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Hermite-Hadamard inequalities for relative semi-convex functions and applications

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Abstract. In this paper, we prove some Hermite-Hadamard inequalities for the class of relative semi-convex functions. Several special cases are also discussed. Thus it is worth mentioning that our results can be viewed as a generalization of previous results. Some applications to special means are also presented. Ideas and techniques of this paper may inspire further research in various branches of pure and applied sciences.

1. Introduction

Convexity plays an important role in the different fields of applied and pure sciences. Over the years, concept of convexity has been extended and generalized in different dimensions using novel and innovative techniques, see [1,2,4,11,15] and the references therein. Youness [15] had given a significant generalization of the convex set and convex function by introducing the concept of relative convex (g-convex) sets and relative convex (g-convex) functions. Noor [9,10,11,12] had shown that the optimality condition for relative convex (g-convex) function on the relative convex (g-convex) set can be characterized by some classes of variational inequality. Inspired by the research work of Youness [15] and Noor [9], Chen [1] introduced and studied a new class, which is called the relative semi-convex function. Niculescu and Persson [8] introduced the concept of relative convexity and had proved various properties and generalizations of classical results for relative convexity. Mercer [7], has also proved some useful results for relative convexity. For recent results see [16, 17, 18].

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then the following double inequality is known as Hermite-Hadamard inequality in the literature [4,8].

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

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In recent decades, researches have payed a lot of attention to this remarkable result by studying and proving its different forms for convex and nonconvex functions. A very useful detail for the interested readers is given in [3,4,6,7,8,13,14]. Noor et al. [13] have extended these double inequalities for relative semi-convex functions.
In this paper, we prove some Hermite-Hadamard inequalities for differentiable relative semi-convex functions. We discuss applications of these results to special means as well. Several special cases are also discussed.

2. Preliminaries

In this section, we recall some basic results and concepts, which are useful in proving our results. Let $\mathbb{R}^n$ be the finite dimensional space, whose innerproduct and norm is denoted by $\langle ., . \rangle$ and $\|.\|$, respectively.

Definition 2.1 [15]. A set $M \subseteq \mathbb{R}^n$ is said to be a relative convex (g-convex) set, if and only if, there exists a function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$(1-t)g(x) + tg(y) \in M, \ \forall x, y \in \mathbb{R}^n : g(x), g(y) \in M, t \in [0,1]. \tag{2.1}$$

Remark 2.1. Recently it has been shown in [5], that if M is a relative convex set then it is possible that it may not be a classical convex set. For example, for $M = [-1, -\frac{1}{2}] \cup [0,1]$ and $g(x) = x^2, \forall x \in \mathbb{R}$. Clearly, this is a relative convex set but not classical convex set. Another possibility may occur that relative convex set may be a classical convex set, for example if $M = [-1,1]$ and $g(x) = \sqrt[4]{|x|}, \forall x \in \mathbb{R}$.

Definition 2.3 [15]. A function f is said to be a relative convex (g-convex) function on a relative convex (g-convex) set M, if and only if, there exists a function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$f((1-t)g(x) + tg(y)) \leq (1-t)f(g(x)) + tf(g(y)), \ \forall x, y \in \mathbb{R}^n : g(x), g(y) \in M, t \in [0,1]. \tag{2.2}$$

Remark 2.4 [15]. Every convex function f on a convex set is a relative convex function. But the converse is not true. There are functions which are relative convex function but may not be a convex function in the classical sense. For example let $M \subset \mathbb{R}$ be given as:

$$M = \{(x, y) \in \mathbb{R}^2 : (x, y) = \lambda_1(0,0) + \lambda_2(0,3) + \lambda_3(2,1)\},$$

where $\lambda_i > 0$, $\sum_{i=1}^{3} \lambda_i = 1$, and function $g : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $g : (x, y) = (0, y)$, then the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} x^3, & if\ y < 1, \\ xy^3, & if\ y \geq 1, \end{cases}$$

is a relative convex function but not a convex function.

