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Differential Subordination and Superordination Results for Certain Subclasses of Analytic Functions by the Technique of Admissible Functions

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Abstract. By investigating appropriate classes of admissible functions, various Differential subordination and superordination results for analytic functions in the open unit disk are obtained using Cho-Kwon-Srivastava operator. As a consequence of these results, Sandwich-type results are also obtained.

1. Introduction and Motivations

Let $\mathcal{H}(\mathbb{U})$ be the class of functions analytic in

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$$

and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots,$$

with $a \in \mathbb{C}$, $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$. Let $\mathcal{A}_p$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in \mathbb{U}) \tag{1}$$

and let $\mathcal{A}_1 := \mathcal{A}$.

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Let f and F be members of $\mathscr{H}(\mathbb{U})$. The function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ with

$$w(0) = 0 \text{ and } |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that $f(z) = F(w(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent, then

$$f(z) \prec F(z) \text{ if and only if } f(0) = F(0)$$

and $f(\mathbb{U}) \subset F(\mathbb{U})$.

For two functions $f(z)$ given by (1) and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$, the Hadamard product(or convolution) of f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k := (g * f)(z). \tag{2}$$

For a function $f \in \mathscr{A}_p$, given by (1.1) and it follows from

$$I_p^{\lambda}(a, c)f(z) = \phi_p^{(+)}(a, c; z) * f(z), z \in \mathbb{U}$$

that for $\lambda > -p$ and $a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-$

$$\begin{aligned} I_p^{\lambda}(a, c)f(z) &= z^p + \sum_{k=1}^{\infty} \frac{(c)_k(\lambda + p)_k}{(a)_k(1)_k} a_{p+k} z^{p+k} \\ &= z^p {}_2F_1(c, \lambda + p, a; z) * f(z) \quad (z \in \mathbb{U}). \end{aligned} \tag{3}$$

where

$$\phi_p(a, c; z) * \phi_p^{+}(a, c; z) = \frac{z^p}{(1 - z)^{\lambda+p}}$$

and

$$\phi_p(a, c; z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k}.$$

From (3), we deduce that

$$z(I_p^{\lambda}(a, c)f(z))' = (\lambda + p)I_p^{\lambda+1}(a, c)f(z) - \lambda I_p^{\lambda}(a, c)f(z) \tag{4}$$

and

$$z(I_p^{\lambda}(a + 1, c)f(z))' = aI_p^{\lambda}(a, c)f(z) - (a - p)I_p^{\lambda}(a + 1, c)f(z). \tag{5}$$

We also note that

$$\begin{aligned} I_p^0(p + 1, 1)f(z) &= p \int_0^z \frac{f(t)}{t} dt, \\ I_p^0(p, 1)f(z) &= I_p^1(p + 1, 1)f(z) = f(z), \\ I_p^1(p, 1)f(z) &= \frac{zf'(z)}{p}, \\ I_p^2(p, 1)f(z) &= \frac{2zf'(z) + z^2 f''(z)}{p(p + 1)}, \\ I_p^2(p + 1, 1)f(z) &= \frac{f(z) + zf'(z)}{p + 1}, \\ I_p^n(a, a)f(z) &= D^{n+p-1} f(z), n \in \mathbb{N}, n > -p. \end{aligned}$$

The Ruscheweyh derivative $D^{n+p-1}f(z)$ was introduced by Y. C. Kim et al. [11] and the operator $I_p^\lambda(a,c)(\lambda > -p, a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-)$ was recently introduced by Cho et al. [6], who investigated (among other things) some inclusion relationships and properties of various subclasses of multivalent functions in $\mathscr{A}_p$, (see, [3, 17, 18]) which were defined by means of the operator $I_p^\lambda(a,c)$. For $\lambda = c = 1$ and $a = n + p$, the Cho-Kwon-Srivastava operator $I_p^\lambda(a,c)$ yields the Noor integral operator $I_p^1(n+p,1) = I_{n,p}\,(n > -p)$ of $(n+p-1)$ the order, studied by Liu and Noor [9](see also the works of [4, 5, 15, 16]). The linear operator $I_1^\lambda(\mu+2,1)$ $(\lambda > -1, \mu > -2)$ was also recently introduced and studied by Choi et al. [8]. For relevant details about further special cases of the Choi-Saigo-Srivastava operator $I_1^\lambda(\mu+2,1)$, the interested reader may refer to the works by Cho et al. [6] and Choi et al.[8] (see also [2, 7, 10]). In an earlier investigation, a sequence of results using differential subordination with convolution for the univalent case has been studied by Shanmugam [19] while sharp coefficient estimates for a certain general class of spirallike functions by means of differential subordination was studied by Xu et al. [25]. A systematic study of the subordination and superordination using certain operators under the univalent case has also been studied by Shanmugam *et al.* [20–23] and by sun et al. [24]. We observe that for these results, many of the investigations have not yet been studied by using appropriate classes of admissible functions.

Motivated by the aforementioned works, we obtain certain differential subordination and superordination results for analytic functions in the open unit disk using Cho-Kwon-Srivastava operator by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained as a consequence of the main results.

2. Preliminaries

To prove our results, we need the following definitions and theorems.

Denote by $\mathscr{L}$ the set of all functions $q(z)$ that are analytic and injective on $\overline{\mathbb{U}}\backslash E(q)$, where

$$E(q) = \{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(q)$. Further let the subclass of $\mathscr{L}$ for which $q(0) = a$ be denoted by $\mathscr{L}(a)$, $\mathscr{L}(0) \equiv \mathscr{L}_0$ and $\mathscr{L}(1) \equiv \mathscr{L}_1$.

