 1$ and

$$F_i = \begin{cases} \sum_{j=1}^{n-2k} \binom{k}{k-j-(i-1)(n-2k)}\binom{n-2k}{n-2k+(i-1)(n-2k)+j} - F_{i-1} & if\ i \geq 3\,, i\ is\ odd\,, \\ \sum_{j=1}^{n-2k} \binom{k}{k-j-(i-1)(n-2k)}\binom{n-k}{(i-1)(n-2k)+j} - F_{i-1} & if\ i \geq 2\,, i\ is\ even. \end{cases}$$

Proof. Since by Lemma 1, $H_{n,k}$ is edge-transitive, we can use Theorem 3 to write

$$PI(H_{n,k}) = \binom{n}{k}\binom{n-k}{k}\left(n_{eu}(e|H_{n,k}) + n_{ev}(e|H_{n,k})\right),$$

where $e = uv$ is a fixed edge of H_k and $u \in X_k$. Since $H_{n,k}$ is a symmetric graph therefore $n_{eu}(e|H_{n,k}) = n_{ev}(e|H_{n,k})$, hence

$$PI(H_{n,k}) = 2\binom{n}{k}\binom{n-k}{k}n_{eu}(e|H_{n,k}).$$

We proceed to calculate $n_{eu}(e|H_{n,k})$.

proof (1) By Lemma 4 and properties of bipartite graphs we define S_i, $i = 0, 1$ to be the number of edges like g in E such that $d(u,g) = i$ and $d(u,g) < d(v,g)$. In fact the number of edges like $f \in E$ such that $d(u,f) = 0$ is equal to the number of vertices like $m \in V$ such that $d(u,m) = 1$, $d(u,m) < d(v,m)$ and also similar to proof Theorem 6 we can define $S_i = E_{i+1}$ where $i = 0, 1$. Therefore $S_0 = \binom{n-k}{k} - 1$ and $S_1 = E_2 = \left(\binom{n}{k} - 1\right) - S_0$. Then we have

$$PI(H_{n,k}) = 2\binom{n}{k}\binom{n-k}{k}\left(\binom{n}{k}-1\right),$$

proof (2) Since by Lemma 1, $H_{n,k}$ is edge-transitive, we can use Theorem 3 and Lemma 5 to write

$$PI(H_{n,k}) = \left(\binom{n}{k}\binom{n-k}{k}\right)^2 - \binom{n}{k}\binom{n-k}{k}N(e)$$

where $e = uv$ is a fixed edge of $H_{n,k}$. First we calculate $N(e)$. In fact it is obvious that by the properties of bipartite graphs we must calculate the number of vertices like w in $E(H_{n,k})$ such that $d(u,w) = d(v,w) = 2i$, $0 \leq i \leq m+1$. Therefore we can define F_i, $0 \leq i \leq 2m+2$, in the same manner as in the proof of Theorem 6. Then we have

$$F_0 = 1,\ F_1 = \binom{k}{k-1} - F_0\ and$$

$$F_i = \begin{cases} \sum_{j=1}^{n-2k} \binom{k}{k-j-(i-1)(n-2k)}\binom{n-2k}{n-2k+(i-1)(n-2k)+j} - F_{i-1} & if\ i \geq 3\,, i\ is\ odd\,, \\ \sum_{j=1}^{n-2k} \binom{k}{k-j-(i-1)(n-2k)}\binom{n-k}{(i-1)(n-2k)+j} - F_{i-1} & if\ i \geq 2\,, i\ is\ even. \end{cases}$$

□

References

[1] M. R. Darafsheh, *Computation of topological indices of some graphs*, Acta Appl. Math., **110** (2010), 1225–1235

[2] M. R. Darafsheh, *The Wiener, Szeged and PI-index of the triangle graph*, In Lee See Keong and Hailiza Kamarul Haili (eds) 5th Asian mathematica conference proceedings (vol.1) June 2009, pp. 81–89, ISBN: 978-967-5417-53-5

[3] J. D. Dixon, and B. Mortimer, *Permutation groups*, Springer-Verlag New York, Inc., 1996

[4] A. A. Dobrynin, R. Entringer, and I. Gutman, *Wiener index of trees: Theory and applications*, Acta Appl. Math. **66** (2001), 211–249

[5] A. A. Dobrynin, and I. Gutman, *Congruence relations for the Szeged index of hexagonal chains*, Univ. Beogr, Publ. Elektrotehn. Fak. Ser. Mat. **8** (1997), 106–113

[6] C. Godsil, and G. Royle, *Algebraic graph theory*, Springer-Verlag New York, Inc., 2001

[7] I. Gutman, *A formula for the Wiener number of trees and its extension to graphs containing cycles*, Graph Theory Notes NY, **27** (1994), 9–15

[8] I. Gutman, and A. A. Dobrynin, *The Szeged index a success story*, Graph Theory Notes NY, **34** (1998), 37–44

[9] H. Hosoya, *Topological index. A newly proposed quality characterizing the topological nature of structural isomers of saturated hydrocarbons*, Bull. Chem. Soc. Jpn. **4** (1971), 2332–2339

[10] P. V. Khadikar, N. V. Deshpande, P. P. Kale, A. A. Dobrynin, and I. Gutman, *The Szeged index and analogy with the Wiener index*, J. Chem. Inf. Comput. Sci., **35** (1995), 547–550

[11] P. V. Khadikar, S. Karmarkar, and V. K. Agrawal, *Relationship and relative correction potential of the Wiener, Szeged and PI indexes*. Natl. Acad. Sci. Lett. **23**, 165–170 (2000)

[12] P. V. Khadikar, *On a novel structural descriptor PI*. Natl. Acad. Sci. Lett. **23**, 113–118 (2000)

[13] R. Mohammadyari, M. R. Darafsheh, *Topological indices of double odd graph* $2O_k$, Creat. Math. Inform. **20** (2011), No. 2, 163 – 170.

[14] H. Wiener, *Structural determination of paraffin boiling points*, J. Am. Chem. Soc. **69** (1947), 17–20

[15] J. Žerovnik, *Szeged index of a symmetric graphs*, J. Chem. Inf. Comput. Sci., **39** (1999), 77–80