Definition 2.5 [1]. A function f is said to be a relative semi-convex function, if and only if there exists a function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$f((1-t)g(x) + tg(y)) \leq (1-t)f(x) + tf(y), \ x, y \in M, t \in [0,1]. \tag{2.3}$$

Theorem 2.6 [1]. If function $f : \mathbb{R}^n \to \mathbb{R}$ is relative semi-convex on an relative convex set $M \subseteq \mathbb{R}^n$. Then $f(g(x)) \leq f(x)$ for each $x \in M$.

Remark 2.7. A relative convex function on relative convex set is not necessary a relative semi-convex function.

Example 2.8 [1]. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $g(x,y) = (1+x,y)$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x,y) = x^2 + y^2$, $\mathbb{R}^2$ is a relative convex set, f is relative convex function on set M. Since $f(g(0,0)) = 1 > f(0,0) = 0$, then from Theorem 2.6, it follows that f is not a relative semi-convex function on set M.

Theorem 2.9 [1]. Suppose a function $f : \mathbb{R}^n \to \mathbb{R}$ is relative convex on relative convex set $M \subseteq \mathbb{R}^n$. Then f is relative semi-convex on set M if and only if $f(g(x)) \le f(x)$ for each $x \in M$.

Remark 2.10. From Theorem 2.9, it follows that relative convex function f on a relative convex set M with the property $f(g(x)) \le f(x)$ for each $x \in M$ is relative semi-convex on set M, but the converse is not true.

Example 2.11 [1]. Let $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$g(x) = \begin{cases} 1, & 1 \le x \le 4, \\ 1 + \frac{2}{\pi}\arctan(1-x), & x < 1, \\ 2 + \frac{4}{\pi}\arctan(x-4), & x > 4. \end{cases}$$

The set is relative convex set, and let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 7, & if\, x < 1, or\, x > 4, \\ x-3, & if\, 1 \le x < 2, \\ 3-x, & if\, 2 \le x \le 3, \\ x-3, & if\, 3 < x \le 4. \end{cases}$$

The function is relative semi-convex on relative convex set. But the function is not relative convex on relative convex set.

Let $M = I = [g(a), g(b)]$ be a relative semi-convex set. We now define relative semi-convex functions on I, which is mainly due to Noor et al. [13].

Definition 2.12 [13]. Let $I = [g(a), g(b)]$, then f is called relative semi-convex function, if and only if,

$$\begin{vmatrix} 1 & 1 & 1 \\ g(a) & g(x) & g(b) \\ f(a) & f(g(x)) & f(b) \end{vmatrix} \ge 0; \quad g(a) \le g(x) \le g(b).$$

One can easily show that the following are equivalent:

1. f is relative semi-convex function on relative convex set.
2. $f(g(x)) \le f(a) + \frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))$.
3. $\frac{f(g(x))-f(a)}{g(x)-g(a)} \le \frac{f(b)-f(a)}{g(b)-g(a)} \le \frac{f(b)-f(g(x))}{g(b)-g(x)}$.
4. $\frac{f(a)}{(g(x)-g(a))(g(b)-g(a))} + \frac{f(g(x))}{(g(b)-g(x))(g(x)-g(a))} + \frac{f(b)}{(g(b)-g(a))(g(b)-g(x))} \ge 0$.
5. $(g(b)-g(x))f(a) + (g(b)-g(a))f(g(x)) + (g(x)-g(a))f(b) \ge 0$,

where $g(x) = (1-t)g(a) + tg(b)$ and $t \in [0,1]$.

For further properties of relative semi-convex functions, see [13].

3. Main Results

In this section, we will discuss our main results for relative semi-convex function. We need the following result, which can be proved using the technique of Kirmaci [6]. This result plays a key part in obtaining our main results.