Definition 2.1. *[13, Definition 2.3a, p.27] Let Ω be a set in $\mathbb{C}$, $q \in \mathscr{L}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(r,s,t;z) \notin \Omega$$

whenever

$$r = q(\zeta), s = k\zeta q'(\zeta),$$

and

$$\mathscr{R}\left\{\frac{t}{s}+1\right\} \geq k\mathscr{R}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}\backslash E(q)$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In particular when $q(z) = M\dfrac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, then

$$q(\mathbb{U}) = \mathbb{U}_M := \{w : |w| < M\}, q(0) = a, E(q) = \emptyset \text{ and } q \in \mathscr{L}.$$

In this case, we set $\Psi_n[\Omega, M, a] := \Psi_n[\Omega, q]$ and in the special case when the set $\Omega = \mathbb{U}_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 2.2. *[14, Definition 3, p.817] Let Ω be a set in $\mathbb{C}, q(z) \in \mathscr{H}[a,n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(r,s,t;\zeta) \in \Omega$$

whenever

$$r = q(z), s = \frac{zq'(z)}{m},$$

and

$$\mathscr{R}\left\{\frac{t}{s}+1\right\} \leq \frac{1}{m}\mathscr{R}\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}$ and $m \geq n \geq 1$.
In particular, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

Theorem 2.3. *[13, Theorem 2.3b, p.28]*
Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function

$$p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$$

satisfies

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \Omega,$$

then

$$p(z) \prec q(z).$$

□

Theorem 2.4. *[14, Theorem 1, p.818]*
Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If the analytic function $p(z) \in \mathscr{L}(a)$ and

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in \mathbb{U}\right\}$$

implies

$$q(z) \prec p(z).$$

□

3. Main Results

Definition 3.1. *Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathscr{L}_0 \cap \mathscr{H}[0,p]$. The class of admissible functions $\Phi_I[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u,v,w;z) \notin \Omega$$

whenever

$$u = q(\zeta), v = \frac{k\zeta q'(\zeta) + \lambda q(\zeta)}{\lambda + p},$$

$$\mathscr{R}\left\{\frac{(\lambda+p)(\lambda+p+1)w - 2\lambda(\lambda+p)v + \lambda(\lambda-1)u}{(\lambda+p)v - \lambda u}\right\} \geq k\mathscr{R}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}\backslash E(q)$ and $k \geq p$.

Theorem 3.2. *Let $\phi \in \Phi_I[\Omega, q]$. If $f(z) \in \mathscr{A}_p$ satisfies*

$$\left\{\phi\left(I_p^{\lambda}(a,c)f(z), I_p^{\lambda+1}(a,c)f(z), I_p^{\lambda+2}(a,c)f(z); z\right) : z \in \mathbb{U}\right\} \subset \Omega \tag{6}$$

then

$$I_p^{\lambda}(a,c)f(z) \prec q(z).$$

Proof. Define the analytic function $p(z)$ in $\mathbb{U}$ by

$$p(z) = I_p^{\lambda}(a,c)f(z). \tag{7}$$

In view of the relation

$$(\lambda + p)I_p^{\lambda+1}(a,c)f(z) = z(I_p^{\lambda}(a,c)f(z))' + \lambda I_p^{\lambda}(a,c)f(z) \tag{8}$$

this implies from (7) we get,

$$I_p^{\lambda+1}(a,c)f(z) = \frac{zp'(z) + \lambda p(z)}{(\lambda + p)}. \tag{9}$$

Further computations show that,

$$I_p^{\lambda+2}(a,c)f(z) = \frac{z^2p''(z) + (2\lambda+1)zp'(z) + \lambda(\lambda+1)p(z)}{(\lambda+p)(\lambda+p+1)}. \tag{10}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, v = \frac{s + \lambda r}{(\lambda + p)}, w = \frac{t + (2\lambda+1)s + \lambda(\lambda+1)r}{(\lambda+p)(\lambda+p+1)}. \tag{11}$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, \frac{s+\lambda r}{\lambda+p}, \frac{t + (2\lambda+1)s + \lambda(\lambda+1)r}{(\lambda+p)(\lambda+p+1)}; z\right). \tag{12}$$

The proof shall make use of Theorem 2.3 Using equations (7), (9) and (10), from (12), we find

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) = \phi\left(I_p^{\lambda}(a,c)f(z), I_p^{\lambda+1}(a,c)f(z), I_p^{\lambda+2}(a,c)f(z); z\right). \tag{13}$$

Hence (6) becomes

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_I[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2.1. Note that

$$\frac{t}{s} + 1 = \frac{(\lambda+p)(\lambda+p+1)w - 2\lambda(\lambda+p)v + \lambda(\lambda-1)u}{(\lambda+p)v - \lambda u},$$

and hence $\psi \in \Psi_p[\Omega, q]$. By Theorem 2.3, $p(z) \prec q(z)$ or

$$I_p^{\lambda}(a,c)f(z) \prec q(z).$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case the class $\Phi_I[h(\mathbb{U}), q]$ can be written as $\Phi_I[h, q]$.
The following result is an immediate consequence of Theorem 3.2.

Theorem 3.3. *Let $\phi \in \Phi_I[h, q]$. If $f(z) \in \mathscr{A}_p$ satisfies*

$$\phi(I_p^{\lambda}(a,c)f(z), I_p^{\lambda+1}(a,c)f(z), I_p^{\lambda+2}(a,c)f(z); z) \prec h(z) \tag{14}$$

then

$$I_p^{\lambda}(a,c)f(z) \prec q(z).$$

□

Our next result is an extension of Theorem 3.3 to the class where the behavior of $q(z)$ on $\partial\mathbb{U}$ is not known.