Lemma 3.1. *Let $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0(interior of I) with $g(a) < g(b)$. If $f' \in L[g(a), g(b)]$, then we have*

$$\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right) = (g(b)-g(a))\left[\int_0^1 \mu(t) f'(tg(a)+(1-t)g(b))dt\right],$$

where

$$\mu(t) = \begin{cases} t, & [0, \frac{1}{2}), \\ t-1, & [\frac{1}{2}, 1]. \end{cases}$$

Proof. The proof is obvious. □

Theorem 3.2. *Let $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0, $g(a), g(b) \in I$ with $g(a) < g(b)$. If $|f'|$ is relative semi-convex on I and $f' \in L[g(a), g(b)]$, then we have*

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right| \leq \frac{g(b)-g(a)}{8}(|f'(a)| + |f'(b)|).$$

Proof. Using Lemma (3.1), we have

$$\begin{aligned}
&\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right| \\
&= \left|(g(b)-g(a))\left[\int_0^1 \mu(t) f'(tg(a)+(1-t)g(b))dt\right]\right| \\
&= \left|(g(b)-g(a))\left[\int_0^{\frac{1}{2}} t f'(tg(a)+(1-t)g(b))dt + \int_{\frac{1}{2}}^1 (t-1) f'(tg(a)+(1-t)g(b))dt\right]\right| \\
&\leq (g(b)-g(a))\left[\int_0^{\frac{1}{2}} |t||f'(tg(a)+(1-t)g(b))|dt + \int_{\frac{1}{2}}^1 |(t-1)||f'(tg(a)+(1-t)g(b))|dt\right] \\
&\leq (g(b)-g(a))\left[\int_0^{\frac{1}{2}} \{t^2|f'(a)| + t(1-t)|f(b)|\}dt + \int_{\frac{1}{2}}^1 \{t(t-1)|f'(a)| + (1-t)^2|f(b)|\}dt\right] \\
&= \frac{g(b)-g(a)}{8}\{|f'(a)| + |f'(b)|\}.
\end{aligned}$$

This completes the proof. □

Remark 3.3. *If we take $g = I$, where I is an identity function, then Theorem* 3.2 *coincides with Theorem* 2.2 *of* [6].

Theorem 3.4. *Let $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0, $g(a), g(b) \in I$ with $g(a) < g(b)$ and $p > 1$. If*

$|f'|^{\frac{p}{p-1}}$ is relative semi-convex on I and $f' \in L[g(a), g(b)]$, then we have

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$\leq \frac{g(b)-g(a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left\{|f'(a)|^{\frac{p}{p-1}} + 3|f'(b)|^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}} + \left\{3|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\right]$$

Proof. Using Lemma (3.1), and well-known Holder's inequality, we have

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$= \left|(g(b)-g(a))\left[\int_0^1 \mu(t)f'(tg(a)+(1-t)g(b))dt\right]\right|$$
$$= \left|(g(b)-g(a))\left[\int_0^{\frac{1}{2}} tf'(tg(a)+(1-t)g(b))dt + \int_{\frac{1}{2}}^1 (t-1)f'(tg(a)+(1-t)g(b))dt\right]\right|$$
$$\leq (g(b)-g(a))\left[\int_0^{\frac{1}{2}} |t||f'(tg(a)+(1-t)g(b))|dt + \int_{\frac{1}{2}}^1 |(t-1)||f'(tg(a)+(1-t)g(b))|dt\right]$$
$$\leq (g(b)-g(a))\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} |f'(tg(a)+(1-t)g(b))|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1 |t-1|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 |f'(tg(a)+(1-t)g(b))|^q dt\right)^{\frac{1}{q}}\right]. \tag{3.1}$$

Since $|f'|$ is relative semi-convex, we have

$$\int_0^{\frac{1}{2}} |f'(tg(a)+(1-t)g(b))|^q dt \leq \int_0^{\frac{1}{2}} [t|f'(a)|^q + (1-t)|f'(b)|^q]dt$$
$$= \frac{|f'(a)|^q + 3|f'(b)|^q}{8}, \tag{3.2}$$

and

$$\int_{\frac{1}{2}}^1 |f'(tg(a)+(1-t)g(b))|^q dt \leq \int_{\frac{1}{2}}^1 [t|f'(a)|^q + (1-t)|f'(b)|^q]dt$$
$$= \frac{3|f'(a)|^q + |f'(b)|^q}{8}. \tag{3.3}$$

Also

$$\int_0^{\frac{1}{2}} t^p dt = \int_{\frac{1}{2}}^{1} |t-1|^p dt = \frac{1}{(p+1)2^{p+1}}. \tag{3.4}$$

Using (3.2-3.4) in (3.1), we have the required result. □

Remark 3.5. *If we take $g = I$, where I is an identity function, then Theorem* 3.4 *coincides with Theorem* 2.3 *of* [6].

Theorem 3.6. *Let $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0, $g(a), g(b) \in I$ with $g(a) < g(b)$ and $p > 1$ if $|f'|^{\frac{p}{p-1}}$ is relative semi-convex on I and $f' \in L[g(a), g(b)]$, then we have*

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right| \le \frac{g(b)-g(a)}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}} \{|f'(a)| + |f'(b)|\}.$$

Proof. From Theorem 3.4, we have

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$\le \frac{g(b)-g(a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left\{|f'(a)|^{\frac{p}{p-1}} + 3|f'(b)|^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}} + \left\{3|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\right].$$

Let $a_1 = |f'(a)|^{\frac{p}{p-1}}$, $b_1 = 3|f'(b)|^{\frac{p}{p-1}}$, $a_2 = 3|f'(a)|^{\frac{p}{p-1}}$ and $b_2 = |f'(b)|^{\frac{p}{p-1}}$. Where $0 \le \frac{p-1}{p} < 1$, for $p > 1$. Then, using the fact,

$$\sum_{i=1}^{n}(a_i + b_i)^s \le \sum_{i=1}^{n} a_i^s + \sum_{i=1}^{n} b_i^s,$$

for $0 \le s < 1$, $a_1, a_2, \ldots, a_n \ge 0$ and $b_1, b_2, \ldots, b_n \ge 0$, we have the required result. □

Remark 3.7. *If we take $g = I$, where I is an identity function, then Theorem* 3.6 *coincides with Theorem* 2.4 *of* [6].

Theorem 3.8. *Let $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0, $g(a), g(b) \in I$ with $g(a) < g(b)$ if $|f'|$ is relative semi-convex on I and $f' \in L[g(a), g(b)]$, then we have*

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$\le \frac{g(b)-g(a)}{8}\left(\frac{1}{3}\right)^{\frac{p-1}{p}}\left[\left\{|f'(a)|^{\frac{p}{p-1}} + 2|f'(b)|^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}} + \left\{2|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\right].$$

Proof. The proof follows from Theorem 3.4. The only difference is, we have utilized power-mean inequality instead of Holder's inequality. □

4. Applications To Special Means

We now recall the following well-known concepts [3,6,12]. For arbitrary real numbers α, β, $\alpha \neq \beta$, we define

1. Harmonic Mean

$$H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}, \quad \alpha,\beta \in \mathbb{R}\backslash\{0\}.$$

2. Arithmetic Mean

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in \mathbb{R}.$$

3. Logarithmic Mean

$$\bar{L}(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta| - \ln|\alpha|}, \quad \alpha,\beta \in \mathbb{R}\backslash\{0\}.$$

4. Generalized Log-Mean

$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \quad n \in N, n \geq 1, \alpha,\beta \in \mathbb{R}, \alpha < \beta.$$

Now we give some applications of the results derived in section 2 to special means of real numbers.

Proposition 4.1. *Let $g(a), g(b) \in \mathbb{R}$ with $g(a) < g(b)$ and $n \in N$, $n \geq 2$, then we have the following result*

$$|L_n^n(g(a),g(b)) - A^n(g(a),g(b))| \leq \frac{n(g(b)-g(a))}{4} A(|a|^{n-1}, |b|^{n-1}). \tag{4.1}$$

Proof. The assertion follows from Theorem 3.2 applied for $f(g(x)) = [g(x)]^n$, $g(x) \in \mathbb{R}$. □

Remark 4.2. *If we make substitution $g(a) \to [g(b)]^{-1}$ and $g(b) \to [g(a)]^{-1}$ in Proposition 4.1, then we have*

Corollary 4.3. *Let $g(a), g(b) \in \mathbb{R}\backslash\{0\}$, with $g(a) < g(b)$ then $[g(b)]^{-1} < [g(a)]^{-1}$. For $n \in N$, $n \geq 2$, we have*

$$|L_n^n([g(b)]^{-1},[g(a)]^{-1}) - H^{-n}(g(b),g(a))| \leq \frac{n([g(a)]^{-1}-[g(b)]^{-1})}{4} H^{-1}(|a|^{n-1}, |b|^{n-1}). \tag{4.2}$$