Corollary 3.4. *Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in $\mathbb{U}, q(0) = 0$. Let $\phi \in \Phi_I[\Omega, q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If $f(z) \in \mathscr{A}_p$ and*

$$\phi(I_p^{\lambda}(a,c)f(z), I_p^{\lambda+1}(a,c)f(z), I_p^{\lambda+2}(a,c)f(z); z) \in \Omega,$$

then

$$I_p^{\lambda}(a,c) \prec q(z).$$

Proof. Theorem 3.2 gives $I_p^{\lambda}(a,c) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$. □

Theorem 3.5. *Let $h(z)$ and $q(z)$ be univalent in $\mathbb{U}$, with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \to \mathbb{C}$ satisfy one of the following conditions:*

1. *$\phi \in \Phi_I[h, q_\rho]$, for some $\rho \in (0,1)$ or*
2. *there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_I[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f(z) \in \mathscr{A}_p$ satisfies (14), then

$$I_p^{\lambda}(a,c)f(z) \prec q(z).$$

Proof. The proof is similar to the proof of [13, Theorem 2.3d, p.30] and is therefore omitted. □

The next Theorem yields the best dominant of the *differential subordination* (14).

Theorem 3.6. *Let $h(z)$ be univalent in $\mathbb{U}$. Let $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z) + \lambda q(z)}{\lambda + p}, \frac{z^2 q''(z) + (2\lambda+1)zq'(z) + \lambda(\lambda+1)q(z)}{(\lambda+p)(\lambda+p+1)}; z\right) = h(z) \tag{15}$$

has a solution $q(z)$ with $q(0) = 0$ and satisfy one of the following conditions:

1. *$q(z) \in \mathscr{L}_0$ and $\phi \in \Phi_I[h, q]$,*
2. *$q(z)$ is univalent in $\mathbb{U}$ and $\phi \in \Phi_I[h, q_\rho]$ for some $\rho \in (0,1)$ or*
3. *$q(z)$ is univalent in $\mathbb{U}$ and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_I[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.*

If $f(z) \in \mathscr{A}_p$ satisfies (14) then

$$I_p^{\lambda}(a,c)f(z) \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Following the same argument in [13, Theorem 2.3e, p.31]. We deduce that $q(z)$ is a dominant from Theorems 3.3 and 3.5. Since $q(z)$ satisfies (15) it is also a solution of (14) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant. □

In the particular case $q(z) = Mz$, $M > 0$ and in view of the Definition 3.1, the class of admissible functions $\Phi_I[\Omega, q]$, denoted by $\Phi_I[\Omega, M]$ is described below.

Definition 3.7. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_I[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ such that*

$$\phi\left(Me^{i\theta}, \frac{k+\lambda}{\lambda+p}Me^{i\theta}, \frac{L + ((2\lambda+1)k + \lambda(\lambda+1))\,Me^{i\theta}}{(\lambda+p)(\lambda+p+1)}; z\right) \notin \Omega \tag{16}$$

whenever $z \in \mathbb{U}, \theta \in \mathbb{R}, \mathscr{R}\left(Le^{-i\theta}\right) \geq (k-1)kM$ for all real θ and $k \geq p$.

Corollary 3.8. *Let $\phi \in \Phi_I[\Omega, M]$. If $f(z) \in \mathscr{A}_p$ satisfies*

$$\phi\left(I_p^\lambda(a,c)f(z), I_p^{\lambda+1}(a,c)f(z), I_p^{\lambda+2}(a,c)f(z); z\right) \in \Omega,$$

then

$$\left|I_p^\lambda(a,c)f(z)\right| < M.$$

In the special case $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$, the class $\Phi_I[\Omega, M]$ is simply denoted by $\Phi_I[M]$.

□

Corollary 3.9. *Let $\phi \in \Phi_I[M]$. If $f(z) \in \mathscr{A}_p$ satisfies*

$$\left|\phi\left(I_p^\lambda(a,c)f(z), I_p^{\lambda+1}(a,c)f(z), I_p^{\lambda+2}(a,c)f(z); z\right)\right| < M,$$

then

$$\left|I_p^\lambda(a,c)f(z)\right| < M.$$

□

Remark 3.10. *when $\Omega = \mathbb{U}, \lambda = a - 1 (a > 0), p = 1$ and $M = 1$, Corollary 3.8 reduces to [6, Theorem2, p.231]. When $\Omega = \mathbb{U}, \lambda = 1, p = 1$ and $M = 1$, Corollary 3.8 reduces to [1, Theorem 1, p. 477].*

Corollary 3.11. *If $M > 0$ and $f(z) \in \mathscr{A}_p$ satisfies*

$$\left|(\lambda+p)(\lambda+p+1)I_p^{\lambda+2}(a,c)f(z) - (\lambda+p)I_p^{\lambda+1}(a,c)f(z) - \lambda(\lambda+1)I_p^\lambda(a,c)f(z)\right| < \left[(2p-1)\lambda + p(p-1)\right]M$$

then

$$\left|I_p^\lambda(a,c)f(z)\right| < M. \tag{17}$$

Proof. This follows from Corollary 3.8 by taking

$$\phi(u,v,w;z) = (\lambda+p)(\lambda+p+1)w - (\lambda+p)v - \lambda(\lambda+1)u$$

and $\Omega = h(\mathbb{U})$ where $h(z) = \left[(2p-1)\lambda + p(p-1)\right]Mz, M > 0$. To use Corollary 3.8, we need to show that $\phi \in \Phi_I[\Omega, M]$, that is the admissible condition (16) is satisfied. This follows since,

$$\begin{aligned}
&\left|\phi\left(Me^{i\theta}, \frac{k+\lambda}{\lambda+p}Me^{i\theta}, \frac{L + ((2\lambda+1)k + \lambda(\lambda+1))Me^{i\theta}}{(\lambda+p)(\lambda+p+1)}; z\right)\right| \\
&= |L + ((2\lambda+1)k + \lambda(\lambda+1))Me^{i\theta} - (k+\lambda)Me^{i\theta} - \lambda(\lambda+1)Me^{i\theta}| \\
&= \left|L + (2k-1)\lambda Me^{i\theta}\right| \\
&\geq (2k-1)\lambda M + \mathscr{R}(Le^{-i\theta}) \\
&\geq (2k-1)\lambda M + k(k-1)M \\
&\geq \left[(2p-1)\lambda + p(p-1)\right]M,
\end{aligned}$$

where $z \in \mathbb{U}, \theta \in \mathbb{R}, \mathscr{R}\left(Le^{\iota\theta}\right) \geq k(k-1)M$ and $k \geq p$.
Hence by Corollary 3.8, we deduce the required result. □

Definition 3.12. *Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathscr{L}_0 \cap \mathscr{H}_0$. The class of admissible functions $\Phi_{I,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), v = \frac{k\zeta q'(\zeta) + (\lambda + p - 1)q(\zeta)}{(\lambda + p)}$$

$$\mathscr{R}\left\{\frac{(\lambda + p)\left[(\lambda + p + 1)w - (2\lambda + 2p - 1)v + 3(\lambda + p - 1)u\right]}{(\lambda + p)v - (\lambda + p - 1)u}\right\} \geq k\mathscr{R}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}\backslash E(q)$ and $k \geq 1$.

Theorem 3.13. *Let $\phi \in \Phi_{I,1}[\Omega, q]$. If $f(z) \in \mathscr{A}_p$ satisfies*

$$\left\{\phi\left(\frac{I_p^{\lambda}(a, c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+1}(a, c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+2}(a, c)f(z)}{z^{p-1}}; z\right) : z \in \mathbb{U}\right\} \subset \Omega \tag{18}$$

then

$$\frac{I_p^{\lambda}(a, c)f(z)}{z^{p-1}} \prec q(z).$$

Proof. Define an analytic function $p(z)$ in $\mathbb{U}$ by

$$p(z) := \frac{I_p^{\lambda}(a, c)f(z)}{z^{p-1}}. \tag{19}$$

By making use of (8), we get,

$$\frac{I_p^{\lambda+1}(a, c)f(z)}{z^{p-1}} = \frac{zp'(z) + (\lambda + p - 1)p(z)}{(\lambda + p)}. \tag{20}$$

Further computations show that,

$$\frac{I_p^{\lambda+2}(a, c)f(z)}{z^{p-1}} = \frac{z^2p''(z) + 2(\lambda + p)zp'(z) + (\lambda + p - 1)(\lambda + p + 1)p(z)}{(\lambda + p)(\lambda + p + 1)}. \tag{21}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, v = \frac{s + (\lambda + p - 1)r}{(\lambda + p)}, w = \frac{t + 2(\lambda + p)s + (\lambda + p - 1)(\lambda + p + 1)r}{(\lambda + p)(\lambda + p + 1)}. \tag{22}$$

Let

$$\psi(r, s, t; z) = \phi(u, v, w; z) = \phi\left(r, \frac{s + (\lambda + p - 1)r}{(\lambda + p)}, \frac{t + 2(\lambda + p)s + (\lambda + p - 1)(\lambda + p + 1)r}{(\lambda + p)(\lambda + p + 1)}; z\right). \tag{23}$$

The proof shall make use of Theorem 2.3 using equations (19), (20) and (21), from (23) we obtain

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) = \phi\left(\frac{I_p^{\lambda}(a, c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+1}(a, c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+2}(a, c)f(z)}{z^{p-1}}; z\right). \tag{24}$$

Hence (18) becomes,

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{I,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2.1.
Note that,

$$\frac{t}{s} + 1 = \frac{(\lambda + p)[(\lambda + p + 1)w - (2\lambda + 2p - 1)v + 3(\lambda + p - 1)u]}{(\lambda + p)v - (\lambda + p - 1)u}$$

and hence, $\psi \in \Psi[\Omega, q]$.
By Theorem 2.3 , $p(z) \prec q(z)$ or

$$\frac{I_p^\lambda(a, c)f(z)}{z^{p-1}} \prec q(z).$$

If $\Omega \neq \mathbb{C}$ is simply connected domain, then $\Omega = h(\mathbb{U})$, for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case the class $\Phi_{I,1}[h(\mathbb{U}), q]$ can be written as $\Phi_{I,1}[h, q]$.
In the particular case $q(z) = Mz, M > 0$, the class of admissible functions $\Phi_{I,1}[\Omega, q]$, denoted by $\Phi_{I,1}[\Omega, M]$. The following result is an immediate consequence of Theorem 3.13. □

Theorem 3.14. *Let $\phi \in \Phi_{I,1}[h, q]$. If $f(z) \in \mathscr{A}_p$ satisfies*

$$\phi\left(\frac{I_p^\lambda(a, c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+1}(a, c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+2}(a, c)f(z)}{z^{p-1}}; z\right) \prec h(z) \tag{25}$$

then

$$\frac{I_p^\lambda(a, c)f(z)}{z^{p-1}} \prec q(z).$$

□

Definition 3.15. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{I,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ such that*

$$\phi\left(Me^{i\theta}, \frac{k + (\lambda + p - 1)}{(\lambda + p)}Me^{i\theta}, \frac{L + [2(\lambda + p)k + (\lambda + p - 1)(\lambda + p + 1)]Me^{i\theta}}{(\lambda + p)(\lambda + p + 1)}; z\right) \notin \Omega \tag{26}$$

whenever $z \in \mathbb{U}, \theta \in \mathbb{R}$,

$$\mathscr{R}\left(Le^{-i\theta}\right) \geq (k - 1)kM$$

$\forall$ real θ and $k \geq 1$.