Proposition 4.4. *Let $g(a), g(b) \in \mathbb{R}$ with $g(a) < g(b)$ and $n \in N$, $n \geq 2$, $p > 1$ then we have the following result*

$$\begin{aligned} &|L_n^n(g(a),g(b)) - A^n(g(a),g(b))| \\ &\leq \frac{n(g(b)-g(a))}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(|a|^{\frac{p(n-1)}{p-1}} + 3|b|^{\frac{p(n-1)}{p-1}}\right)^{\frac{p-1}{p}} + \left(3|a|^{\frac{p(n-1)}{p-1}} + |b|^{\frac{p(n-1)}{p-1}}\right)^{\frac{p-1}{p}}\right]. \end{aligned} \tag{4.3}$$

Proof. The assertion follows from Theorem 3.4 applied for $f(g(x)) = [g(x)]^n$, $g(x) \in \mathbb{R}$. □

Remark 4.5. *If we make substitution $g(a) \to [g(b)]^{-1}$ and $g(b) \to [g(a)]^{-1}$ in Proposition 4.4, then we have*

Corollary 4.6. *Let $g(a), g(b) \in \mathbb{R}\backslash\{0\}$, with $g(a) < g(b)$ then $[g(b)]^{-1} < [g(a)]^{-1}$. For $n \in N$, $n \geq 2$, $\forall p > 1$, we have*

$$\begin{aligned} &|L_n^n([g(b)]^{-1},[g(a)]^{-1}) - H^{-n}(g(b),g(a))| \\ &\leq \frac{n([g(a)]^{-1}-[g(b)]^{-1})}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(|b|^{\frac{p(1-n)}{p-1}} + 3|a|^{\frac{p(1-n)}{p-1}}\right)^{\frac{p-1}{p}} + \left(3|b|^{\frac{p(1-n)}{p-1}} + |a|^{\frac{p(1-n)}{p-1}}\right)^{\frac{p-1}{p}}\right]. \end{aligned} \tag{4.4}$$

Proposition 4.7. *Let $g(a), g(b) \in \mathbb{R}$ with $g(a) < g(b)$ and $n \in N$, $n \geq 2$, $p > 1$, then we have the following result*

$$|L_n^n(g(a), g(b)) - A^n(g(a), g(b))| \leq n\left(\frac{g(b) - g(a)}{2}\right)\left(\frac{4}{p+1}\right)^{\frac{1}{p}} A(|a|^{n-1}, |b|^{n-1}). \tag{4.5}$$

Proof. The assertion follows from Theorem 3.6 applied for $f(g(x)) = [g(x)]^n$, $g(x) \in \mathbb{R}$. □

Remark 4.8. *If we make substitution $g(a) \to [g(b)]^{-1}$ and $g(b) \to [g(a)]^{-1}$ in Proposition 4.7, then we have*

Corollary 4.9. *Let $g(a), g(b) \in \mathbb{R}\backslash\{0\}$, with $g(a) < g(b)$ then $[g(b)]^{-1} < [g(a)]^{-1}$. For $n \in N$, $n \geq 2$, $\forall p > 1$, we have*

$$|L_n^n([g(b)]^{-1}, [g(a)]^{-1}) - H^{-n}(g(b), g(a))| \leq n\left(\frac{[g(a)]^{-1} - [g(b)]^{-1}}{2}\right)\left(\frac{4}{p+1}\right)^{\frac{1}{p}} H^{-1}(|a|^{n-1}, |b|^{n-1}). \tag{4.6}$$

Proposition 4.10. *Let $g(a), g(b) \in \mathbb{R}$ with $g(a) < g(b)$ and $n \in N$, $n \geq 2$, $p > 1$, then we have the following result*

$$\begin{aligned}&|L_n^n(g(a), g(b)) - A^n(g(a), g(b))|\\ &\leq \frac{n(g(b) - g(a))}{8}\left(\frac{1}{3}\right)^{\frac{p-1}{p}}\left[\left(|a|^{\frac{p(n-1)}{p-1}} + 2|b|^{\frac{p(n-1)}{p-1}}\right)^{\frac{p-1}{p}} + \left(2|a|^{\frac{p(n-1)}{p-1}} + |b|^{\frac{p(n-1)}{p-1}}\right)^{\frac{p-1}{p}}\right].\end{aligned} \tag{4.7}$$