Corollary 3.16. *Let $\phi \in \Phi_{I,1}[\Omega, M]$. If $f(z) \in \mathscr{A}_p$ satisfies*

$$\phi\left(\frac{I_p^\lambda(a, c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+1}(a, c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+2}(a, c)f(z)}{z^{p-1}}; z\right) \in \Omega,$$

then

$$\left|\frac{I_p^\lambda(a, c)f(z)}{z^{p-1}}\right| < M.$$

In the special case $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$, the class $\Phi_{I,1}[\Omega, M]$ is simply denoted by $\Phi_{I,1}[M]$.

□

Corollary 3.17. *Let $\phi \in \Phi_{I,1}[M]$. If $f(z) \in \mathscr{A}_p$ satisfies*

$$\left|\phi\left(\frac{I_p^\lambda(a, c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+1}(a, c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+2}(a, c)f(z)}{z^{p-1}}; z\right)\right| < M,$$

then

$$\left|\frac{I_p^\lambda(a, c)f(z)}{z^{p-1}}\right| < M.$$

□

Remark 3.18. *When* $\Omega = \mathbb{U}, \lambda = a - 1(a > 0), p = 1$ *and* $M = 1$. *Corollary 3.16 reduces to [12, Theorem 2, p.231]. When* $\Omega = \mathbb{U}, \lambda = 1, p = 1$ *and* $M = 1$, *Corollary 3.16 reduces to [1, Theorem 1, p.477].*

Corollary 3.19. *If* $f(z) \in \mathscr{A}_p$, *then*

$$\left|\frac{I_p^{\lambda+1}(a,c)f(z)}{z^{p-1}}\right| < M \Rightarrow \left|\frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}}\right| < M.$$

This follows from Corollary 3.17 by taking $\phi(u, v, w; z) = v$.

□

Corollary 3.20. *If* $M > 0$ *and* $f(z) \in \mathscr{A}_p$ *satisfies*

$$\left|(\lambda+p)(\lambda+p-1)\frac{I_p^{\lambda+2}(a,c)f(z)}{z^{p-1}} + (\lambda+p)\frac{I_p^{\lambda+1}(a,c)f(z)}{z^{p-1}} - \left((\lambda+p)^2-1\right)\frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}}\right| < [3(\lambda+p)]M$$

then

$$\left|\frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}}\right|. \tag{27}$$

Proof. This follows from Corollary 3.16 by taking

$$\phi(u, v, w; z) = (\lambda+p)(\lambda+p-1)w + (\lambda+p)v - (\lambda+p-1)(\lambda+p+1)u$$

and $\Omega = h(\mathbb{U})$ where $h(z) = 3(\lambda+p)Mz, M > 0$. To use Corollary 3.16, we need to show that $\phi \in \Phi_{I,1}[\Omega, M]$, that is the admissible condition (26) is satisfied. This follows since

$$\left|\phi\left(Me^{i\theta}, \frac{k+(\lambda+p-1)}{(\lambda+p)}Me^{i\theta}, \frac{L+[2(\lambda+p)k+(\lambda+p-1)(\lambda+p+1)]Me^{i\theta}}{(\lambda+p)(\lambda+p+1)}; z\right)\right|$$

$$\begin{aligned}
&= |L + [2(\lambda+p)k + (\lambda+p-1)(\lambda+p+1)]Me^{i\theta} + (k+\lambda+p-1)Me^{i\theta} - (\lambda+p-1)(\lambda+p+1)Me^{i\theta}| \\
&= L + \{2(\lambda+p)k + (k+\lambda+p-1)\}Me^{i\theta} \\
&\geq [2(\lambda+p)k + (k+p+\lambda-1)]M + \mathscr{R}\left[Le^{-i\theta}\right] \\
&\geq [2(\lambda+p)k + (k+p+\lambda-1)]M + k(k-1)M \\
&\geq 3(\lambda+p)M,
\end{aligned}$$

where $z \in \mathbb{U}, \theta \in \mathbb{R}, \mathscr{R}\left(Le^{-i\theta}\right) \geq k(k-1)M$ and $k \geq 1$.
Hence by Corollary 3.16, we deduce the required result. □

Definition 3.21. *Let* Ω *be a set in* $\mathbb{C}$ *and* $q(z) \in \mathscr{L}_1 \cap \mathscr{H}$. *The class of admissible functions* $\Phi_{I,2}[\Omega, q]$*consists of those functions* $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ *that satisfy the admissibility condition*

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), v = \frac{1}{(\lambda+p+1)}\left[1 + (\lambda+p)q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)}\right], (q(\zeta) \neq 0),$$

$$k\mathscr{R}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\} \leq \mathscr{R}\left\{\frac{(\lambda+p+1)(\lambda+p+2)u^2w + (\lambda+p+1)uv - (\lambda+p)(\lambda+p+3)u^3 - (\lambda+p+4)u^2 + 4}{(\lambda+p+1)uv - (\lambda+p)u^2 - u}\right\},$$

where $\in \mathbb{U}, \zeta \in \partial\mathbb{U}\backslash E(q)$ *and* $k \geq 1$.

Theorem 3.22. *Let $\phi \in \Phi_{I,2}[\Omega, q]$ and $I_p^{\lambda}(a,c)f(z) \neq 0$. If $f(z) \in \mathscr{A}_p$ satisfies*

$$\left\{\phi\left(\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}, \frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda+1}(a,c)f(z)}, \frac{I_p^{\lambda+3}(a,c)f(z)}{I_p^{\lambda+2}(a,c)f(z)}; z\right) : z \in \mathbb{U}\right\} \subset \Omega, \tag{28}$$

then

$$\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)} \prec q(z).$$

Proof. Define an analytic function $p(z)$ in $\mathbb{U}$ by

$$p(z) := \frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}. \tag{29}$$

By making use of (8) and (29) we get,

$$\frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda+1}(a,c)f(z)} = \frac{1}{(\lambda+p+1)}\left[1 + (\lambda+p)p(z) + \frac{zp'(z)}{p(z)}\right]. \tag{30}$$

Further computations show that

$$\frac{I_p^{\lambda+3}(a,c)f(z)}{I_p^{\lambda+2}(a,c)f(z)} = \frac{1}{(\lambda+p+2)}\left[\frac{zp'(z)}{p(z)} + (\lambda+p)p(z) + 2 + \frac{(\lambda+p)zp'(z) + \frac{z^2p''(z)}{p(z)} + \frac{zp'(z)}{p(z)} - \left(\frac{zp'(z)}{p(z)}\right)^2}{1 + (\lambda+p)p(z) + \frac{zp'(z)}{p(z)}}\right]. \tag{31}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, v = \frac{1}{(\lambda+p+1)}\left[1 + (\lambda+p)r + \left(\frac{s}{r}\right)\right]$$

$$w = \frac{1}{(\lambda+p+2)}\left[\left(\frac{s}{r}\right) + (\lambda+p)r + 2 + \frac{(\lambda+p)s + \left(\frac{s}{r}\right) - \left(\frac{s}{r}\right)^2 + \left(\frac{t}{r}\right)}{1 + (\lambda+p)r + \left(\frac{s}{r}\right)}\right]. \tag{32}$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) \tag{33}$$

$$= \phi\left(r, \frac{1}{\lambda+p+1}\left[1 + (\lambda+p)r + \left(\frac{s}{r}\right)\right], \frac{1}{(\lambda+p+2)}\left[\left(\frac{s}{r}\right) + (\lambda+p)r + 2 + \frac{(\lambda+p)s + \left(\frac{s}{r}\right) - \left(\frac{s}{r}\right)^2 + \left(\frac{t}{r}\right)}{1 + (\lambda+p)r + \left(\frac{s}{r}\right)}\right]; z\right)$$

The proof shall make use of Theorem 2.3.
Using equations (29), (30) and (31), from (33) we obtain

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) = \phi\left(\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}, \frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda+1}(a,c)f(z)}, \frac{I_p^{\lambda+3}(a,c)f(z)}{I_p^{\lambda+2}(a,c)f(z)}; z\right) \tag{34}$$

Hence (28) becomes,

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{I,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2.1.

Note that,

$$\frac{t}{s}+1=\frac{(\lambda+p+1)(\lambda+p+2)u^2w+(\lambda+p+1)uv-(\lambda+p)(\lambda+p+3)u^3-(\lambda+p+4)u^2+u}{(\lambda+p+1)uv-(\lambda+p)u^2-u}$$

and hence $\psi\in\Psi[\Omega,q]$. By Theorem 2.3, $p(z)\prec q(z)$ or

$$\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}\prec q(z).$$

□

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(\mathbb{U})$, for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case the class $\Phi_{I,2}[h(\mathbb{U}),q]$ is written as $\Phi_{I,2}[h,q]$.
In the particular case $q(z)=1+Mz, M>0$, the class of admissible functions $\Phi_{I,2}[\Omega,q]$ becomes the class $\Phi_{I,2}[\Omega,M]$.
Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.22.

Theorem 3.23. *Let $\phi\in\Phi_{I,2}[h,q]$. If $f(z)\in\mathscr{A}_p$ satisfies*

$$\phi\left(\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)},\frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda+1}(a,c)f(z)},\frac{I_p^{\lambda+3}(a,c)f(z)}{I_p^{\lambda+2}(a,c)f(z)};z\right)\prec h(z) \tag{35}$$

then

$$\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}\prec q(z).$$

□

The dual problem of differential subordination, that is, differential superordination of Cho-Kwon-Srivastava operator is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

Definition 3.24. *Let Ω be a set in $\mathbb{C}$ and $q(z)\in\mathscr{H}[0,p]$ with $zq'(z)\neq 0$. The class of admissible functions $\Phi_I'[\Omega,q]$ consists of those functions $\phi:\mathbb{C}^3\times\overline{\mathbb{U}}\to\mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u,v,w;\zeta)\in\Omega$$

whenever

$$u=q(z), v=\frac{\frac{zq'(z)}{m}+\lambda q(z)}{(\lambda+p)},$$

$$\mathscr{R}\left\{\frac{(\lambda+p)(\lambda+p+1)w-2\lambda(\lambda+p)v+\lambda(\lambda-1)u}{(\lambda+p)v-\lambda u}\right\}\leq\frac{1}{m}\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

$z\in\mathbb{U},\zeta\in\partial\mathbb{U}$ and $m\geq p$.

Theorem 3.25. *Let $\phi\in\Phi_I'[\Omega,q]$. If $f(z)\in\mathscr{A}_p, I_p^{\lambda}(a,c)f(z)\in\mathscr{L}_0$ and*

$$\phi\left(I_p^{\lambda}(a,c)f(z),I_p^{\lambda+1}(a,c)f(z),I_p^{\lambda+2}(a,c)f(z);z\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega\subset\left\{\phi\left(I_p^{\lambda}(a,c)f(z),I_p^{\lambda+1}(a,c)f(z),I_p^{\lambda+2}(a,c)f(z);z\right):z\in\mathbb{U}\right\} \tag{36}$$

implies

$$q(z)\prec I_p^{\lambda}(a,c)f(z).$$

Proof. From (13) and (36), we have

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in \mathbb{U}\right\}.$$

From (11), we see that the admissibility condition for $\phi \in \Phi_I'[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2.2. Hence $\psi \in \Psi_p'[\Omega, q]$ and by Theorem 2.4, $q(z) \prec p(z)$ or

$$q(z) \prec I_p^\lambda(a,c)f(z).$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case the class $\Phi_I'[h(\mathbb{U}), q]$ can be written as $\Phi_I'[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.25.