Proof. The assertion follows from Theorem 3.8 applied for $f(g(x)) = [g(x)]^n$, $g(x) \in \mathbb{R}$. □

Remark 4.11. *If we make substitution $g(a) \to [g(b)]^{-1}$ and $g(b) \to [g(a)]^{-1}$ in Proposition 4.10, then we have*

Corollary 4.12. *Let $g(a), g(b) \in \mathbb{R}\backslash\{0\}$, with $g(a) < g(b)$ then $[g(b)]^{-1} < [g(a)]^{-1}$. For $n \in N$, $n \geq 2$, $\forall p > 1$, we have*

$$\begin{aligned}&|L_n^n([g(b)]^{-1}, [g(a)]^{-1}) - H^{-n}(g(b), g(a))|\\ &\leq \frac{n([g(a)]^{-1} - [g(b)]^{-1})}{8}\left(\frac{1}{3}\right)^{\frac{p-1}{p}}\left[\left(|b|^{\frac{p(1-n)}{p-1}} + 2|a|^{\frac{p(1-n)}{p-1}}\right)^{\frac{p-1}{p}} + \left(2|b|^{\frac{p(1-n)}{p-1}} + |a|^{\frac{p(1-n)}{p-1}}\right)^{\frac{p-1}{p}}\right].\end{aligned} \tag{4.8}$$

Proposition 4.13. *Let $g(a), g(b) \in \mathbb{R}$ with $g(a) < g(b)$ and $0 \neq [g(a), g(b)]$, then we have the following result*

$$|\bar{L}^{-1}(g(a), g(b)) - A^{-1}(g(a), g(b))| \leq \frac{g(b) - g(a)}{4} A(|a|^{-2}, |b|^{-2}). \tag{4.9}$$

Proof. The assertion follows from Theorem 3.2 applied for $f(g(x)) = \frac{1}{g(x)}$, $g(x) \in [g(a), g(b)]$. □

Remark 4.14. *If we make substitution $g(a) \to [g(b)]^{-1}$ and $g(b) \to [g(a)]^{-1}$ in Proposition 4.13, then we have*

Corollary 4.15. *Let $g(a), g(b) \in \mathbb{R}\backslash\{0\}$, with $g(a) < g(b)$ then $[g(b)]^{-1} < [g(a)]^{-1}$. For $n \in N$, $n \geq 2$, we have*

$$|\bar{L}^{-1}([g(b)]^{-1}, [g(a)]^{-1}) - H(g(b), g(a))| \leq \frac{[g(a)]^{-1} - [g(b)]^{-1}}{4} H^{-1}(|a|^{-2}, |b|^{-2}). \tag{4.10}$$

Proposition 4.16. *Let $g(a), g(b) \in \mathbb{R}$ with $g(a) < g(b)$ and $0 \neq [g(a), g(b)]$, $p > 1$, then we have the following result*

$$\begin{aligned}&|\bar{L}^{-1}(g(a), g(b)) - A^{-1}(g(a), g(b))|\\ &\leq \left(\frac{g(b) - g(a)}{16}\right)\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(|a|^{\frac{-2p}{p-1}} + 3|b|^{\frac{-2p}{p-1}}\right)^{\frac{p-1}{p}} + \left(3|a|^{\frac{-2p}{p-1}} + |b|^{\frac{-2p}{p-1}}\right)^{\frac{p-1}{p}}\right].\end{aligned} \tag{4.11}$$

Proof. The assertion follows from Theorem 3.4 applied for $f(g(x)) = \frac{1}{g(x)}$, $g(x) \in [g(a), g(b)]$. □

Remark 4.17. *If we make substitution $g(a) \to [g(b)]^{-1}$ and $g(b) \to [g(a)]^{-1}$ in Proposition 4.16, then we have*

Corollary 4.18. *Let $g(a), g(b) \in \mathbb{R}\backslash\{0\}$, with $g(a) < g(b)$ then $[g(b)]^{-1} < [g(a)]^{-1}$. For $n \in N$, $n \geq 2$, $\forall p > 1$, we have*

$$|\bar{L}^{-1}([g(b)]^{-1}, [g(a)]^{-1}) - H(g(b), g(a))| \\ \leq \left(\frac{[g(a)]^{-1} - [g(a)]^{-1}}{16}\right)\left(\frac{4}{p+1}\right)^{\frac{1}{p}} \left[\left(|b|^{\frac{2p}{p-1}} + 3|a|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}} + \left(3|b|^{\frac{2p}{p-1}} + |a|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}\right]. \tag{4.12}$$

Proposition 4.19. *Let $g(a), g(b) \in \mathbb{R}$ with $g(a) < g(b)$ and $0 \neq [g(a), g(b)]$, $p > 1$, then we have the following result*

$$|\bar{L}^{-1}(g(a), g(b)) - A^{-1}(g(a), g(b))| \leq \left(\frac{g(b) - g(a)}{2}\right)\left(\frac{4}{p+1}\right)^{\frac{1}{p}} A(|a|^{-2}, |b|^{-2}). \tag{4.13}$$

Proof. The assertion follows from Theorem 3.6 applied for $f(g(x)) = \frac{1}{g(x)}$, $g(x) \in [g(a), g(b)]$. □

Remark 4.20. *If we make substitution $g(a) \to [g(b)]^{-1}$ and $g(b) \to [g(a)]^{-1}$ in Proposition 4.19, then we have*

Corollary 4.21. *Let $g(a), g(b) \in \mathbb{R}\backslash\{0\}$, with $g(a) < g(b)$ then $[g(b)]^{-1} < [g(a)]^{-1}$. For $n \in N$, $n \geq 2$, $\forall p > 1$, we have*

$$|\bar{L}^{-1}([g(b)]^{-1}, [g(a)]^{-1}) - H(g(b), g(a))| \leq \left(\frac{[g(a)]^{-1} - [g(b)]^{-1}}{2}\right)\left(\frac{4}{p+1}\right)^{\frac{1}{p}} H^{-1}(|a|^{-2}, |b|^{-2}). \tag{4.14}$$

Proposition 4.22. *Let $g(a), g(b) \in \mathbb{R}$ with $g(a) < g(b)$ and $0 \neq [g(a), g(b)]$, $p > 1$, then we have the following result*

$$|\bar{L}^{-1}(g(a), g(b)) - A^{-1}(g(a), g(b))| \\ \leq \left(\frac{g(b) - g(a)}{8}\right)\left(\frac{1}{3}\right)^{\frac{p-1}{p}} \left[\left(|a|^{\frac{-2p}{p-1}} + 2|b|^{\frac{-2p}{p-1}}\right)^{\frac{p-1}{p}} + \left(2|a|^{\frac{-2p}{p-1}} + |b|^{\frac{-2p}{p-1}}\right)^{\frac{p-1}{p}}\right]. \tag{4.15}$$

Proof. The assertion follows from Theorem 3.8 applied for $f(g(x)) = \frac{1}{g(x)}$, $g(x) \in [g(a), g(b)]$. □

Remark 4.23. *If we make substitution $g(a) \to [g(b)]^{-1}$ and $g(b) \to [g(a)]^{-1}$ in Proposition 4.22, then we have*

Corollary 4.24. *Let $g(a), g(b) \in \mathbb{R}\backslash\{0\}$, with $g(a) < g(b)$ then $[g(b)]^{-1} < [g(a)]^{-1}$. For $n \in N$, $n \geq 2$, $\forall p > 1$, we have*

$$|\bar{L}^{-1}([g(b)]^{-1}, [g(a)]^{-1}) - H(g(b), g(a))| \\ \leq \left(\frac{[g(a)]^{-1} - [g(a)]^{-1}}{8}\right)\left(\frac{1}{3}\right)^{\frac{p-1}{p}} \left[\left(|b|^{\frac{2p}{p-1}} + 2|a|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}} + \left(2|b|^{\frac{2p}{p-1}} + |a|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}\right]. \tag{4.16}$$

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