Theorem 3.26. *Let $q(z) \in \mathscr{H}[0,p]$, $h(z)$ is analytic on $\mathbb{U}$ and $\phi \in \Phi_I'[h,q]$.If $f(z) \in \mathscr{A}_p$, $I_p^\lambda(a,c)$, $f(z) \in \mathscr{L}_0$ and $\phi\left(I_p^\lambda(a,c)f(z), I_p^{\lambda+1}(a,c)f(z), I_p^{\lambda+2}(a,c)f(z); z\right)$ is univalent in $\mathbb{U}$, then*

$$h(z) \prec \phi\left(I_p^\lambda(a,c)f(z), I_p^{\lambda+1}(a,c)f(z), I_p^{\lambda+2}(a,c)f(z); z\right) \tag{37}$$

implies

$$q(z) \prec I_p^\lambda(a,c)f(z).$$

□

Theorem 3.25 and Theorem 3.26 can only be used to obtain subordinants of differential superordination of the form (36) and (37). The following theorem proves the existence of the best subordinant of (37) for certain ϕ.

Theorem 3.27. *Let $h(z)$ be analytic in $\mathbb{U}$ and $\phi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z) + \lambda q(z)}{(\lambda + p)}, \frac{z^2q''(z) + (2\lambda+1)zq'(z) + \lambda(\lambda+1)q(z)}{(\lambda+p)(\lambda+p+1)}; z\right) = h(z) \tag{38}$$

has a solution $q(z) \in \mathscr{L}_0$. If $\phi \in \Phi_I'[h,q]$, $f(z) \in \mathscr{A}_p$, $I_p^\lambda(a,c)f(z) \in \mathscr{L}_0$ and

$$\phi\left(I_p^\lambda(a,c)f(z), I_p^{\lambda+1}(a,c)f(z), I_p^{\lambda+2}(a,c)f(z); z\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(I_p^\lambda(a,c)f(z), I_p^{\lambda+1}(a,c)f(z), I_p^{\lambda+2}(a,c)f(z); z\right)$$

implies

$$q(z) \prec I_p^\lambda(a,c)f(z)$$

and $q(z)$ is the best subordinant.

Proof. The proof is similar to the proof of Theorem 3.6 and is therefore omitted. □

Definition 3.28. *Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathscr{H}_0$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi_{I,1}'[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z), v = \frac{\frac{zq'(z)}{m} + (\lambda + p - 1)q(z)}{(\lambda + p)},$$

$$\mathscr{R}\left\{(\lambda+p)\frac{[(\lambda+p+1)w - (2\lambda+2p-1)v + 3(\lambda+p-1)u]}{(\lambda+p)v - (\lambda+p-1)u}\right\} \leq \frac{1}{m}\mathscr{R}\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}$ and $m \geq 1$.

Now we will give the dual result of Theorem 3.13 for differential superordination.

Theorem 3.29. *Let $\phi \in \Phi'_{I,1}[\Omega, q]$. If $f(z) \in \mathscr{A}_p$, $\dfrac{I_p^\lambda(a,c)f(z)}{z^{p-1}} \in \mathscr{L}_0$ and*

$$\phi\left(\frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+1}(a,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+2}(a,c)f(z)}{z^{p-1}}\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+2}(a,c)f(z)}{z^{p-1}}; z\right) : z \in \mathbb{U}\right\} \tag{39}$$

implies

$$q(z) \prec \frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}.$$

Proof. From (24) and (40), we have

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathbb{U}\right\}.$$

From (22), we see that the admissibility condition for $\phi \in \Phi'_{I,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2.2. Hence $\psi \in \Psi'[\Omega, q]$ and by Theorem 2.4,

$$q(z) \prec p(z) \text{ or } q(z) \prec \frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}.$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω and the class $\Phi'_{I,1}[h(\mathbb{U}), q]$ cam be written as $\Phi'_{I,1}[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.29.

Theorem 3.30. *Let $q(z) \in \mathscr{H}_0, h(z)$ is analytic on $\mathbb{U}$ and $\phi \in \Phi'_{I,1}[h, q]$. If $f(z) \in \mathscr{A}_p$, $I_p^\lambda(a,c)f(z) \in \mathscr{L}_0$ and*

$$\phi\left(\frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+1}(a,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+2}(a,c)f(z)}{z^{p-1}}; z\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(\frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+1}(a,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+2}(a,c)f(z)}{z^{p-1}}; z\right) \tag{40}$$

implies

$$q(z) \prec \frac{I_p^\lambda(a,c)f(z)}{z^{p-1}}.$$

□

Now, we will give the dual result of Theorem 3.22 for the differential superordination.

Definition 3.31. *Let Ω be a set in $\mathbb{C}, q(z) \neq 0, zq'(z) \neq 0$ and $q(z) \in \mathscr{H}$. The class of admissible functions $\Phi'_{l,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(\zeta), v = \frac{1}{(\lambda + p + 1)}\left[1 + (\lambda + p)q(z) + \frac{zq'(z)}{mq(z)}\right],$$

$$\mathscr{R}\left\{\frac{(\lambda + p + 1)(\lambda + p + 2)u^2w + (\lambda + p + 1)uv - (\lambda + p)(\lambda + p + 3)u^3 - (\lambda + p + 4)u^2 + 4}{(\lambda + p + 1)uv - (\lambda + p)u^2 - u}\right\} \leq \frac{1}{m}\mathscr{R}\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

where $z \in \mathbb{U}, \zeta \in \partial\mathbb{U}$ and $m \geq 1$.

Theorem 3.32. *Let $\phi \in \Phi'_{l,2}[\Omega, q]$. If $f(z) \in \mathscr{A}_p$, $\dfrac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)} \in \mathscr{L}_1$ and*

$$\phi\left(\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}, \frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda+1}(a,c)f(z)}, \frac{I_p^{\lambda+3}(a,c)f(z)}{I_p^{\lambda+2}(a,c)f(z)}; z\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}, \frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda+1}(a,c)f(z)}, \frac{I_p^{\lambda+3}(a,c)f(z)}{I_p^{\lambda+2}(a,c)f(z)}; z\right) : z \in \mathbb{U}\right\} \tag{41}$$

implies

$$q(z) \prec \frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}.$$

Proof. From (34) and (42), we have

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathbb{U}\right\}.$$

From (33), we see that the admissibility condition for $\phi \in \Phi'_{l,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 2.4, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}.$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case the class $\Phi'_{l,2}[h(\mathbb{U}), q]$ can be written as $\Phi'_{l,2}[h, q]$.

The following result is an immediate consequence of Theorem 3.32.

Theorem 3.33. *Let $h(z)$ be analytic in $\mathbb{U}$ and $\phi \in \Phi'_{l,2}[h, q]$. If $f(z) \in \mathscr{A}_p$, $\dfrac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)} \in \mathscr{L}_1$ and*

$$\phi\left(\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}, \frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda+1}(a,c)f(z)}, \frac{I_p^{\lambda+3}(a,c)f(z)}{I_p^{\lambda+2}(a,c)f(z)}; z\right)$$

is univalent in $\mathbb{U}$*, then*

$$h(z) \prec \phi\left(\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}, \frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda+1}(a,c)f(z)}, \frac{I_p^{\lambda+3}(a,c)f(z)}{I_p^{\lambda+2}(a,c)f(z)}; z\right) \tag{42}$$

implies

$$q(z) \prec \frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}.$$

□

4. Further Corollaries and Observations

Combining Theorem 3.3 and Theorem 3.26, we obtain the following sandwich theorem.

Corollary 4.1. *Let* $h_1(z)$ *and* $q_1(z)$ *be analytic functions in* $\mathbb{U}, h_2(z)$ *be univalent in* $\mathbb{U}.q_2(z) \in \mathscr{L}_0$ *with* $q_1(0) = q_2(0) = 0$ *and* $\phi \in \Phi_I[h_2, q_2] \cap \Phi_I'[h_1, q_1]$*. If* $f(z) \in \mathscr{A}_p, I_p^{\lambda}(a,c)f(z) \in \mathscr{H}[0,p] \cap \mathscr{L}_0$ *and*

$$\phi\left(I_p^{\lambda}(a,c)f(z), I_p^{\lambda+1}(a,c)f(z), I_p^{\lambda+2}(a,c)f(z); z\right) \prec h_2(z),$$

implies

$$q_1(z) \prec I_p^{\lambda}(a,c)f(z) \prec q_2(z).$$

□

Combining Theorem 3.14 and Theorem 3.30 we obtain the following sandwich theorem.

Corollary 4.2. *Let* $h_1(z)$ *and* $q_1(z)$ *be analytic functions in* $\mathbb{U}$*,* $h_2(z)$ *be univalent function in* $\mathbb{U}, q_2(z) \in \mathscr{L}_0$ *with* $q_1(0) = q_2(0) = 0$ *and* $\phi \in \Phi_{I,1}[h_2, q_2] \cap \Phi_{I,1}'[h_1, q_1]$*. If* $f(z) \in \mathscr{A}_p, \dfrac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}} \in \mathscr{H}_0 \cap \mathscr{L}_0$ *and*

$$\phi\left(\frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+1}(a,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+2}(a,c)f(z)}{z^{p-1}}; z\right)$$

is univalent in $\mathbb{U}$*, then*

$$h_1(z) \prec \phi\left(\frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+1}(a,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda+2}(a,c)f(z)}{z^{p-1}}; z\right) \prec h_2(z),$$

implies

$$q_1(z) \prec \frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}} \prec q_2(z).$$

□

Combining Theorem 3.23 and Theorem 3.33, we obtain the following sandwich theorem.

Corollary 4.3. *Let* $h_1(z)$ *and* $q_1(z)$ *be analytic functions in* $\mathbb{U}$*,* $h_2(z)$ *be univalent function in* $\mathbb{U}$*,* $q_2(z) \in \mathscr{L}_1$ *with* $q_1(0) = q_2(0) = 1$ *and* $\phi \in \Phi_{I,2}[h_2, q_2] \cap \Phi_{I,2}'[h_1, q_1]$*. If* $f(z) \in \mathscr{A}_p, \dfrac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)} \in \mathscr{H} \cap \mathscr{L}_1, I_p^{\lambda}(a,c)f(z) \neq 0$ *and*

$$\phi\left(\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}, \frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda+1}(a,c)f(z)}, \frac{I_p^{\lambda+3}(a,c)f(z)}{I_p^{\lambda+2}(a,c)f(z)}; z\right)$$

is univalent in $\mathbb{U}$, *then*

$$h_1(z) \prec \phi\left(\frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)}, \frac{I_p^{\lambda+2}(a,c)f(z)}{I_p^{\lambda+1}(a,c)f(z)}, \frac{I_p^{\lambda+3}(a,c)f(z)}{I_p^{\lambda+2}(a,c)f(z)}; z\right) \prec h_2(z),$$

implies

$$q_1(z) \prec \frac{I_p^{\lambda+1}(a,c)f(z)}{I_p^{\lambda}(a,c)f(z)} \prec q_2(z).$$

□

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